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Tribonacci Numbers and the Brocard-Ramanujan Equation

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Abstract

Let $(T_n)_{n\geq 0}$ be the Tribonacci sequence defined by the recurrence $T_{n+2} = T_{n+1} + T_n + T_{n-1}$, with $T_0 = 0$ and $T_1 = T_2 = 1$. In this short note, we prove that there are no integer solutions (u, m) to the Brocard-Ramanujan equation $m! + 1 = u^2$ where u is a Tribonacci number.

1 Introduction

In the past few years, several authors have considered Diophantine equations involving factorial numbers. For instance, Erdős and Selfridge [6] proved that $n!$ is a perfect power only when $n = 1$. However, the most famous among such equations was posed by Brocard [5] in 1876 and independently by Ramanujan ([17], [18, p. 327]) in 1913. The Diophantine equation

$$m! + 1 = u^2 \tag{1}$$

is now known as *Brocard-Ramanujan Diophantine equation*.

It is a simple matter to find the three known solutions, namely $m = 4$, 5 and 7. Recently, Berndt and Galway [2] showed that there are no further solutions with $m \leq 10^9$. An interesting contribution to the problem is due to Overholt [15], who showed that the equation

(1) has only finitely many solutions if we assume a weak version of the abc conjecture. However, the Brocard-Ramanujan equation is still an open problem.

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* (sequence A000045 in the OEIS [19]) given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$.

A number of mathematicians have been interested in Diophantine equations that involve both factorial and Fibonacci numbers. For example, Grossman and Luca [8] showed that if k is fixed, then there are only finitely many positive integers n such that

$$F_n = m_1! + m_2! + \cdots + m_k!$$

holds for some positive integers $m_1, \ldots, m_k$. Also, all the solutions for the case $k \leq 2$ were determined. Later, Bollman, Hernández and Luca [3] solved the case $k = 3$. In a recent paper, Luca and Siksek [11] found all factorials expressible as the sum of at least three Fibonacci numbers.

In 1999, Luca [10] proved that F_n is a product of factorials only when $n = 1, 2, 3, 6$ and 12. Also, Luca and Stănică [12] showed that the largest product of distinct Fibonacci numbers which is a product of factorials is $F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$.

In 2012, Marques [13] proved that $(m, u) = (4, 5)$ is the only solution of Eq. (1) where u is a Fibonacci number. His proof depends on the primitive divisor theorem together with factorizations formulas for $F_n \pm 1$.

Among the several generalizations of Fibonacci numbers, one of the best known is the *Tribonacci* sequence $(T_n)_{n\geq 0}$ (sequence A000073 in the OEIS). This is defined by the recurrence $T_{n+1} = T_n + T_{n-1} + T_{n-2}$, with initial values $T_0 = 0$ and $T_1 = T_2 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705.$$

Tribonacci numbers have a long history. They were first studied in 1914 by Agronomof [1] and subsequently by many others. The name Tribonacci was coined in 1963 by Feinberg [7].

Here, we are interested in solutions (m, u) of the Brocard-Ramanujan equation where u is a Tribonacci number. We point out that in this we have neither a primitive divisor theorem for T_n nor a factorization formula for $T_n \pm 1$.

More precisely, we shall prove the following theorem.

Theorem 1. *There is no solution* (m, u) *for the Brocard-Ramanujan equation* (1)*, where* u *is a Tribonacci number.*

The idea behind the proof is as follows. The equation we are interested in solving is $m! = (T_n - 1)(T_n + 1)$. The 2-adic valuation of $m!$ is $m + O(\log m)$. We show that the 2-adic valuation of $(T_n - 1)(T_n + 1)$ is $\lessapprox 8 \log n / \log 2$. Thus $m \lessapprox 8 \log n / \log 2$. This forces $m!$ to be smaller than $(T_n - 1)(T_n + 1)$, for m and n sufficiently large, which allows us to complete the proof.

2 The proof of Theorem 1

2.1 A key lemma

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. The p-adic order of a Fibonacci number was completely characterized by Lengyel [9]. Also, very recently the 2-adic order of Tribonacci numbers was made explicit by Lengyel and Marques [14]. Here, we shall prove the following key result which will play an important role in the proof of Theorem 1.

Lemma 2. *We have*

$$\nu_2(T_n+1)=\begin{cases}15, & \text{if } n=61;\\ 0, & \text{if } n\equiv 0,3 \pmod 4;\\ 1, & \text{if } n\equiv 1,2,6 \pmod 8;\\ 3, & \text{if } n\equiv 5 \pmod{16};\\ \nu_2((n+3)^2)-3, & \text{if } n\equiv 13,29,45 \pmod{64};\\ \nu_2((n-61)(n+3))-3, & \text{if } n>61 \text{ and } n\equiv 61 \pmod{64}.\end{cases}$$

and, for $n \geq 5$,

$$\nu_2(T_n-1)=\begin{cases}0, & \text{if } n\equiv 0,3 \pmod 4;\\ 1, & \text{if } n\equiv 5 \pmod 8;\\ \nu_2(n+2)-1, & \text{if } n\equiv 6 \pmod 8;\\ \nu_2(n-2)-1, & \text{if } n\equiv 2 \pmod 8;\\ \nu_2((n-1)(n+7))-3, & \text{if } n\equiv 1 \pmod 8.\end{cases}$$

The case T_n-1:

First, note that Lengyel and Marques [14] proved that T_n-1 is odd for every $n\equiv 0,3 \pmod 4$, which proves the first case. Now, note that, in order to prove the second case, it suffices to prove that $T_n\equiv 3 \pmod 4$. In this case, we have $n=8k+5$, with $k\geq 0$. Then we proceed on induction on k. For $k=0$, it follows directly, since $T_5-1=7-1=6=2\cdot 3$. So, we suppose that $T_{8k+5}\equiv 3 \pmod 4$. Using the sum formula for T_n (proved by Feng [16]), we have that

$$\begin{aligned} T_{8(k+1)+5} &= T_{(8k+5)+8}\\ &= T_6T_{8k+5}+(T_6+T_5)T_{8k+6}+T_7T_{8k+7}\\ &= 13T_{8k+5}+20T_{8k+6}+24T_{8k+7}\\ &\equiv 3 \pmod 4. \end{aligned}$$

In the third case, for $t \geq 6$ and $s \geq 1$ odd, we write $n = 2^{t-3}s + 2$. Now, by a Lengyel and Marques result [14, Lemma 3.1], we have that

$$\begin{aligned} T_{2^{t-3}s+2} &= T_{2^{t-3}s+1} + T_{2^{t-3}s} + T_{2^{t-3}s-1} \\ &\equiv 1 + 2^{t-4} + 0 \pmod{2^{t-3}} \\ &\equiv 1 + 2^{t-4} \pmod{2^{t-3}}. \end{aligned}$$

This yields $\nu_2(T_n - 1) = t - 4 = \nu_2(2^{t-3}s) - 1 = \nu_2(n-2) - 1$.

The fourth case follows by proceeding in the same way as the third one. For $t \geq 6$ and $s \geq 1$ odd, we write $n = 2^{t-3}s - 2$. Then, by the Lengyel and Marques result [14, Lemma 3.1], we have that

$$\begin{aligned} T_{2^{t-3}s-2} &= T_{2^{t-3}s+1} - T_{2^{t-3}s} - T_{2^{t-3}s-1} \\ &\equiv 1 - 2^{t-4} - 0 \pmod{2^{t-3}} \\ &\equiv 1 + 2^{t-4} \pmod{2^{t-3}}. \end{aligned}$$

This yields $\nu_2(T_n - 1) = t - 4 = \nu_2(2^{t-3}s) - 1 = \nu_2(n+2) - 1$.

Now, for the last case, we know that 16 divides exactly one among $n - 1$ and $n + 7$. Suppose that $16|(n+a)$, for some $a \in \{-1, 7\}$. Then $\nu_2(n+b) = 3$ for $b \in \{-1, 7\} \setminus \{a\}$. So, we desire to prove that

$$\nu_2(T_n - 1) = \nu_2(n + a).$$

For that, we write $n = 2^{t-2}s - a$, for $t \geq 5$ and $s \geq 1$ odd, and proceed as in Lengyel and Marques [14, Lemma 3.1] to prove that

$$T_{2^{t-2}s-a} - 1 \equiv 2^{t-2} \pmod{2^{t-1}}.$$

Therefore

$$\nu_2(T_n - 1) = t - 2 = \nu_2(n + a) + 1,$$

and this completes the proof.

□

The case $T_n + 1$:

The first two cases are trivial. The third and the fourth cases follow in the same way. Note that, in order to prove them, it suffices to show that $T_n \equiv 1 \pmod 4$ when $n \equiv 1, 2, 6 \pmod 8$ and to show that $T_n = 7 \pmod{16}$ when $n \equiv 5 \pmod{16}$. In order to avoid unnecessary repetitions, we shall prove only one of these cases. So, let us write $n = 8k + 6$ and apply induction on $k \geq 0$. For $k = 0$, it follows directly, since $T_6 + 1 = 13 + 1 = 14 = 2 \cdot 7$. Now, suppose that $T_{8k+6} \equiv 1 \pmod 4$. Then, we have that

$$\begin{aligned} T_{8(k+1)+6} &= T_{(8k+6)+8} \\ &= T_6 T_{8k+6} + (T_6 + T_5) T_{8k+7} + T_7 T_{8k+8} \\ &= 13 T_{8k+6} + 20 T_{8k+7} + 24 T_{8k+8} \\ &\equiv 1 \pmod 4. \end{aligned}$$

Now, for the the fifth case, note that, if $n = 64k + 13$,

$$\begin{aligned}\nu_2((n+3)^2) - 3 &= 2\nu_2(n+3) - 3 \\ &= 2\nu_2(64k+13+3) - 3 = 2\nu_2(16(4k+1)) - 3 = 2\cdot 4 - 3 \\ &= 5.\end{aligned}$$

So, it suffices to prove that $T_n \equiv 31 \pmod{64}$. Again, we proceed on induction. First, observe that $T_{13} = 927 \equiv 31 \pmod{64}$. Now, we have that

$$\begin{aligned}T_{64(k+1)+13} &= T_{(64k+13)+64} \\ &= T_{62}T_{64k+13} + (T_{62} + T_{61})T_{64k+14} + T_{63}T_{64k+15} \\ &\equiv -1 + 32T_{64k+14} \pmod{64}.\end{aligned}$$

But, from the previous case, we have that $T_{64k+14} \equiv 1 \pmod 4$. Then,

$$\begin{aligned}T_{64(k+1)+13} &\equiv -1 + 32T_{64k+14} \pmod{64} \\ &\equiv 32 - 1 \pmod{64} \\ &\equiv 31 \pmod{64}.\end{aligned}$$

When $n \equiv 29, 45 \pmod{64}$, we proceed in the same way.

For the last case, we proceed as for the last case of the previous theorem. Note that 128 divides exactly one among $n-61$ and $n+3$. Suppose that $128|(n+a)$, for some $a \in \{-61, 3\}$. Then $\nu_2(n+b) = 6$ for $b \in \{-61, 3\} \setminus \{a\}$. So, we desire to prove that

$$\nu_2(T_n + 1) = \nu_2(n+a) + 3.$$

For that, we write $n = 2^{t-2}s - a$, for $t \geq 8$ and $s \geq 1$ odd, and proceed as in Lengyel and Marques [14, Lemma 3.1] to prove that

$$T_{2^{t-2}s-a} + 1 \equiv 2^{t+1} \pmod{2^{t+2}}.$$

Therefore

$$\nu_2(T_n + 1) = t + 1 = \nu_2(n+a) + 3.$$

This completes the proof. □

Now, we are ready to deal with the proof of the main theorem.

2.2 The proof

If $n \leq 61$, a straightforward search shows that there are no solutions. So we shall suppose that $n > 61$. Then $m \geq 30$. Next we use the fact that $\nu_2(m!) \geq m - \lfloor \log m / \log 2 \rfloor - 1$ (which is a consequence of the De Polignac's formula) together with Lemma 2. Then

$$\begin{aligned}m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 1 &\leq \nu_2(m!) = \nu_2(T_n - 1) + \nu_2(T_n + 1) \\ &< \nu_2((n+2)(n-2)(n-1)(n+7)(n+3)^3(n-61)) + 5 \\ &\leq 8\nu_2(n+\omega) + 5,\end{aligned}$$

for some $\omega \in \{-61, -2, -1, 2, 3, 7\}$. Thus $\nu_2(n+\omega) \geq (m - \lfloor \log m/\log 2 \rfloor - 6)/8$. Therefore, $2^{\lfloor (m-\lfloor \log m/\log 2\rfloor -6)/8\rfloor} \mid (n+\omega)$. In particular, $2^{\lfloor (m-\lfloor \log m/\log 2\rfloor -6)/8\rfloor} \leq |n+\omega| \leq n+61$ (here we used that $n+\omega \neq 0$). By applying the log function, we obtain

$$\left\lfloor \frac{1}{8}\left(m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 6\right) \right\rfloor \leq \frac{\log(n+61)}{\log 2}. \tag{2}$$

On the other hand, $(1.83)^{2n-4} < T_n^2 = m! + 1 < 2(m/2)^m$ (the first inequality was proved by Bravo and Luca [4]). So $n < 0.9m\log(m/2) + 2.6$. Substituting this in equation (2), we obtain

$$\left\lfloor \frac{1}{8}\left(m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 6\right) \right\rfloor \leq \frac{\log(0.9m\log(m/2) + 63.6)}{\log 2}.$$

This inequality yields $m \leq 78$. Then $n < 0.9 \cdot 78 \log(78/2) + 2.6 = 259.782\ldots$. Now, we use a simple routine written in *Mathematica* which does not return any solution in the range $30 \leq m \leq 78$ and $62 \leq n \leq 259$. The proof is complete. $\square$

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