Journal of Integer Sequences, Vol. 19 (2016),
Article 16.8.8

Mode and Edgeworth Expansion for the Ewens Distribution and the Stirling Numbers

Zakhar Kabluchko
Institut für Mathematische Statistik
Universität Münster
Orléans–Ring 10
48149 Münster
Germany
zakhar.kabluchko@uni-muenster.de

Alexander Marynych
Faculty of Cybernetics
Taras Shevchenko National University of Kyiv
01601 Kyiv
Ukraine
marynych@unicyb.kiev.ua

Henning Sulzbach
School of Computer Science
McGill University
3480 University Street
Montréal, QC H3A 0E9
Canada
henning.sulzbach@gmail.com

Abstract

We provide asymptotic expansions for the Stirling numbers of the first kind and, more generally, the Ewens (or Karamata-Stirling) distribution. Based on these expansions, we obtain some new results on the asymptotic properties of the mode and the maximum of the Stirling numbers and the Ewens distribution. For arbitrary $\theta > 0$ and for all sufficiently large $n \in \mathbb{N}$, the unique maximum of the Ewens probability mass function

$$\mathbb{L}_n(k) = \frac{\theta^k}{\theta(\theta+1)\cdots(\theta+n-1)} \begin{bmatrix} n \\ k \end{bmatrix}, \quad k = 1, \dots, n,$$

is attained at $k = \lfloor a_n \rfloor$ or $\lceil a_n \rceil$, where $a_n = \theta \log n - \theta \Gamma'(\theta)/\Gamma(\theta) - 1/2$. We prove that the mode is the nearest integer to a_n for a set of n's of asymptotic density 1, yet this formula is not true for infinitely many n's.

1 Introduction and statement of results

1.1 Introduction

The (unsigned) *Stirling numbers* of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ are defined, for $n \in \mathbb{N}$ and $1 \le k \le n$, by the formula

$$x^{(n)} := x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad x \in \mathbb{R}. \tag{1}$$

For $n \in \mathbb{N}$, a random variable $K_n(\theta)$ is said to have the *Ewens distribution* with parameter $\theta > 0$ if its probability mass function is given by the formula

$$\mathbb{P}(K_n(\theta) = k) = \frac{\theta^k}{\theta^{(n)}} \begin{bmatrix} n \\ k \end{bmatrix}, \quad k = 1, \dots, n.$$

Bingham [2] called this distribution the *Karamata-Stirling law*. One can interpret $K_n(\theta)$ as the number of blocks in a random partition of $\{1, \dots, n\}$ distributed according to the Ewens sampling formula, or, equivalently, the number of different alleles in the infinite alleles model, the number of tables in a Chinese restaurant process, or the number of colors in the Hoppe urn. The Ewens sampling formula plays an important role in population genetics [6], [4, Section 1.3]. There is a distributional representation of $K_n(\theta)$ as a sum of independent random variables

$$K_n(\theta) \stackrel{d}{=} \xi_1 + \cdots + \xi_n, \text{ where } \xi_i \sim \mathrm{Bern}(\theta/(\theta+i-1))$$

and $\mathrm{Bern}(p)$ denotes the Bernoulli distribution with parameter p. In the special case $\theta = 1$, classical results going back at least to Feller [7] and Rényi [21] state that the random variable $K_n(1)$ has the same distribution as the number of cycles in a uniformly chosen random permutation of n objects, or the number of records in a sample of n i.i.d. variables from a continuous distribution. It follows easily from Lindeberg's theorem that $K_n(\theta)$ satisfies a central limit theorem of the form

$$\frac{K_n(\theta) - \theta \log n}{\sqrt{\theta \log n}} \xrightarrow[n\to\infty]{d} \mathrm{N}(0,1)$$

known as Goncharov's CLT in the case $\theta = 1$.

Asymptotic expansions, as $n \to \infty$, of the Stirling numbers $\left[{n \atop k}\right]$ in various regions of k were provided in numerous works [12, 13, 18, 20, 23, 24]. Most notably, Hwang [13, Theorem 2] (and Theorem 14 on page 108 of his dissertation [12] for a more general result) gave an asymptotic expansion valid uniformly in the domain $2 \le k \le \eta \log n$, for any fixed $\eta > 0$. Louchard [18, Theorem 2.1] computed three non-trivial terms of the asymptotic expansion in the central regime $k = \log n + O(\sqrt{\log n})$ which is similar to the classical Edgeworth expansion in the central limit theorem.

In this short note we start by deriving a full Edgeworth expansion, as $n \to \infty$, for the sequence of probability mass functions $k \mapsto \mathbb{P}(K_n(\theta) = k)$ which is uniform both in $\theta \in [1/\eta, \eta]$ (where $\eta > 1$) and in $k \in \{1, \dots, n\}$; see Theorem 1. Our result is an application of the general Edgeworth expansion for deterministic or random profiles which the authors [16] recently obtained. Using this asymptotic expansion we derive some new results on the mode and the maximum of the Ewens distribution. In the case $\theta = 1$ the mode can be interpreted as the most probable number of cycles in a random permutation of n objects. It was investigated in the works of Hammersley [10, 11] and Erdős [5]. Our results on the mode and the maximum will be stated in Theorems 5 and 7 below.

1.2 Asymptotic expansion of the Ewens distribution

Before stating our main result we need to recall some notions. The (complete) *Bell polynomials* $B_j(z_1, \dots, z_j)$ are defined by the formal identity

$$\exp\left(\sum_{j=1}^{\infty} \frac{x^j}{j!} z_j\right) = \sum_{j=0}^{\infty} \frac{x^j}{j!} B_j(z_1, \dots, z_j).$$

Therefore $B_0 = 1$ and, for $j \in \mathbb{N}$,

$$B_j(z_1, \dots, z_j) = \sum{}' \frac{j!}{i_1! \cdots i_j!} \left(\frac{z_1}{1!}\right)^{i_1} \cdots \left(\frac{z_j}{j!}\right)^{i_j}, \tag{2}$$

where the sum $\sum{}'$ is taken over all $i_1, \dots, i_j \in \mathbb{N}_0$ satisfying $1i_1 + 2i_2 + \dots + ji_j = j$. For example, the first three Bell polynomials are given by

$$B_1(z_1) = z_1, \quad B_2(z_1, z_2) = z_1^2 + z_2, \quad B_3(z_1, z_2, z_3) = z_1^3 + 3z_1z_2 + z_3. \tag{3}$$

Further, we will use the "probabilist" *Hermite polynomials* $\mathrm{He}_n(x)$ defined by

$$\mathrm{He}_n(x) = e^{\frac{1}{2}x^2} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{-\frac{1}{2}x^2}, \quad n \in \mathbb{N}_0. \tag{4}$$

The first few Hermite polynomials needed for the first three terms of the expansion are

$$\begin{aligned}&\mathrm{He}_0(x)=1,\quad \mathrm{He}_1(x)=x,\quad \mathrm{He}_2(x)=x^2-1,\quad \mathrm{He}_3(x)=x^3-3x,\\&\mathrm{He}_4(x)=x^4-6x^2+3,\quad \mathrm{He}_6(x)=x^6-15x^4+45x^2-15.\end{aligned}$$

Theorem 1. *Fix $r\in\mathbb{N}_0$ and a compact subset $L\subset(0,\infty)$. Uniformly over $\theta\in L$ we have*

$$\lim_{n\to\infty}(\log n)^{\frac{r+1}{2}}\sup_{k=1,\dots,n}\left|\mathbb{P}(K_n(\theta)=k)-\frac{e^{-\frac{1}{2}x_n^2(k,\theta)}}{\sqrt{2\pi\theta\log n}}\sum_{j=0}^{r}\frac{H_j(x_n(k,\theta))}{(\theta\log n)^{j/2}}\right|=0.$$

Here, $x_n(k,\theta)=\frac{k-\theta\log n}{\sqrt{\theta\log n}}$ and $H_j(x)$ is a polynomial of degree $3j$ given by

$$H_j(x):=H_j(x,\theta)=\frac{(-1)^j}{j!}\,e^{\frac{1}{2}x^2}B_j(\widetilde{D_1},\dots,\widetilde{D_j})e^{-\frac{1}{2}x^2},\tag{5}$$

where B_j is the j-th Bell polynomial and $\widetilde{D_1},\widetilde{D_2},\dots$ are differential operators given by

$$\widetilde{D_j}:=\widetilde{D_j}(\theta)=\frac{1}{(j+1)(j+2)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j+2}+\widetilde{\chi_j}(0)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j}\tag{6}$$

with $\widetilde{\chi_j}(\beta)=-\left(\frac{\mathrm{d}}{\mathrm{d}\beta}\right)^j\log\Gamma(\theta e^\beta)$ and Γ denoting the Euler gamma function.

Remark 2. It follows from (3), (5) and (6) that the first three coefficients of the expansion are

$$\begin{aligned}H_0(x)&=1,\\H_1(x)&=-\frac{\Gamma'(\theta)}{\Gamma(\theta)}\theta x+\frac{1}{6}\mathrm{He}_3(x),\\H_2(x)&=\left(\theta^2\frac{\Gamma'^2(\theta)}{\Gamma^2(\theta)}-\frac{\theta^2\Gamma''(\theta)+\theta\Gamma'(\theta)}{2\Gamma(\theta)}\right)\mathrm{He}_2(x)+\left(\frac{1}{24}-\frac{\Gamma'(\theta)}{6\Gamma(\theta)}\theta\right)\mathrm{He}_4(x)\\&\quad+\frac{1}{72}\mathrm{He}_6(x).\end{aligned}$$

An expression for $\widetilde{\chi_j}(0)$ involving polygamma functions and Stirling numbers of the second kind will be given below in (13). The tilde in $\widetilde{D_j}$ and $\widetilde{\chi_j}$ is needed to keep the notation consistent with our more general work [16]. It is easy to check that $H_j(-x)=(-1)^jH_j(x)$ [16, Remark 2.4].

To compute $H_j(x)$ one can proceed as follows. First, express $\frac{1}{j!}B_j(\widetilde{D_1},\dots,\widetilde{D_j})$ as a polynomial in $D:=\frac{\mathrm{d}}{\mathrm{d}x}$ (and note that only even/odd powers of D are present if j is even/odd). Then replace each occurrence of D^l by $\mathrm{He}_l(x)$; see (4) for justification.

Remark 3. It is possible to choose the value of θ as a function of k. One natural choice is $\theta = 1$ which provides a full version of Louchard's expansion [18, Theorem 2.1] (although he used a slightly different normalization in his analogue of $x_n(k, 1)$ and his term $-355x^3/144$ should be replaced by $-47x^3/144$). Another possible choice is $\theta = k/\log n$ (so that $x_n(k, \theta) = 0$), which gives a large-deviation-type expansion valid uniformly in the region $\eta^{-1} \log n < k < \eta \log n$, for fixed $\eta > 1$ and $q \in \mathbb{N}_0$:

$$\frac{(k/\log n)^k}{(k/\log n)^{(n)}} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{\sqrt{2\pi k}} \sum_{s=0}^{q} \frac{H_{2s}(0, k/\log n)}{k^s} + o\left(\frac{1}{(\log n)^{q+1}}\right).$$

Observe that the terms with half-integer powers of k are not present in the sum because $H_{2j+1}(0) = 0$. Using the formula

$$\frac{\Gamma(n+\theta)}{n!} = n^{\theta-1}\left(1 + O\left(\frac{1}{n}\right)\right)$$

yields the expansion

$$\frac{1}{n!}\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{\Gamma(\theta)} n^{\theta - \theta\log\theta - 1}\left(\frac{1}{\sqrt{2\pi k}} \sum_{s=0}^{q} \frac{H_{2s}(0, \theta)}{k^s} + o\left(\frac{1}{(\log n)^{q+1}}\right)\right) \tag{7}$$

valid as $n \to \infty$ uniformly over k in the region $\theta = k/\log n \in (\eta^{-1}, \eta)$. In this region, this expansion must be equivalent to Hwang's result [13, Theorem 2]. It is not easy to rigorously verify this equivalence by a direct comparison, but we checked using Mathematica 9 that the first three non-trivial terms coincide. Note a misprint in the formula for the remainder term $Z_\mu(m, n)$ in Hwang [13, Theorem 2]: $(\log n)^m/(m!n)$ should be replaced by $(\log n)/(mn)$. Expansion (7) could be also deduced from the work of Féray et al. [8, Theorem 3.4].

Taking sums over k in Theorem 1 and using the Euler-Maclaurin formula to approximate Riemann sums by integrals, one obtains that

$$\mathbb{P}\left(\frac{K_n(\theta) - \theta\log n}{\sqrt{\theta\log n}} \le x\right) = \Phi(x) + \frac{e^{-x^2/2}}{\sqrt{2\pi\theta\log n}}\left(\frac{1}{2} - \frac{x^2-1}{6} + \theta\frac{\Gamma'(\theta)}{\Gamma(\theta)}\right) + O\left(\frac{1}{\log n}\right),$$

uniformly in $x \in (\theta\log n)^{-1/2}(\mathbb{Z} - \theta\log n)$, where $\Phi(x)$ is the standard normal distribution function. The proof follows Grübel and Kabluchko [9, Proposition 2.5] and is therefore omitted. Yamato [25] recently stated a slightly incorrect version of this expansion missing the term $1/2$ which comes from the Euler-Maclaurin formula. Similarly, one can obtain further terms in the expansion of the distribution function of $(K_n(\theta) - \theta\log n)/\sqrt{\theta\log n}$.

Remark 4. Since the set $L \subseteq (0, \infty)$ in Theorem 1 has to be chosen compact, our results do not yield asymptotic expansions for $\mathbb{P}(K_n(\theta) = k)$ in the regime $k = o(\log n)$ of the same precision as Hwang's [13, Theorems 1 and 2]. Also, they do not extend straightforwardly to the case $k = n - O(n^\alpha)$ for $0 < \alpha < 1$ treated by Louchard [18, Section 3]. A generalization of our approach to cover these regions will be content of future work.

1.3 Mode and maximum of the Ewens distribution

Theorem 1 allows us to deduce various results on the *mode* and the *maximum* of the Ewens distribution. A mode is any value $k \in \{1, \dots, n\}$ maximizing $\mathbb{P}(K_n(\theta) = k)$, while the maximum $M_n(\theta)$ is defined by

$$M_n(\theta) = \max_{1 \le k \le n} \mathbb{P}(K_n(\theta) = k).$$

Let $u_n(\theta)$ denote the least mode. In this context, it is important to note that, for all $\theta > 0$, the function $k \mapsto \mathbb{P}(K_n(\theta) = k)$ is log-concave by a theorem attributed to Newton [11, 22], and

$$\begin{aligned} \mathbb{P}(K_n(\theta) = 1) < \ldots < \mathbb{P}(K_n(\theta) = u_n(\theta)) \\ \ge \mathbb{P}(K_n(\theta) = u_n(\theta) + 1) > \ldots > \mathbb{P}(K_n(\theta) = n). \end{aligned} \tag{8}$$

In particular, there are at most two modes. For $\theta = 1$, Erdős [5], proving a conjecture of Hammersley [11], showed that the mode is unique for all $n \ge 3$. By (8), uniqueness also holds for irrational θ; however, for rational θ, the mode need not be unique since, for example,

$$\frac{2}{3}\begin{bmatrix}3\\1\end{bmatrix} = \left(\frac{2}{3}\right)^2 \begin{bmatrix}3\\2\end{bmatrix} > \left(\frac{2}{3}\right)^3 \begin{bmatrix}3\\3\end{bmatrix}.$$

Theorem 5. *Fix $\theta > 0$. There exists $N_1 \in \mathbb{N}$ such that for $n \ge N_1$, $u_n(\theta)$ is the unique mode of the Ewens distribution with parameter θ. The mode $u_n(\theta)$ equals one of the numbers $\lfloor u_n^*(\theta) \rfloor$ or $\lceil u_n^*(\theta) \rceil$, where*

$$u_n^*(\theta) = \theta \log n - \frac{\theta \Gamma'(\theta)}{\Gamma(\theta)} - \frac{1}{2}$$

and $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ denote the floor and the ceiling functions, respectively. Write $\delta_n(\theta) := \min_{k \in \mathbb{Z}} |u_n^*(\theta) - k|$. For the maximum $M_n(\theta)$, we have

$$\sqrt{2\pi\theta \log n}\, M_n(\theta) = 1 + \frac{\theta(\log \Gamma)'(\theta) + \theta^2 (\log \Gamma)''(\theta) + 1/12 - \delta_n^2(\theta)}{2\theta \log n} + o\left(\frac{1}{\log n}\right).$$

In the case $\theta = 1$, Hammersley [11] and Erdős [5] derived related results for the mode. Cramer [3] discusses statistical applications and Mező [19] provides an overview and generalizations. Theorem 5 states that the mode is one of the numbers $\lfloor \log n + \gamma - \frac{1}{2} \rfloor$ or $\lceil \log n + \gamma - \frac{1}{2} \rceil$, for sufficiently large n. In fact, this holds for all $n \in \mathbb{N}$.

Proposition 6. $u_n(1) \in \{\lfloor \log n + \gamma - \frac{1}{2} \rfloor, \lceil \log n + \gamma - \frac{1}{2} \rceil\}$ *for all $n \in \mathbb{N}$.*

The proof uses the following formula of Hammersley [11]:

$$u_n(1) = \left\lfloor \log n + \gamma + \frac{\zeta(2) - \zeta(3)}{\log n + \gamma - \frac{3}{2}} + \frac{h(n)}{(\log n + \gamma - \frac{3}{2})^2} \right\rfloor, \tag{9}$$

for some $-1.098011 < h(n) < 1.430089$. Hwang [12, Section 5.7.9] gives a more precise expansion. Erdős [5] observed that, for $n > 189$, Hammersley's formula implies that the mode is one of the numbers $\lfloor \log(n-1) + \frac{1}{2} \rfloor$ or $\lfloor \log(n-1) + 1 \rfloor$. Note that his $\Sigma_{n,s}$ equals $\left[{n+1 \atop n+1-s} \right]$ and his $n - f(n)$ is $u_{n+1}(1) - 1$ in our notation.

The next theorem provides more precise information about the behavior of the mode. Recall that a set $A \subset \mathbb{N}$ has *asymptotic density* $\alpha \in [0,1]$ if

$$\lim_{n\to\infty} \frac{\#(A \cap \{1, \ldots, n\})}{n} = \alpha.$$

For $x \in \mathbb{R}$, let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Let $\mathrm{nint}(x)$ be the integer closest to x (if $\{x\} = 1/2$, we agree to take $\mathrm{nint}(x) = \lceil x \rceil$). That is,

$$\mathrm{nint}(x) := \arg\min_{k\in\mathbb{Z}} |x - k| = \left\lfloor x + \frac{1}{2} \right\rfloor.$$

Theorem 7. *Fix $\theta > 0$. The mode $u_n(\theta)$ of the Ewens distribution with parameter θ has the following properties:*

(i) *there exists a constant $C_0 > 0$ such that, for all $n \in \mathbb{N}$ satisfying*

$$\left| \{u_n^*(\theta)\} - \frac{1}{2} \right| > \frac{C_0}{\log n},$$

the mode $u_n(\theta)$ equals

$$\mathrm{nint}(u_n^*(\theta)) = \left\lfloor \theta \log n - \frac{\theta \Gamma'(\theta)}{\Gamma(\theta)} \right\rfloor;$$

(ii) *there are arbitrarily long intervals of consecutive n's for which $u_n(\theta) = \lceil u_n^*(\theta) \rceil$; similarly, there are arbitrarily long intervals of consecutive n's for which $u_n(\theta) = \lfloor u_n^*(\theta) \rfloor$;*

(iii) *the set of $n \in \mathbb{N}$ such that $u_n(\theta) = \mathrm{nint}(u_n^*(\theta))$ has asymptotic density one;*

(iv) *there are infinitely many $n \in \mathbb{N}$ such that $u_n(\theta) \neq \mathrm{nint}(u_n^*(\theta))$.*

The proof of part (iv) uses five terms in the Edgeworth expansion, where the first two terms influence the form of $u_n^*(\theta)$, while the remaining terms are needed for technical reasons. The idea is that the formula $u_n(\theta) = \mathrm{nint}(u_n^*(\theta))$ becomes wrong if the fractional part of $u_n^*(\theta)$ is slightly below $\frac{1}{2}$, so that higher order terms in the Edgeworth expansion decide which of the values $\lfloor u_n^*(\theta) \rfloor$ and $\lceil u_n^*(\theta) \rceil$ is the mode. Using even more terms in the expansion, it is possible to replace $u_n^*(\theta)$ by some more complicated expressions involving higher-order corrections in inverse powers of $\theta \log n$ [12, Section 5.7.9]. However, it seems that there is no formula of the form

$$u_n(1) = \mathrm{nint}\left(\log n + a_0 + \frac{a_1}{\log n} + \cdots + \frac{a_r}{(\log n)^r} \right)$$

which is valid for all sufficiently large n.

Finally, we would like to mention that one can easily obtain counterparts of the above results for the B- and D-analogues of Stirling numbers of the first kind. These are defined as the coefficients of $(x+1)(x+3)\cdots(x+2n-1)$ and $((x+1)(x+3)\cdots(x+2n-3))(x+n-1)$, respectively. They appear, for example, in the study of intrinsic volumes of Weyl chambers [14].

2 Proofs

Proof of Theorem 1. The proof follows from the general Edgeworth expansion for random or deterministic profiles [16, Theorem 2.1]. We consider the sequence of "profiles"

$$\mathbb{L}_n(k) := \mathbb{P}(K_n(\theta) = k) = \frac{\theta^k}{\theta^{(n)}} \begin{bmatrix} n \\ k \end{bmatrix} \mathbb{1}_{\{k\in\{1,\dots,n\}\}},$$

and define

$$w_n := \theta \log n, \quad \varphi(\beta) := e^{\beta} - 1, \quad (\beta_-, \beta_+) = \mathbb{R}, \quad \mathscr{D} = \{z \in \mathbb{C} \colon |\mathrm{Im}\, z| < \pi\}.$$

In order to apply [16, Theorem 2.1], we need to check Conditions A1–A4 given in the beginning of Section 2 of the cited paper. Note that

$$\begin{aligned} W_n(\beta) &:= e^{-\varphi(\beta) w_n} \sum_{k\in\mathbb{Z}} e^{\beta k} \mathbb{L}_n(k) = n^{-\theta(e^{\beta}-1)} \sum_{k=1}^{n} e^{\beta k} \frac{\theta^k}{\theta^{(n)}} \begin{bmatrix} n \\ k \end{bmatrix} \\ &= n^{-\theta(e^{\beta}-1)} \frac{(\theta e^{\beta})^{(n)}}{\theta^{(n)}} = n^{-\theta(e^{\beta}-1)} \frac{\Gamma(\theta e^{\beta} + n)\Gamma(\theta)}{\Gamma(\theta e^{\beta})\Gamma(\theta+n)} \underset{n\to\infty}{\longrightarrow} \frac{\Gamma(\theta)}{\Gamma(\theta e^{\beta})} =: W_\infty(\beta) \end{aligned}$$

locally uniformly in $\beta \in \mathscr{D}$ with a rate of convergence which is polynomial in n^{-1}. Hence Conditions A1–A3 are satisfied. In order to check A4 it is enough to show that for every $a > 0$, $r \in \mathbb{N}$ and every compact subset K_1 of $\mathbb{R}$

$$\sup_{\beta\in K_1} \sup_{a\le u\le\pi} \left(n^{-\theta(e^{\beta}-1)} \left| \frac{\Gamma(\theta e^{\beta+iu} + n)\Gamma(\theta)}{\Gamma(\theta+n)\Gamma(\theta e^{\beta+iu})} \right| \right) = o(\log^{-r} n), \quad n \to \infty.$$

But this easily follows from

$$\begin{aligned} &\sup_{\beta\in K_1} \left(n^{-\theta(e^{\beta}-1)} \sup_{a\le u\le\pi} \left| \frac{\Gamma(\theta e^{\beta+iu} + n)\Gamma(\theta)}{\Gamma(\theta+n)\Gamma(\theta e^{\beta+iu})} \right| \right) \\ &\le C \sup_{\beta\in K_1} \left(n^{-\theta(e^{\beta}-1)} \sup_{a\le u\le\pi} \left| \frac{\Gamma(\theta e^{\beta+iu} + n)}{\Gamma(\theta+n)} \right| \right) \le C_1 \sup_{\beta\in K_1} n^{\theta e^{\beta}(\cos a - 1)}, \end{aligned}$$

with constants C, C_1 depending on K_1, θ and a. Therefore, Theorem 2.1 of [16] is applicable for the Ewens distribution with arbitrary fixed $\theta > 0$. In particular, for $\theta = 1$, we obtain

$$(\log n)^{\frac{r+1}{2}} \sup_{\beta \in K} \sup_{1 \le k \le n} \left| \frac{\Gamma(e^\beta) e^{\beta k}}{n^{e^\beta - 1} n!} \begin{bmatrix} n \\ k \end{bmatrix} - \frac{e^{-\frac{1}{2} x_n^2(k, e^\beta)}}{\sqrt{2\pi e^\beta \log n}} \sum_{j=0}^{r} \frac{G_j(x_n(k, e^\beta); \beta)}{(\log n)^{j/2}} \right| \underset{n\to\infty}{\longrightarrow} 0, \tag{10}$$

where K is a compact subset of $\mathbb{R}$, and the polynomials $G_0, G_1, \ldots$ are defined as in Theorem 2.1 of [16]: for $j \in \mathbb{N}_0$, we have

$$G_j(x; \beta) = \frac{(-1)^j}{j!} e^{\frac{1}{2}x^2} B_j(D_1, \ldots, D_j) e^{-\frac{1}{2}x^2} \tag{11}$$

with the differential operators

$$D_j := D_j(\beta) = e^{-\frac{1}{2}\beta j} \left(\frac{1}{(j+1)(j+2)} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{j+2} + \chi_j(\beta) \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{j} \right), \tag{12}$$

where

$$\chi_j(\beta) = -\left(\frac{\mathrm{d}}{\mathrm{d}\beta} \right)^j \log \Gamma(e^\beta).$$

Now, if $L \subseteq (0, \infty)$ is compact, then $K := \log L$ is compact in $\mathbb{R}$. Applying (10) with $K = \log L$ and $\beta = \log \theta \in K$, we obtain

$$(\log n)^{\frac{r+1}{2}} \sup_{\theta \in L} \sup_{1 \le k \le n} \left| \frac{\Gamma(\theta)\theta^k}{n^{\theta - 1} n!} \begin{bmatrix} n \\ k \end{bmatrix} - \frac{e^{-\frac{1}{2} x_n^2(k, \theta)}}{\sqrt{2\pi \theta \log n}} \sum_{j=0}^{r} \frac{G_j(x_n(k, \theta); \log \theta)}{(\log n)^{j/2}} \right| \underset{n\to\infty}{\longrightarrow} 0.$$

By Stirling's formula, uniformly in $\theta \in L$, $n \in \mathbb{N}$ and $1 \le k \le n$, we have

$$\frac{\Gamma(\theta)\theta^k}{n^{\theta-1} n!} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{\theta^k}{\theta^{(n)}} \begin{bmatrix} n \\ k \end{bmatrix} (1 + O(n^{-1})) = \frac{\theta^k}{\theta^{(n)}} \begin{bmatrix} n \\ k \end{bmatrix} + O(n^{-1}).$$

We conclude the proof by noting that $G_j(x; \log \theta) = \theta^{-j/2} H_j(x)$ which follows directly from $\widetilde{\chi_j}(0) = \chi_j(\log \theta)$. Indeed, by comparing (6) and (12), we obtain

$$D_j(\log \theta) = \theta^{-j/2} \widetilde{D_j}(\theta),$$

which implies that

$$B_j(D_1(\log \theta), \ldots, D_j(\log \theta)) = \theta^{-j/2} B_j(\widetilde{D_1}(\theta), \ldots, \widetilde{D_j}(\theta))$$

since $B_j(z_1, \ldots, z_j)$ is a sum of terms of the form $c \cdot z_1^{i_1} z_2^{i_2} \cdots z_j^{i_j}$ with $1i_1 + 2i_2 + \cdots + j i_j = j$; see (2). Comparing (5) and (11), we obtain the required identity $G_j(x; \log \theta) = \theta^{-j/2} H_j(x)$.

To see that $\widetilde{\chi}_j(0) = \chi_j(\log\theta)$, one can easily show by induction over $j \geq 1$ that, both

$$\chi_j(\beta) = -\sum_{\ell=1}^{j} \left\{ {j \atop \ell} \right\} \psi^{(\ell-1)}(e^\beta) e^{\ell\beta},$$

and

$$\widetilde{\chi}_j(\beta) = -\sum_{\ell=1}^{j} \left\{ {j \atop \ell} \right\} \psi^{(\ell-1)}(\theta e^\beta)(\theta e^\beta)^\ell. \tag{13}$$

Here $\psi^{(j)}(x) = (\log\Gamma(x))^{(j+1)}$ denotes the polygamma function and $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind satisfying the recurrence

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\}, \quad 1 \leq k \leq n, \quad n \in \mathbb{N},$$

with initial conditions $\left\{ {0 \atop 0} \right\} = 1$, $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$. $\square$

Proof of Theorem 5. It follows from Theorems 2.10 in [16] that for sufficiently large n, the maximizers of the function $k \mapsto \mathbb{P}(K_n(\theta) = k)$ must be of the form $\lfloor u_n^* \rfloor$ or $\lceil u_n^* \rceil$.

Next we prove that the maximizer is unique (for sufficiently large n) by following a method of Erdős [5] who considered the case $\theta = 1$. Thanks to (8), the uniqueness is evident if θ is irrational. Hence, we assume that $\theta = Q_1/Q_2$ is rational with Q_1, Q_2 being integer. We have, by (1),

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{1 \leq a_1 < \ldots < a_{n-k} \leq n-1} a_1 \cdots a_{n-k}.$$

Put $k_n = \lceil u_n^*(\theta) \rceil = \theta \log n + O(1)$ as $n \to \infty$. By (8), it is sufficient to show that

$$\theta^{k_n} \begin{bmatrix} n \\ k_n \end{bmatrix} \neq \theta^{k_n - 1} \begin{bmatrix} n \\ k_n - 1 \end{bmatrix}. \tag{14}$$

By Erdős' argument relying on the prime number theorem with an appropriate error term [5, p. 233], for all sufficiently large n, there is a prime number p satisfying $(n-1)/k_n < p < (n-1)/(k_n - 1)$. Then,

$$\begin{bmatrix} n \\ k_n \end{bmatrix} \not\equiv 0 \pmod{p}, \qquad \begin{bmatrix} n \\ k_n - 1 \end{bmatrix} \equiv 0 \pmod{p}$$

because in the representation of the former Stirling number all products except one are divisible by p, whereas in the latter all products are divisible by p. If n is large, p is not among the prime factors of Q_1 and Q_2. Hence (14) follows and the mode of $K_n(\theta)$ is unique. Finally, the formula for M_n follows from Theorem 2.13 of [16]. $\square$

Proof of Proposition 6. Recall Hammersley's formula (9):

$$u_n(1) = \left\lfloor \log n + \gamma + \frac{\zeta(2)-\zeta(3)}{\log n + \gamma - \frac{3}{2}} + \frac{h(n)}{(\log n + \gamma - \frac{3}{2})^2} \right\rfloor$$

with some $-1.1 < h(n) < 1.44$. It is easy to check that

$$\frac{\zeta(2)-\zeta(3)}{x} - \frac{1.1}{x^2} > -\frac{1}{2} \text{ and } \frac{\zeta(2)-\zeta(3)}{x} + \frac{1.44}{x^2} < \frac{1}{2}$$

for $x > 2.5$. Hence, the proposition is true for $\log n + \gamma - \frac{3}{2} > 2.5$, that is for $n \geq 31$. For $n = 1, 2, \ldots, 30$ the statement is easy to verify using Mathematica 9. $\square$

Proof of Theorem 7 (i) and (ii). Part (i) follows essentially from Theorem 2.10 in [16] and its proof. Namely, by [16, Equation (90)], for $k = k(n) = u_n^*(\theta) + g \in \mathbb{Z}$ with $g = O(1)$, we have

$$\sqrt{2\pi\theta \log n}\,(\mathbb{P}(K_n(\theta) = k+1) - \mathbb{P}(K_n(\theta) = k)) = -\frac{2g+1}{2\theta \log n} + o\left(\frac{1}{\log n}\right).$$

The same relation, but with a better remainder term $O(\frac{1}{\log^2 n})$, follows from (16) which we shall prove below. Taking $g = -\{u_n^*(\theta)\}$, so that $k = \lfloor u_n^*(\theta) \rfloor$ and $k+1 = \lceil u_n^*(\theta) \rceil$, yields

$$\begin{aligned} &\mathbb{P}(K_n(\theta) = \lceil u_n^*(\theta) \rceil) - \mathbb{P}(K_n(\theta) = \lfloor u_n^*(\theta) \rfloor) \\ &\qquad = \frac{1}{\sqrt{2\pi\theta \log n}} \left(\frac{\{u_n^*(\theta)\} - \frac{1}{2}}{\theta \log n} + O\left(\frac{1}{\log^2 n}\right)\right). \end{aligned}$$

It follows that there is a sufficiently large constant $C_0 > 0$ such that, if $\{u_n^*(\theta)\} > \frac{1}{2} + \frac{C_0}{\log n}$, then the right-hand side is positive, and the mode equals $\lceil u_n^*(\theta) \rceil$. Similarly, if $\{u_n^*(\theta)\} < \frac{1}{2} - \frac{C_0}{\log n}$, then the right-hand side is negative, and the mode equals $\lfloor u_n^*(\theta) \rfloor$.

The proof of part (ii) follows immediately from part (i) and the fact that, for every fixed $L > 0$, we have $\log(n+L) - \log n \to 0$ as $n \to \infty$. $\square$

Proof of Theorem 7 (iii). In view of part (i) it suffices to show that

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\#\{1 \leq k \leq n : \mathrm{dist}(u_k^*(\theta), \mathbb{Z} + 1/2) < \varepsilon\}}{n} = 0,$$

which, in turn, follows from the fact that

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\#\{1 \leq k \leq n : \mathrm{dist}(\log k, \alpha\mathbb{Z} + \beta) < \varepsilon\}}{n} = 0, \tag{15}$$

for all $\alpha > 0$ and $\beta \in \mathbb{R}$. Equation (15) would be true if the sequence of fractional parts of $\alpha^{-1}\log k$, $k \in \mathbb{N}$, were uniformly distributed on $[0,1]$. However, the latter claim is unfortunately not true [17, Examples 2.4 and 2.5, pp. 8–9]. Let us prove (15). We have, assuming that $\varepsilon < \alpha/2$,

$$\begin{aligned}
\#\{1 \le k \le n : \operatorname{dist}(\log k, \alpha\mathbb{Z} + \beta) < \varepsilon\} &= \sum_{k=1}^{n} \#\{j \in \mathbb{Z} : \operatorname{dist}(\log k, \alpha j + \beta) < \varepsilon\} \\
&= \sum_{j\in\mathbb{Z}} \#\{1 \le k \le n : e^{\alpha j + \beta - \varepsilon} < k < e^{\alpha j + \beta + \varepsilon}\} \\
&\le \sum_{j\in\mathbb{Z}} \#\{k \in \mathbb{N} : e^{\alpha j + \beta - \varepsilon} \vee 1 \le k \le e^{\alpha j + \beta + \varepsilon} \wedge n\}.
\end{aligned}$$

The summand on the right-hand side is the number of integers in the interval $[e^{\alpha j+\beta-\varepsilon} \vee 1, e^{\alpha j+\beta+\varepsilon} \wedge n]$ (which is empty if either $e^{\alpha j+\beta-\varepsilon} > n$ or $e^{\alpha j+\beta+\varepsilon} < 1$). Hence, it is bounded from above by $\left(e^{\alpha j+\beta+\varepsilon} \wedge n - e^{\alpha j+\beta-\varepsilon} \vee 1 + 1\right)_+$. Therefore,

$$\#\{1 \le k \le n : \operatorname{dist}(\log k, \alpha\mathbb{Z} + \beta) < \varepsilon\} \le \sum_{j\in\mathbb{Z}} \left(e^{\alpha j+\beta+\varepsilon} \wedge n - e^{\alpha j+\beta-\varepsilon} \vee 1 + 1\right)_+ .$$

Further,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \left(e^{\alpha j+\beta+\varepsilon} \wedge n - e^{\alpha j+\beta-\varepsilon} \vee 1 + 1\right)_+ \\
&= \sum_{j\in\mathbb{Z}} e^{\alpha j+\beta+\varepsilon} \mathbb{1}_{\{\alpha j+\beta+\varepsilon<0\}} + \sum_{j\in\mathbb{Z}} e^{\alpha j+\beta+\varepsilon} \mathbb{1}_{\{\alpha j+\beta-\varepsilon<0, 0\le\alpha j+\beta+\varepsilon<\log n\}} \\
&+ \sum_{j\in\mathbb{Z}} n \mathbb{1}_{\{\alpha j+\beta-\varepsilon<0, \log n\le\alpha j+\beta+\varepsilon\}} \\
&+ \sum_{j\in\mathbb{Z}} \left(e^{\alpha j+\beta+\varepsilon} - e^{\alpha j+\beta-\varepsilon} + 1\right) \mathbb{1}_{\{\alpha j+\beta-\varepsilon\ge 0, \alpha j+\beta+\varepsilon<\log n\}} \\
&+ \sum_{j\in\mathbb{Z}} \left(n - e^{\alpha j+\beta-\varepsilon} + 1\right)_+ \mathbb{1}_{\{\alpha j+\beta-\varepsilon\ge 0, \log n\le\alpha j+\beta+\varepsilon\}}.
\end{aligned}$$

Note that the first series converges, the second contains at most one summand since we assume $\varepsilon < \alpha/2$, and the third vanishes for n large enough. It can be checked that

$$\sum_{j\in\mathbb{Z}} \left(e^{\alpha j+\beta+\varepsilon} - e^{\alpha j+\beta-\varepsilon} + 1\right) \mathbb{1}_{\{\alpha j+\beta-\varepsilon\ge 0, \alpha j+\beta+\varepsilon<\log n\}} \le C(\alpha,\beta)(e^{\beta+\varepsilon} - e^{\beta-\varepsilon})n$$

with an absolute constant $C(\alpha, \beta)$. Further, for n sufficiently large, we have

$$\sum_{j\in\mathbb{Z}} \left(n - e^{\alpha j+\beta-\varepsilon} + 1\right)_+ \mathbb{1}_{\{\alpha j+\beta-\varepsilon\ge 0, \log n\le\alpha j+\beta+\varepsilon\}} \le n(1 - e^{-2\varepsilon}) + 1.$$

Putting pieces together gives (15). $\square$

Proof of Theorem 7 (iv). Recall the notation $w_n = \theta \log n$ and $x_n(k) = x_n(k,\theta) = (k - w_n)/\sqrt{w_n}$. Using Theorem 1 with $r = 4$, we obtain

$$\sqrt{2\pi w_n}\,\mathbb{P}(K_n(\theta) = k) = e^{-\frac{1}{2}x_n^2(k)} \times \left(1 + \frac{H_1(x_n(k))}{w_n^{1/2}} + \frac{H_2(x_n(k))}{w_n} + \frac{H_3(x_n(k))}{w_n^{3/2}} + \frac{H_4(x_n(k))}{w_n^2} + o\left(\frac{1}{\log^2 n}\right)\right),$$

as $n \to \infty$ uniformly in $1 \le k \le n$. Now let $k = \theta \log n + a$, where $a = O(1)$ as $n \to \infty$, so that $x_n(k) = a/w_n^{1/2}$. We have

$$\begin{aligned}
H_1(x_n(k)) &= A_{11}(\theta)\frac{a}{w_n^{1/2}} + A_{12}(\theta)\frac{a^3}{w_n^{3/2}},\\
H_2(x_n(k)) &= A_{21}(\theta) + A_{22}(\theta)\frac{a^2}{w_n} + o\left(\frac{1}{w_n}\right),\\
H_3(x_n(k)) &= A_{31}(\theta)\frac{a}{w_n^{1/2}} + o\left(\frac{1}{w_n^{1/2}}\right),\\
H_4(x_n(k)) &= A_{41}(\theta) + o(1),
\end{aligned}$$

where $A_{11}(\theta), \dots, A_{41}(\theta)$ are some polynomials in $\widetilde{\chi_1}(0), \widetilde{\chi_2}(0), \widetilde{\chi_3}(0)$ and $\widetilde{\chi_4}(0)$; see Remark 2. Plugging these expressions into the asymptotic expansion above and using the expansion $e^y = 1 + y + y^2/2 + o(y^2)$, as $y \to 0$, yields

$$\sqrt{2\pi w_n}\,\mathbb{P}(K_n(\theta) = k) = 1 - \left(\frac{a^2}{2} - A_{11}(\theta)a - A_{21}(\theta)\right)\frac{1}{w_n} + \frac{P_\theta(a)}{w_n^2} + o\left(\frac{1}{\log^2 n}\right),$$

where

$$P_\theta(a) := \frac{1}{8}a^4 + \left(A_{12}(\theta) - \frac{1}{2}A_{11}(\theta)\right)a^3 + \left(A_{22}(\theta) - \frac{1}{2}A_{21}(\theta)\right)a^2 + A_{31}(\theta)a + A_{41}(\theta).$$

Now let us write $k = \theta \log n + a^* + g$, where $a^* := A_{11}(\theta) = -\frac{\theta\Gamma'(\theta)}{\Gamma(\theta)} - \frac{1}{2}$, yielding

$$\sqrt{2\pi w_n}\,\mathbb{P}(K_n(\theta) = k) = 1 - \left(\frac{g^2 - (a^*)^2}{2} - A_{21}(\theta)\right)\frac{1}{w_n} + \frac{P_\theta(a^* + g)}{w_n^2} + o\left(\frac{1}{\log^2 n}\right). \quad (16)$$

We are interested in g being either $\lfloor u_n^*(\theta)\rfloor - u_n^*(\theta) =: g_n'$ or $\lceil u_n^*(\theta)\rceil - u_n^*(\theta) =: g_n''$. Let M be the set of natural numbers n with $\{u_n^*(\theta)\} < 1/2 < \{u_{n+1}^*(\theta)\}$. Note that M has infinitely many elements because $\log n \to \infty$ and $\log(n+1) - \log n \to 0$. In the remainder of the proof, we always consider $n \in M$. Since $u_{n+1}^*(\theta) - u_n^*(\theta) = O(n^{-1})$, we have

$$g_n' = -1/2 + O(n^{-1}), \quad g_n'' = 1/2 + O(n^{-1}).$$

Putting $k = \lfloor u_n^*(\theta) \rfloor$ into (16) yields

$$\begin{aligned}
&\sqrt{2\pi w_n}\, \mathbb{P}(K_n(\theta) = \lfloor u_n^*(\theta) \rfloor) \\
&= 1 - \left(\frac{(g_n')^2 - (a^*)^2}{2} - A_{21}(\theta) \right) \frac{1}{w_n} + \frac{P_\theta(a^* + g_n')}{w_n^2} + o\left(\frac{1}{\log^2 n} \right) \\
&= 1 - \left(\frac{1 - 4(a^*)^2}{8} - A_{21}(\theta) \right) \frac{1}{w_n} + \frac{P_\theta(a^* - 1/2)}{w_n^2} + o\left(\frac{1}{\log^2 n} \right).
\end{aligned}$$

Analogously, putting $k = \lceil u_n^*(\theta) \rceil$ gives

$$\begin{aligned}
&\sqrt{2\pi w_n}\, \mathbb{P}(K_n(\theta) = \lceil u_n^*(\theta) \rceil) \\
&= 1 - \left(\frac{1 - 4(a^*)^2}{8} - A_{21}(\theta) \right) \frac{1}{w_n} + \frac{P_\theta(a^* + 1/2)}{w_n^2} + o\left(\frac{1}{\log^2 n} \right).
\end{aligned}$$

For sufficiently large n the mode $u_n(\theta)$ equals either $\lfloor u_n^*(\theta) \rfloor$ or $\lceil u_n^*(\theta) \rceil$ depending on the sign of

$$s^*(\theta) := P_\theta(a^* + 1/2) - P_\theta(a^* - 1/2).$$

In the following we shall show that $s^*(\theta) > 0$, hence $u_n(\theta) = \lceil u_n^*(\theta) \rceil$, while $\mathrm{nint}(u_n(\theta)) = \lfloor u_n^*(\theta) \rfloor$, so that $u_n(\theta) \neq \mathrm{nint}(u_n^*(\theta))$. Recalling the polygamma function $\psi^{(m)}(\theta) = (\log \Gamma(\theta))^{(m+1)}$, the authors [15] checked with the help of Mathematica 9 that

$$s^*(\theta) = \frac{\theta^2}{2} \left(2\psi^{(1)}(\theta) + \theta \psi^{(2)}(\theta) \right).$$

Using the well-known formula for the polygamma function [1, 6.4.10]

$$\psi^{(m)}(\theta) = (\log \Gamma(\theta))^{(m+1)} = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(\theta + k)^{m+1}}, \quad -\theta \notin \mathbb{N}_0, \quad m \geq 1,$$

we finally obtain

$$s^*(\theta) = \theta^2 \sum_{k=1}^{\infty} \frac{k}{(\theta + k)^3}, \quad \theta > 0,$$

yielding positivity of $s^*(\theta)$ for all $\theta > 0$. The proof of part (iv), as well as of the whole theorem, is complete. □

Remark 8. For $\theta = 1$ we have $s^*(1) = \zeta(2) - \zeta(3)$, a term appearing in Hammersley's formula (9). In fact, in the special case $\theta = 1$ part (iv) could be deduced directly from (9).

3 Acknowledgements

The work of Alexander Marynych was supported by a Humboldt Research Fellowship of the Alexander von Humboldt Foundation. The work of Henning Sulzbach was supported by a Feodor Lynen Research Fellowship of the Alexander von Humboldt Foundation.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, volume 55 of National Bureau of Standards Applied Mathematics Series, 1964.

[2] N. H. Bingham, Tauberian theorems for Jakimovski and Karamata-Stirling methods, *Mathematika* **35** (1988), 216–224.

[3] E. Cramer, Asymptotic estimators of the sample size in a record model, *Statist. Papers* **41** (2000), 159–171.

[4] R. Durrett, *Probability Models for DNA Sequence Evolution*, Probability and its Applications, Springer, second edition, 2008.

[5] P. Erdős, On a conjecture of Hammersley, *J. London Math. Soc.* **28** (1953), 232–236.

[6] W. Ewens and S. Tavaré, Multivariate Ewens distribution. In N. L. Johnson, S. Kotz, and N. Balakrishnan, editors, *Discrete Multivariate Distributions*, Wiley Series in Probability and Statistics: Applied Probability and Statistics. John Wiley & Sons, 1997.

[7] W. Feller, The fundamental limit theorems in probability, *Bull. Amer. Math. Soc.* **51** (1945), 800–832.

[8] V. Féray, P.-L. Méliot, and A. Nikeghbali, Mod–ϕ convergence, I: Normality zones and precise deviations, preprint, 2015, http://arxiv.org/abs/1304.2934.

[9] R. Grübel and Z. Kabluchko, Edgeworth expansions for profiles of lattice branching random walks, *to appear in Ann. Inst. H. Poincaré*, 2015, http://arxiv.org/abs/1503.04616.

[10] J. M. Hammersley, On estimating restricted parameters, *J. Roy. Statist. Soc. Ser. B.* **12** (1950), 192–229, discussion, 230–240.

[11] J. M. Hammersley, The sums of products of the natural numbers, *Proc. London Math. Soc. (3)* **1** (1951), 435–452.

[12] H.-K. Hwang, Théorèmes limites pour les structures combinatories et les fonctions arithmétiques, Ph.D. thesis, 1994, http://algo.stat.sinica.edu.tw/hk/?p=80.

[13] H.-K. Hwang, Asymptotic expansions for the Stirling numbers of the first kind, *J. Combin. Theory Ser. A* **71** (1995), 343–351.

[14] Z. Kabluchko, V. Vysotsky, and D. Zaporozhets, Convex hulls of random walks, hyperplane arrangements, and Weyl chambers, preprint, 2015, http://arxiv.org/abs/1510.04073.

[15] Z. Kabluchko, A. Marynych, and H. Sulzbach, Mathematica notebook, 2016, http://www.math.uni-muenster.de/statistik/kabluchko/files/stirling.pdf and http://www.math.uni-muenster.de/statistik/kabluchko/files//stirling.nb.

[16] Z. Kabluchko, A. Marynych, and H. Sulzbach, General Edgeworth expansions with applications to profiles of random trees, preprint, 2016, http://arxiv.org/abs/1606.03920, version with full proofs at http://www.math.uni-muenster.de/statistik/kabluchko/files/edgeworth_full.pdf.

[17] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Pure and Applied Mathematics, Wiley-Interscience, 1974.

[18] G. Louchard, Asymptotics of the Stirling numbers of the first kind revisited: a saddle point approach, *Discrete Math. Theor. Comput. Sci.* **12** (2010), 167–184.

[19] I. Mező, On the maximum of r-Stirling numbers, *Adv. in Appl. Math.* **41** (2008), 293–306.

[20] L. Moser and M. Wyman, Asymptotic development of the Stirling numbers of the first kind, *J. London Math. Soc.* **33** (1958), 133–146.

[21] A. Rényi, On the outliers of a series of observations, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **12** (1962), 105–121.

[22] M. Sibuya, Log-concavity of Stirling numbers and unimodality of Stirling distributions, *Ann. Inst. Statist. Math.* **40** (1988), 693–714.

[23] N. M. Temme, Asymptotic estimates of Stirling numbers, *Stud. Appl. Math.* **89** (1993), 233–243.

[24] H. S. Wilf, The asymptotic behavior of the Stirling numbers of the first kind, *J. Combin. Theory Ser. A* **64** (1993), 344–349.

[25] H. Yamato, Edgeworth expansions for the number of distinct components associated with the Ewens sampling formula, *J. Japan Statist. Soc.* **43** (2013), 17–28.

2010 *Mathematics Subject Classification*: Primary 11B73; Secondary 60C05, 41A60, 60F05, 60F10.

Keywords: asymptotic expansion, Ewens distribution, Stirling number of the first kind, mode of a distribution.

(Concerned with sequences A008275, A048994, A132393.)

Received revised version received Published in *Journal of Integer Sequences*,

Return to Journal of Integer Sequences home page.