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On a Ramanujan-type Congruence for Bipartitions with 5-Cores

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Abstract

In this short note, we prove a Ramanujan-type congruence modulo 5^α $(\alpha \geq 1)$ for $A_5(n)$, which counts the number of 5-core bipartitions of n.

1 Introduction

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition. For example, the partitions of 4 are $4, 3+1, 2+2, 2+1+1, 1+1+1+1$.

Given a partition $[\lambda] = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ of n, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, the *Ferrers-Young diagram* of $[\lambda]$ is an array of nodes with λ_i nodes in the i^{th} row. The (i,j) hook is the set of nodes directly below, together with the set of nodes directly to the right of the (i,j) nodes, as well as the (i,j) node itself. The hook number of (i,j), denoted by $H(i,j)$, is the total number of nodes in the (i,j) hook. For a positive integer $t \geq 2$, a partition of n is said to be *t-core* if it has no hook numbers that are multiples of t.

Example 1. The Ferrers-Young diagram of the partition $\lambda = 5 + 3 + 2$ of 10 is

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array}$$

The nodes $(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(3,1)$ and $(3,2)$ have hook numbers $7,6,4,2,1,4,3,1,2$ and 1, respectively. Therefore λ is 5-core. Note that λ is t-core for $t \geq 8$.

A *bipartition* of n is a pair of partitions (λ, μ) such that the sum of all the parts of λ and μ equals n. A bipartition with t-core of n is a bipartition (λ, μ) of n such that λ and μ are both t-cores. Let $A_t(n)$ denote the number of bipartitions with t-cores of n. The generating function for $A_t(n)$ is given by

$$\sum_{n=0}^{\infty} A_t(n) q^n = \frac{(q^t;q^t)_\infty^{2t}}{(q;q)_\infty^2}. \tag{1}$$

Here and throughout this note, we assume that $|q| < 1$ and we follow standard q-series notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

Motivated by the work of Ramanujan on congruences for unrestricted partition function $p(n)$, many mathematicians considered the function $A_t(n)$ and studied its congruence properties. For example, Lin [3] established several congruences for $A_3(n)$. Soon after, Xia [8] and Yao [9] extended the list of congruences for $A_3(n)$.

The main aim of this note is to prove the following Ramanujan-type congruence modulo 5^α ($\alpha \geq 1$) for $A_5(n)$:

$$A_5(5^\alpha n + 5^\alpha - 2) \equiv 0 \pmod{5^\alpha}, \quad \alpha \geq 1.$$

The following 5-dissection formula for $(q;q)_\infty$ was first stated by Ramanujan [4, p. 212] without proof.

Lemma 2. *[4, p. 212] We have*

$$(q;q)_\infty = (q^{25};q^{25})_\infty \Big(\frac{1}{R(q^5)} - q - q^2 R(q^5) \Big), \tag{2}$$

where $R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$.

Watson [5] presented a proof of (2) using the quintuple product identity.

Lemma 3. *[2, eq. (7.4.14), p. 165] We have*

$$\begin{aligned} \frac{1}{(q;q)_\infty} =& \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \Big(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) \\ &+ 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \Big). \end{aligned} \tag{3}$$

The following lemmas are useful to prove our main congruence for $A_5(n)$:

Lemma 4. *Let* $\sum_{n=0}^{\infty} a(n)q^n = \frac{q}{(q;q)_\infty^2}$. *Then*

$$\sum_{n=0}^{\infty} a(5n+4)q^n = 125q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}} + 10\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^6}. \tag{4}$$

Proof. In view of (3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n =& \frac{(q^{25};q^{25})_\infty^{10}}{(q^5;q^5)_\infty^{12}}\Big(\frac{q}{R(q^5)^8} + \frac{2q^2}{R(q^5)^7} + \frac{5q^3}{R(q^5)^6} + \frac{10q^4}{R(q^5)^5} + \frac{20q^5}{R(q^5)^4} + \frac{16q^6}{R(q^5)^3}\\ &+ \frac{27q^7}{R(q^5)^2} + \frac{20q^8}{R(q^5)} + 15q^9 - 20q^{10}R(q^5) + 27q^{11}R(q^5)^2 - 16q^{12}R(q^5)^3\\ &+ 20q^{13}R(q^5)^4 - 10q^{14}R(q^5)^5 + 5q^{15}R(q^5)^6 - 2q^{16}R(q^5)^7 + q^{17}R(q^5)^8\Big).\end{aligned} \tag{5}$$

Extracting the terms involving q^{5n+4} in (5), dividing by q^4 and replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} a(5n+4)q^n =& \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}}\Big(\frac{10}{R(q)^5} + 15q - 10q^2R(q)^5\Big)\\ =& \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}}\Big(10\Big(\frac{1}{R(q)^5} - 11q - q^2R(q)^5\Big) + 125q\Big).\end{aligned} \tag{6}$$

Berndt [2, Thm. 7.4.4] proved the following identity:

$$\frac{1}{R(q)^5} - 11q - q^2R(q)^5 = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}. \tag{7}$$

Employing (7) in (6), we obtain (4). ∎

In a similar way, we have the following:

Lemma 5. *Let* $\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2}$. *Then*

$$\sum_{n=0}^{\infty} b(5n+3)q^n = 125q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}} + 10\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^6}.$$

Lemma 6. *Let* $\sum_{n=0}^{\infty} c(n)q^n = (q;q)_\infty^4$. *Then*

$$\sum_{n=0}^{\infty} c(5n+4)q^n = -5(q^5;q^5)_\infty^4.$$

Theorem 7. *Let α be a integer ≥ 1. Then*

$$\sum_{n=0}^{\infty} \mathrm{A}_5(5^{\alpha}n + 5^{\alpha} - 2)q^n = 5^{3\alpha} q \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2} + 5^{\alpha}\Big(\frac{25^{\alpha} - (-1)^{\alpha}}{13}\Big)(q;q)_\infty^4 (q^5;q^5)_\infty^4. \tag{8}$$

Proof. From (1) and Lemma 5, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{A}_5(5n+3)q^n =&(q;q)_\infty^{10}\Big(125q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}} + 10\frac{(q^5;q^5)_\infty^{4}}{(q;q)_\infty^{6}}\Big)\\ =&10(q;q)_\infty^4(q^5;q^5)_\infty^4 + 125q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2},\end{aligned}$$

which is same as (8) with $\alpha = 1$. Suppose that (8) holds for some $\alpha \geq 1$. From (8), Lemmas 4 and 6, we deduce

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathrm{A}_5(5^{\alpha+1}n + 5^{\alpha+1} - 2)q^n\\ &= 5^{3\alpha}(q;q)_\infty^{10}\Big(125q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}} + 10\frac{(q^5;q^5)_\infty^{4}}{(q;q)_\infty^{6}}\Big) - 5^{\alpha+1}\Big(\frac{25^{\alpha} - (-1)^{\alpha}}{13}\Big)(q;q)_\infty^4(q^5;q^5)_\infty^4\\ &= 5^{3\alpha+3}q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2} + \Big(10\cdot 5^{3\alpha} - 5^{\alpha+1}\Big(\frac{25^{\alpha} - (-1)^{\alpha}}{13}\Big)\Big)(q;q)_\infty^4(q^5;q^5)_\infty^4\\ &= 5^{3\alpha+3}q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2} + 5^{\alpha+1}\Big(\frac{25^{\alpha+1} - (-1)^{\alpha+1}}{13}\Big)(q;q)_\infty^4(q^5;q^5)_\infty^4.\end{aligned}$$

That is, (8) holds for $\alpha + 1$. This completes the proof by induction of (8). □

From (8), we have the following congruence relation:

Theorem 8. *For all integers $n \geq 0$ and $\alpha \geq 1$,*

$$\mathrm{A}_5(5^{\alpha}n + 5^{\alpha} - 2) \equiv 0 \pmod{5^{\alpha}}. \tag{9}$$

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