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The Number of Support-Tilting Modules for a Dynkin Algebra

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Abstract

The Dynkin algebras are the hereditary artin algebras of finite representation type. The paper exhibits the number of support-tilting modules for any Dynkin algebra. Since the support-tilting modules for a Dynkin algebra of Dynkin type Δ correspond bijectively to the generalized non-crossing partitions of type Δ, the calculations presented here may also be considered as a categorification of results concerning the generalized non-crossing partitions. In the Dynkin case $\mathbb{A}$, we obtain the Catalan triangle, in the cases $\mathbb{B}$ and $\mathbb{C}$ the increasing part of the Pascal triangle, and finally in the case $\mathbb{D}$ an expansion of the increasing part of the Lucas triangle.

1 Introduction

Let Λ be a hereditary artin algebra. Here we consider left Λ-modules of finite length and call them just modules. The category of all modules will be denoted by $\operatorname{mod}\Lambda$. We let $n = n(\Lambda)$ denote the *rank* of Λ; by definition, this is the number of simple modules (when counting numbers of modules of a certain kind, we always mean the number of isomorphism classes). Following earlier considerations of Brenner and Butler, tilting modules were defined by Happeland Ringel [15]. In the present setting, a *tilting module* M is a module without self-extensions with precisely n isomorphism classes of indecomposable direct summands, and we will assume, in addition, that M is multiplicity-free. The endomorphism ring of a tilting module is said to be a tilted algebra. There is a wealth of papers devoted to tilted algebras, and the *Handbook of Tilting Theory* [1] can be consulted for references.

The present paper deals with the Dynkin algebras: these are the connected hereditary artin algebras which are representation-finite, thus their valued quivers are of Dynkin type $\Delta_n = \mathbb{A}_n, \mathbb{B}_n, \ldots, \mathbb{G}_2$ (see [8]). Its aim is to discuss the number of tilting modules for such an algebra. The corresponding tilted algebras were classified by various authors in the eighties. It seems to be clear that a first step of such a classification result was the determination of all tilting modules, however there are only few traces in the literature (also the *Handbook* [1] is of no help). Apparently, the relevance of the number of tilting modules was seen at that time only in special cases. The tilting modules for a linearly ordered quiver of type $\mathbb{A}_n$ were exhibited in [16] and Gabriel [13] pointed out that here we encounter one of the numerous appearances of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. For the cases $\mathbb{D}_n$, the number of tilting modules was determined by Bretscher-Läser-Riedtmann [6] in their study of self-injective representation-finite algebras.

Given a module M, we let $\Lambda(M)$ denote its *support algebra;* this is the factor algebra of Λ modulo the ideal which is generated by all idempotents e with $eM = 0$ and is again a hereditary artin algebra (but usually not connected, even if Λ is connected). The rank of the support algebra of M will be called the *support-rank* of M. A module T is said to be *support-tilting* provided M considered as a $\Lambda(M)$-module is a tilting module. It may be well-known that the number of tilting modules of a Dynkin algebra depends only on its Dynkin type; at least for path algebras of quivers we can refer to Ladkani [22]. Section 4 of the present paper provides a proof in general. It follows that the number of support-tilting modules with support-rank s also depends only on the type Δ_n; we let $a_s(\Delta_n)$ denote the number of support-tilting Λ-modules with support-rank s, where Λ is of type Δ_n. Of course, $a_n(\Delta_n)$ is just the number of tilting modules, and we denote by $a(\Delta_n)$ the number of all support-tilting modules; thus $a(\Delta_n) = \sum_{s=0}^{n} a_s(\Delta_n)$.

The present paper presents the numbers $a(\Delta_n)$ and $a_s(\Delta_n)$ for $0 \le s \le n$ in a unified way. Of course, the exceptional cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$ can be treated with a computer (but actually, also by hand); thus our main interest lies in the series $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$. In the case $\mathbb{A}$, we obtain in this way the **Catalan triangle** A009766, in the case $\mathbb{B}$ and $\mathbb{C}$ the increasing part of the **Pascal triangle,** and finally in the case $\mathbb{D}$ an **expansion** of the increasing part of the **Lucas triangle** (see Section 2; an outline will be given later in the introduction).

1.1 The numbers

All the numbers which are presented here for the cases $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ are related to the binomial coefficients $\binom{s}{t}$ and they coincide for $\mathbb{B}_n$ and $\mathbb{C}_n$ (as we will show in Section 4); thus it is sufficient to deal with the cases $\mathbb{A}, \mathbb{B}, \mathbb{D}$. For $\mathbb{B}$, the binomial coefficients themselves will play a dominant role. For the cases $\mathbb{A}$ and $\mathbb{D}$, suitable multiples are relevant. In case $\mathbb{A}$, these are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, as well as related numbers. For the case $\mathbb{D}$, it will be convenient to use the notation $\left[\begin{smallmatrix} t \\ s \end{smallmatrix}\right] = \frac{s+t}{t}\binom{t}{s}$ as proposed by Bailey [5], since the relevant numbers in case $\mathbb{D}$ can be written in this way.

Hubery and Krause [18] have pointed out that the numbers $a(\Delta)$ for the simply laced diagrams Δ were discussed already in 1987 by Gabriel and de la Peña [14], but let us quote "although they have the correct number for $\mathbb{E}_8$, their numbers for $\mathbb{E}_6$ and $\mathbb{E}_7$ are slightly wrong".

Theorem 1. *The numbers $a(\Delta_n)$ and $a_s(\Delta_n)$ for $0 \le s \le n$:*

Δ_n	$\mathbb{A}_n$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$a_n(\Delta_n)$	$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n-1}{n-1}$	$\left[\begin{smallmatrix} 2n-2 \\ n-2 \end{smallmatrix}\right]$	418	2 431	17 342	66	5
$a_s(\Delta_n)$ $0 \le s < n$	$\frac{n-s+1}{n+1}\binom{n+s}{s}$	$\binom{n+s-1}{s}$	$\left[\begin{smallmatrix} n+s-2 \\ s \end{smallmatrix}\right]$	..................see Section 3				
$a(\Delta_n)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\left[\begin{smallmatrix} 2n-1 \\ n-1 \end{smallmatrix}\right]$	833	4 160	25 080	105	8

Remark 2. By analogy with the Bailey notation $\left[\begin{smallmatrix} t \\ s \end{smallmatrix}\right]$ one may be tempted to introduce the following notation for the Catalan triangle: $\left]\begin{smallmatrix} t \\ s \end{smallmatrix}\right[= \frac{t-2s+1}{t-s+1}\binom{t}{s}$. Then the numbers for the case $\mathbb{A}$ are written as follows:

$$a_n(\mathbb{A}_n) = \left]\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right[, \ a_s(\mathbb{A}_n) = \left]\begin{smallmatrix} n+s \\ s \end{smallmatrix}\right[, \ a(\mathbb{A}) = \left]\begin{smallmatrix} 2n+2 \\ n+1 \end{smallmatrix}\right[.$$

Remark 3. The reader should observe that for $\mathbb{A}_n$ and $\mathbb{B}_n$, the formula given for $a_s(\Delta_n)$ and $0 \le s < n$ works also for $s = n$. This is not the case for $\mathbb{D}_n$: whereas $\binom{2n-2}{n-2} = \binom{2n-2}{n}$, the numbers $\left[\begin{smallmatrix} 2n-2 \\ n-2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2n-2 \\ n \end{smallmatrix}\right]$ are different (the difference will be highlighted at the end of Section 2). The Lucas triangle consists of the numbers $\left[\begin{smallmatrix} t \\ s \end{smallmatrix}\right]$ for all $0 \le s \le t$; it therefore uses the numbers $\left[\begin{smallmatrix} 2n-2 \\ n \end{smallmatrix}\right]$ at the positions, whereas the $\mathbb{D}$-triangle (which we will now consider) uses the numbers $\left[\begin{smallmatrix} 2n-2 \\ n-2 \end{smallmatrix}\right]$.

1.2 The triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$

The non-zero numbers $a_s(\Delta_n)$ for $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}$ yield three triangles having similar properties. We will exhibit them in Section 2; see the triangles 2.1, 2.2, 2.3. The triangle 2.1 of type $\mathbb{A}$ is the Catalan triangle itself; this is A009766 in Sloane's OEIS [29]. The triangle 2.2 of type $\mathbb{B}$ is the triangle A059481, corresponding to the increasing part of the Pascal triangle (thus it consists of the binomial coefficients $\binom{t}{s}$ with $2s \le t+1$). The triangle 2.3 of type $\mathbb{D}$ is an expansion of the increasing part of the Lucas triangle A029635. Taking the increasing part of the rows in the Lucas triangle (thus the numbers $\left[\begin{smallmatrix} t \\ s \end{smallmatrix}\right]$ with $2s \le t+1$), we obtain numbers which occur in the triangle of type $\mathbb{D}$, namely the numbers $a_s(\mathbb{D}_n)$ with $0 \le s < n$; the numbers $a_n(\mathbb{D}_n)$ on the diagonal however are given by a similar, but deviating formula (they are listed as the sequence A129869). The Lucas triangle is A029635, but the triangle $\mathbb{D}$ itself was, at the time of the writing, not yet recorded in OEIS; now it is A241188.

We see that the entries $a_s(n)$ of the triangles $\mathbb{A}$ and $\mathbb{B}$, as well as those of the lower triangular part of the triangle $\mathbb{D}$ can be obtained in a unified way from three triangles with entries $z_s(t)$ which satisfy the following recursion formula

$$z_s(t) = z_{s-1}(t-1) + z_s(t-1)$$

(they are exhibited in Section 2 as triangles S 2.1, S 2.2, S 2.3 using the shearing $a_s(n) = z_s(n+s-1)$). The recursion formula can be rewritten as $z_s(t) = \sum_{i=0}^{s} z_i(t-s+i+1)$ (sometimes called the hockey stick formula). A consequence of the hockey stick formula is the fact that summing up the rows of any of the three triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$, we again obtain numbers which appear in the triangle.

Let us provide further details on the triangles to be sheared. Consider first the case $\mathbb{B}$. Here we start with the Pascal triangle; thus we deal with the triangle with numbers $z_s(t) = \binom{t}{s}$ and the initial conditions are $z_0(t) = z_t(t) = 1$ for all $t \ge 0$. In case $\mathbb{D}$, we start with the Lucas triangle with numbers $z_s(t) = \left[\begin{smallmatrix} t \\ s \end{smallmatrix}\right]$, and the initial conditions are $z_0(t) = 1,\ z_t(t) = 2$ for all $t \ge 1$ (these initial conditions are the reason for calling the Lucas triangle also the $(1,2)$-triangle). In the case $\mathbb{A}$ we start with a sheared Catalan triangle, and here the initial conditions are $z_0(t) = 1$ and $z_{t+1}(2t) = 0$ for all $t \ge 0$.

1.3 Related results

Let us repeat that in this paper $a_n(\Delta_n)$ denotes the number of tilting modules, $a(\Delta_n)$ the number of support-tilting modules, for Λ of Dynkin type Δ_n. As we have mentioned, the relevance of the numbers $a_n(\Delta_n)$ and $a(\Delta_n)$ was not fully realized in the eighties. It became apparent through the work of Fomin and Zelevinksy when dealing with cluster algebras and the corresponding cluster complexes (see in particular [12] and [11]): the numbers $a_n(\Delta_n)$ and $a(\Delta_n)$ appear in [12] as the numbers $N(\Delta_n)$ of clusters and $N^+(\Delta_n)$ of positive clusters, respectively (see Propositions 3.8 and 3.9 of [12]). For the numbers $a_s(\Delta_n)$ in general, see Chapoton [7] in case $\mathbb{A}$ and $\mathbb{B}$ and Krattentaler [20] in case $\mathbb{D}$. A conceptual proof of the equalities $a(\Delta_n) = N(\Delta_n)$ and $a_n(\Delta_n) = N^+(\Delta_n)$ has been given by Ingalls and Thomas [19] in case Δ_n is simply laced (thus of type $\mathbb{A}, \mathbb{D}$ of $\mathbb{E}$). The considerations of Ingalls and Thomas have been extended by the authors [25] to the non-simply laced cases. The papers [19] and [25] show in which way the representation theory of hereditary artin algebras can be used in order to categorify the cluster complex of Fomin and Zelevinsky: this is the reason for the equalities. Another method to relate clusters and support tilting modules is due to Marsh, Reineke and Zelevinsky [23]. Finally, let us stress that also the Coxeter diagrams $\mathbb{H}_3$ and $\mathbb{H}_4$ can be treated in a similar way, using hereditary artinian rings which are not artin algebras; this will be shown in [11].

The main result of the present paper is the direct calculation of the numbers $a_n(\Delta_n)$ in the case $\Delta = \mathbb{B}$; see Section 5. Of course, using [25], this calculation can be replaced by referring to the determination of the corresponding cluster numbers by Fomin and Zelevinsky in [12]. On the other hand, we hope that our proof is of interest in itself.

There is an independent development which has to be mentioned, namely the theory of generalized non-crossing partitions (see for example [2]). It is the Ingalls-Thomas paper [19] (and [25]; see also the survey [28]) which provides the basic setting for using the representation theory of a hereditary artin algebra Λ in order to deal with non-crossing partitions. It turns out that there is a large number of counting problems for $\operatorname{mod}\Lambda$ which yield the same answer, namely the numbers $a(\Delta_n)$ and $a_s(\Delta_n)$. For example, $a_s(\Delta)$ is also the number of antichains in $\operatorname{mod}\Lambda$ of size s: an *antichain* $A = \{A_1, \dots, A_t\}$ in $\operatorname{mod}\Lambda$ is a set of pairwise orthogonal bricks (a brick is a module whose endomorphism ring is a division ring, and two bricks A_1, A_2 are said to be orthogonal provided $\operatorname{Hom}(A_1, A_2) = 0 = \operatorname{Hom}(A_2, A_1)$; antichains are called discrete subsets in [14] and Hom-free subsets in [18]).

Since the support-tilting modules for a Dynkin algebra of Dynkin type Δ correspond bijectively to the non-crossing partitions of type Δ, the calculations presented here may be considered as a categorification of results concerning non-crossing partitions (for a general outline see Hubery-Krause [18]). Finally, let us mention that there is a corresponding discussion of the number of ad-nilpotent ideals of a Borel subalgebra of a simple Lie algebra; see Panyushev [26].

1.4 Outline of the paper

Let us stress again that there is an inductive procedure using the hook formula (Proposition 19) and a modified hook formula (Proposition 21) in order to obtain the numbers $a_s(\mathbb{A}_n)$ for $0 \le s \le n$, as well as the numbers $a_s(\Delta_n)$ for $\Delta = \mathbb{B}, \mathbb{D}$ for $0 \le s < n$, provided we know the numbers $a_n(\Delta_n)$. As we have mentioned, for the numbers $a_n(\mathbb{D}_n)$ we may refer to [6]. In Section 4, we will show that the numbers $a_n(\mathbb{B}_n)$ and $a_n(\mathbb{C}_n)$ coincide; thus it remains to determine the numbers $a_n(\mathbb{B}_n)$. This will be done in Section 5. In Section 7, we calculate $a(\Delta_n)$ for $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}$.

Section 2 presents the triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$ as well as the corresponding Catalan, Pascal, and Lucas triangles, and some observations concerning repetition of numbers in the triangles are recorded. Section 3 provides the numbers $a_s(\Delta_n)$ for the exceptional cases $\Delta_n = \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$.

2 The triangles

2.1. The triangle of type $\mathbb{A}$; this is A009766

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	sum
0	1										1
1	1	1									2
2	1	2	2								5
3	1	3	5	5							14
4	1	4	9	14	14						42
5	1	5	14	28	42	42					132
6	1	6	20	48	90	132	132				429
7	1	7	27	75	165	297	429	429			1430
8	1	8	35	110	275	572	1001	1430	1430		4862
9	1	9	44	154	429	1001	2002	3432	4862	4862	16796

$$a_s(\mathbb{A}_n) = \frac{n-s+1}{n+1}\binom{n+s}{s}$$

2.2. The triangle of type $\mathbb{B}$; this is A059481

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	sum
0	1										1
1	1	1									2
2	1	2	3								6
3	1	3	6	10							20
4	1	4	10	20	35						70
5	1	5	15	35	70	126					252
6	1	6	21	56	126	252	462				924
7	1	7	28	84	210	462	924	1716			3432
8	1	8	36	120	330	792	1716	3432	6435		12870
9	1	9	45	165	495	1287	3003	6435	12870	24310	48620

$$a_s(\mathbb{B}_n) = \binom{n+s-1}{s}$$

2.3. The triangle of type $\mathbb{D}$; this is now A241188

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	sum
0	.										.
1	.	.									.
2	1	2	1								4
3	1	3	5	5							14
4	1	4	9	16	20						50
5	1	5	14	30	55	77					182
6	1	6	20	50	105	196	294				672
7	1	7	27	77	182	378	714	1122			2508
8	1	8	35	112	294	672	1386	2640	4290		9438
9	1	9	44	156	450	1122	2508	5148	9867	16445	35750

$$a_s(\mathbb{D}_n) = \begin{cases} \left[{n+s-2 \atop s} \right] & \text{for } 0 \le s < n; \\ \left[{2n-2 \atop n-2} \right] & \text{for } s = n. \end{cases}$$

S 2.1. The sheared Catalan triangle A008315

$t \backslash s$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2								
3	1	3	2							
4	1	4	5							
5	1	5	9	5						
6	1	6	14	14						
7	1	7	20	28	14					
8	1	8	27	48	42					
9	1	9	35	75	90	42				

$$\binom{t}{s} - \binom{t}{s-1} = \frac{t-2s+1}{t-s-1}\binom{t}{s}$$

S 2.2. The Pascal triangle A007318, left of the staircase line is the increasing part

$t \backslash s$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

$$\binom{t}{s}$$

S 2.3. The Lucas triangle A029635, left of the staircase line is the increasing part

$t \backslash s$	0	1	2	3	4	5	6	7	8	9
0	·									
1	1	2								
2	1	3	2							
3	1	4	5	2						
4	1	5	9	7	2					
5	1	6	14	16	9	2				
6	1	7	20	30	25	11	2			
7	1	8	27	50	55	36	13	2		
8	1	9	35	77	105	91	49	15	2	
9	1	10	44	112	182	196	140	64	17	2

$$\begin{bmatrix} t \\ s \end{bmatrix}$$

Remark 4. In the triangle 2.3 of type $\mathbb{D}$ and in the corresponding Lucas triangle S 2.3 some values are left open (this is indicated by a dot).

In the Lucas triangle S 2.3, this concerns the value at the position $(0, 0)$ which could be denoted as $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. The value should be one of the numbers 1 or 2 (in OEIS A029635, the number is chosen to be 2). Note that here we deal with the product $\frac{0}{0}\binom{0}{0}$: whereas $\binom{0}{0} = 1$ is well-defined, there is the ambiguous fraction $\frac{0}{0}$.

In the triangle 2.3 of type $\mathbb{D}$, the positions $(0,0),(0,1),(1,1)$ are left open, since the series of Dynkin diagrams $\mathbb{D}_n$ starts with $n = 2$ (but see A129869); by definition $\mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1$ and $\mathbb{D}_3 = \mathbb{A}_3$. As a consequence, also the corresponding entries in the summation sequence are missing.

2.4 Some observations concerning the triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$

Proposition 5. *The sum sequence occurs as a diagonal.*

In the $\mathbb{A}$-triangle, the sum sequence is the same sequence as the main diagonal (and these are just the Catalan numbers):

$$a(\mathbb{A}_n) = a_{n+1}(\mathbb{A}_{n+1}).$$

In the $\mathbb{B}$-triangle, the sum sequence is the same sequence as the second diagonal:

$$a(\mathbb{B}_n) = a_n(\mathbb{B}_{n+1}).$$

In the $\mathbb{D}$-triangle, the sum sequence is the same sequence as the fourth diagonal:

$$a(\mathbb{D}_n) = a_{n-1}(\mathbb{D}_{n+2}).$$

Proposition 6. *The main diagonal uses the same sequence as one of the other diagonals.*

In the $\mathbb{A}$-triangle, this concerns the main diagonal and the second diagonal:

$$a_n(\mathbb{A}_n) = a_{n-1}(\mathbb{A}_n).$$

In the $\mathbb{B}$-triangle, this concerns the main diagonal and the second diagonal:

$$a_n(\mathbb{B}_n) = a_{n-1}(\mathbb{B}_{n+1}).$$

In the $\mathbb{D}$-triangle, this concerns the main diagonal and the fifth diagonal:

$$a_n(\mathbb{D}_n) = a_{n-2}(\mathbb{D}_{n+2}).$$

It may be of interest to exhibit explicit bijections between the corresponding sets of support-tilting modules. It seems that only in the case $\mathbb{A}$, this can be done easily (see Remark 15).

2.5 Comparison between the Lucas triangle and the $\mathbb{D}$-triangle

The difference between the number $\begin{bmatrix} 2n-2 \\ n \end{bmatrix}$ and $\begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix}$ seems to be of interest:

$$\begin{bmatrix} 2n-2 \\ n \end{bmatrix} - \begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix} = \frac{1}{n}\binom{2n-2}{n-1}.$$

This means the following:

Proposition 7.

$$\begin{bmatrix} 2n-2 \\ n \end{bmatrix} - a_n(\mathbb{D}_n) = a_{n-1}(\mathbb{A}_{n-1}).$$

Proof. We show that

$$\frac{3n-4}{n}\binom{2n-2}{n-2} + \frac{1}{n}\binom{2n-2}{n-1} \;=\; \frac{3n-2}{2n-2}\binom{2n-2}{n}$$

We rewrite

$$\begin{aligned} \binom{2n-2}{n-2} &= \frac{n}{2n-2}\binom{2n-2}{n}, \\ \binom{2n-2}{n-1} &= \frac{n}{n-1}\binom{2n-2}{n}. \end{aligned}$$

The assertion now follows from the equality

$$\frac{3n-4}{n}\cdot\frac{n}{2n-2} \;+\; \frac{1}{n}\cdot\frac{n}{n-1} \;=\; \frac{3n-2}{2n-2}.$$

□

Here is a table of these numbers

n	$\begin{bmatrix} 2n-2 \\ n \end{bmatrix}$	$\begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix}$	$\frac{1}{n}\binom{2n-2}{n-1}$
2	2	1	1
3	7	5	2
4	25	20	5
5	91	77	14
6	336	294	42
7	1254	1122	132
8	4719	4290	429
9	17875	16445	1430

Remark 8. Proposition 7 is essentially the modified hook formula for type $\mathbb{D}$ which will be presented in Proposition 21: the Lucas triangle uses the hook formula for the whole triangle, whereas the triangle $\mathbb{D}$ uses the modified hook formula on the subdiagonal.

3 The exceptional cases

Here are the numbers $a_s(\Delta_n)$ and $a(\Delta_n)$ in the exceptional cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4$, and $\mathbb{G}_2$ (we add some suitable additional rows in order to stress the induction scheme):

$\diagdown^s$	0	1	2	3	4	5	6	7	8	sum
$\mathbb{E}_3 = \mathbb{A}_2 \sqcup \mathbb{A}_1$	1	3	4	2						10
$\mathbb{E}_4 = \mathbb{A}_4$	1	4	9	14	14					42
$\mathbb{E}_5 = \mathbb{D}_5$	1	5	14	30	55	77				182
$\mathbb{E}_6$	1	6	20	50	110	228	418			833
$\mathbb{E}_7$	1	7	27	77	187	429	1001	2431		4160
$\mathbb{E}_8$	1	8	35	112	299	728	1771	4784	17342	25080
$\mathbb{B}_3$	1	3	6	10						20
$\mathbb{F}_4$	1	4	10	24	66					105
$\mathbb{G}_2$	1	2	5							8

4 Hereditary artin algebras

4.1 The basic setting

Let Λ be a hereditary artin algebra. Since by assumption $\mathrm{Ext}^i_\Lambda = 0$ for $i \geq 2$, we write $\mathrm{Ext}(M, M')$ instead of $\mathrm{Ext}^1_\Lambda(M, M')$. The vertices of the quiver $Q(\Lambda)$ are the isomorphism classes $[S]$ of the simple Λ-modules S and there is an arrow $[S] \to [S']$ provided $\mathrm{Ext}(S, S') \neq 0$. Note that $Q(\Lambda)$ is finite and directed (the latter means that the simple modules can be labeled $S(i)$ such that the existence of an arrow $[S(i)] \to [S(j)]$ implies that $i > j$). We endow $Q(\Lambda)$ with a valuation as follows: given an arrow $[S] \to [S']$, consider $\mathrm{Ext}(S, S')$ as a left $\mathrm{End}(S)^{\mathrm{op}}$-module and also as a left $\mathrm{End}(S')$-module and put

$$v([S], [S']) = (\dim_{\mathrm{End}(S)} \mathrm{Ext}(S, S'))(\dim_{\mathrm{End}(S')^{\mathrm{op}}} \mathrm{Ext}(S, S'))$$

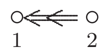
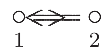
provided $v([S], [S']) > 1$. Given a vertex i of $Q(\Lambda)$, we let $S(i) = S_\Lambda(i), P(i) = P_\Lambda(i), I(i) = I_\Lambda(i)$, respectively, denote a simple, an indecomposable projective or injective module corresponding to the vertex i.

If M is a module, the set of vertices of the quiver $Q(\Lambda(M))$ will be called the *support* of M and M is said to be *sincere* provided any vertex of $Q(\Lambda)$ belongs to the support of M (thus provided the only idempotent $e \in \Lambda$ with $eM = 0$ is $e = 0$).

We also will be interested in the corresponding valued graph $\overline{Q}(\Lambda)$ which is obtained from the valued quiver $Q(\Lambda)$ by replacing the arrows by edges: one says that one *forgets the orientation* of the quiver.

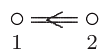
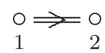
In the special case where $v([S], [S']) = v$ with $v = 2$ or $v = 3$, it is usual to replace the arrow $[S] \longrightarrow [S']$ by a double arrow $[S] \Longrightarrow [S']$ (if $v = 2$) or a similar triple arrow (if $v = 3$). Using the bimodule $\mathrm{Ext}(S, S')$ one obtains an embedding either of $\mathrm{End}\, S$ into $\mathrm{End}\, S'$, or of

End S' into End S; thus one of the division rings is a subring of the other, with index equal to v. One marks the relative size of the endomorphism rings by an additional arrowhead drawn in the middle of the edge, pointing from the larger endomorphism ring to the smaller one (it should be stressed that these inner arrowheads must not be confused with the outer ones). For example, in case there are two simple modules labeled 1 and 2 with an arrow $1 \leftarrow 2$ and $v(1,2) = 2$, there are the following two possibilities:

On the left we see that End $S(1)$ is a division subring of End $S(2)$. On the right, End $S(2)$ is a division subring of End $S(1)$. (Let us exhibit corresponding algebras: let $K : k$ be a field extension of degree 2 and consider the algebras $\Lambda = \left[\begin{smallmatrix} k & K \\ 0 & K \end{smallmatrix}\right]$ and $\Lambda' = \left[\begin{smallmatrix} K & K \\ 0 & k \end{smallmatrix}\right]$; the left quiver shown above is $Q(\Lambda)$, and the right quiver is $Q(\Lambda')$.)

Here are the corresponding valued graphs, which are obtained by forgetting the orientation (thus deleting the outer arrowheads, but not the inner ones):

They are called $\mathbb{B}_2$ and $\mathbb{C}_2$, respectively (observe that there is a difference between $\mathbb{B}_2$ and $\mathbb{C}_2$ only if they occur as subgraphs of larger graphs).

We recall the following [8]. *A connected hereditary artin algebra Λ is representation-finite if and only if $\overline{Q}(\Lambda)$ is one of the Dynkin diagrams*

$$\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$$

and in this case the indecomposable Λ-modules correspond bijectively to the positive roots.

4.2 Change of orientation

We want to show that the number of basic tilting modules is independent of the orientation. We recall that a module is said to be *basic* provided it is a direct sum of pairwise non-isomorphic indecomposable modules; an artin algebra Λ is *basic* provided the regular representation ${}_\Lambda\Lambda$ is basic. In case Λ is the path algebra of a quiver, we may refer to Ladkani [22]. In the case of the tensor algebra of a species (in particular in the case of the path algebra of a quiver), any change of orientation is obtained by applying a sequence of BGP-reflection functors; see [9]. For a general hereditary artin algebra Λ, we have to deal with APR-tilting functors as defined by Auslander, Platzeck and Reiten [3]. In order to do so, we may assume that Λ is basic. We start with a simple projective module S, write ${}_\Lambda\Lambda = S \oplus P$ with a projective module P, and consider $W = P \oplus \tau^- S$ (where $\tau = \tau_\Lambda$ is the Auslander-Reiten translation in mod Λ) and $\Lambda' = (\mathrm{End}\, W)^{\mathrm{op}}$. Note that W is a tilting module (called an APR-tilting module) and the quiver $Q(\Lambda')$ is obtained from the quiver $Q(\Lambda)$ by changing the orientation of all the arrows which involve the vertex $\omega = [S]$. We let $\Lambda'' = (\mathrm{End}\, P)^{\mathrm{op}}$ denote the restriction of Λ to the quiver Q'' obtained from $Q(\Lambda)$ by deleting the vertex ω and the arrows ending in ω. Of course, Q'' is also a subquiver of $Q(\Lambda')$ and Λ'' is the restriction of Λ' to Q'' (thus Λ is a one-point coextension of Λ'', whereas Λ' is a one-point extension of Λ''). We let S' denote the simple Λ'-module with support ω.

Proposition 9. *Let Λ be a hereditary artin algebra and S a simple projective module. Let W be the APR-tilting module defined by S and $\Lambda' = (\operatorname{End} W)^{op}$. Then there is a canonical bijection η between the basic tilting Λ-modules and the basic tilting Λ'-modules.*

Proof. In order to define η, we distinguish two cases.

First, if T is a basic tilting module such that S is not a direct summand of T, let $\eta(T) = \operatorname{Hom}(W, T)$; this is a basic tilting Λ'-module and S' is not a direct summand of $\eta(T)$.

Second, consider a basic tilting Λ-module of the form $S \oplus T$. Let $T'' = T/U$, where U is the sum of the images of all the maps $S \to T$. Obviously, T'' is a basic tilting Λ''-module which we may consider as a Λ'-module. We form the universal extension T' of T'' using copies of S'. Then $T' \oplus S'$ is a basic tilting Λ'-module (and S' is a direct summand). □

Remark 10. We may identify the Grothendieck groups $K_0(\Lambda)$ and $K_0(\Lambda')$, using the common factor algebra Λ'' and identifying the dimension vectors of S and S'. Then, in the first case, the dimension vector of $\eta(T)$ is obtained from the dimension vector of T by applying the reflection σ defined by S. In the second case, the dimension vectors of T and $\eta(T)$ coincide. Actually, here we use twice the internal reflection defined by S in [27], first in the category $\operatorname{mod}\Lambda$, second in the category $\operatorname{mod}\Lambda'$.

4.3 The combinatorial backbone

Let Λ be a Dynkin algebra and assume that the vertices of $Q(\Lambda)$ are labeled $1 \leq i \leq n$. Let $P(i) = P_\Lambda(i)$ be indecomposable projective. Since we assume that Λ is a Dynkin algebra, there is a natural number $q(i) = q(P(i))$ such that $\tau^{-q(i)}P(i)$ is indecomposable injective; the modules $M(i, u) = \tau^{-u}P(i)$ with $0 \leq u \leq q(i)$ and $1 \leq i \leq n$ furnish a complete list of the indecomposable Λ-modules.

Proposition 11. *Let Λ, Λ' be Dynkin algebras and assume that the simple modules of both algebras are indexed by $1 \leq i \leq n$. Assume that $q(P_\Lambda(i)) = q(P_{\Lambda'}(i)) = q(i)$ for all $1 \leq i \leq n$. If the support of $M(u, i) = \tau_\Lambda^{-u}P_\Lambda(i)$ and $M'(i, u) = \tau_{\Lambda'}^{-u}P_\Lambda(i)$ coincide for all $0 \leq u \leq q(i)$ and $1 \leq i \leq n$, then $a_s(\Lambda) = a_s(\Lambda')$ for all s.*

Proof. We may interpret the numbers $a_s(\Lambda)$ and $a_s(\Lambda')$ as the number of antichains in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$, respectively, which have support-rank s. Note that the support of a module M is the set of numbers $1 \leq i \leq n$ such that $\operatorname{Hom}(P(i), M) \neq 0$.

Note that $\operatorname{Hom}(M(i, u), M(j, v)) = 0$ if and only if $\operatorname{Hom}(M'(i, u), M'(j, v)) = 0$. Namely, if $u \leq v$, the Auslander-Reiten translation (see for example [4]) furnishes a group isomorphism

$$\operatorname{Hom}(M(i, u), M(j, v)) \;\simeq\; \operatorname{Hom}(M(i, 0), M(j, v - u))$$

$$= \quad \mathrm{Hom}(P_\Lambda(i), M(j, v-u)),$$

and similarly we have $\mathrm{Hom}(M'(i,u), M'(j,v)) \simeq \mathrm{Hom}(P_{\Lambda'}(i), M'(j, v-u))$. It follows that $\mathrm{Hom}(M(i,u), M(j,v)) = 0$ if and only if i is not in the support of $M(j, v-u)$ if and only if i is not in the support of $M'(j, v-u)$ if and only if $\mathrm{Hom}(M'(i,u), M'(j,v)) = 0$.

If $u > v$, then

$$\mathrm{Hom}(M(i,0), M(j,v)) \simeq \mathrm{Hom}(M(i, u-v), M(j,0)) = 0,$$

since $M(i, u-v)$ is indecomposable and non-projective, whereas $M(j,0)$ is projective. Similarly, we also have $\mathrm{Hom}(M'(i,0), M'(j,v)) = 0$.

As a consequence we see that given an antichain $A = \{A_1, \ldots, A_t\}$ in $\mathrm{mod}\,\Lambda$, the function $M(i,u) \mapsto M'(i,u)$ yields an antichain $A' = \{A'_1, \ldots, A'_t\}$ in $\mathrm{mod}\,\Lambda'$. Of course, the support-rank of A and A' are the same. This completes the proof. $\square$

Corollary 12. *The numbers $a_s(\Lambda)$ depend only on the Dynkin type of $Q(\Lambda)$, not on Λ itself.*

Proof. According to Proposition 11, the numbers $a_s(\Lambda)$ depend only on $Q(\Lambda)$. According to Proposition 9, the orientation of $Q(\Lambda)$ does not play a role. $\square$

Thus, is Λ is of Dynkin type Δ, we write $a_s(\Delta)$ instead of $a_s(\Lambda)$.

Corollary 13. *For all $0 \le s \le n$, we have $a_s(\mathbb{B}_n) = a_s(\mathbb{C}_n)$.*

Proof. Apply the Proposition 11 to the algebras Λ and Λ' with valued quivers

$$\begin{array}{ccccccccccccc} \circ & \longleftarrow & \circ & \longleftarrow & \circ & \leftarrow & \cdots & - & \circ & \longleftarrow & \circ & \Leftarrow\!\!\Rightarrow & \circ \\ 1 & & 2 & & 3 & & & & n-2 & & n-1 & & n \end{array}$$

$$\begin{array}{ccccccccccccc} \circ & \longleftarrow & \circ & \longleftarrow & \circ & \leftarrow & \cdots & - & \circ & \longleftarrow & \circ & \Leftarrow\!\!\Leftarrow & \circ \\ 1 & & 2 & & 3 & & & & n-2 & & n-1 & & n \end{array}$$

respectively; the first valued quiver is of type $\mathbb{B}_n$, and the second is of type $\mathbb{C}_n$. It is well-known (and easy to see) that $q(P_\Lambda(i)) = n-1 = q(P_{\Lambda'}(i))$ for all $1 \le i \le n$ and that the modules $M(i,u)$ and $M'(i,u)$ for $1 \le i \le n$ and $0 \le u \le n-1$ have the same support. $\square$

5 The tilting modules for $\mathbb{B}_n$

We are going to determine the number of tilting modules for the Dynkin algebras of type $\mathbb{B}_n$; namely we will show that $a_n(\mathbb{B}_n) = \binom{2n-1}{n-1}$. By induction, we assume knowledge about the representation theory of $\mathbb{B}_i$ with $i < n$, as well as the calculation of $a_s(\mathbb{B}_n)$ for $s < n$ as shown in Section 6. We consider a Dynkin algebra Λ with quiver

$$\begin{array}{ccccccccccccc} \circ & \longleftarrow & \circ & \longleftarrow & \circ & \leftarrow & \cdots & - & \circ & \longleftarrow & \circ & \Leftarrow\!\!\Rightarrow & \circ \\ 1 & & 2 & & 3 & & & & n-2 & & n-1 & & n \end{array}$$

We interpret $a_n(\mathbb{B}_n)$ as the number of sincere antichains (by definition, an antichain $A = \{A_1, \ldots, A_t\}$ is *sincere* provided the module $\bigoplus A_i$ is sincere) and write it as the sum

$$a_n(\mathbb{B}_n) = u(\mathbb{B}_n) + v(\mathbb{B}_n)$$

where $u(\mathbb{B}_n)$ is the number of antichains with a sincere element, whereas $v(\mathbb{B}_n)$ is the number of sincere antichains without a sincere element. These two numbers will be calculated separately.

5.1 The calculation of $u(\mathbb{B}_n)$

We let $w(\mathbb{B}_n)$ denote the number of antichains which do not contain any injective module.

Lemma 14.

$$w(\mathbb{B}_n) = a_n(\mathbb{B}_n).$$

Proof. Let $\mathcal{W}$ be the set of antichains without injective modules and $\mathcal{S}$ the set of sincere antichains. We want to construct a bijection $\eta : \mathcal{S} \to \mathcal{W}$. Note that an element of $\mathcal{S}$ contains at most one injective module, since the injective modules are pairwise comparable with respect to Hom. If $A \in \mathcal{S}$ contains no injective module, then let $\eta(A) = A$. If $A \in \mathcal{S}$ contains the injective module $I(i)$, let $\eta(A)$ be obtained from A by deleting $I(i)$ and note that $\eta(A)$ is no longer sincere (since all the modules A_j in $\eta(A)$ satisfy $\mathrm{Hom}(A_j, I(i)) = 0$). It follows that η is an injective map. In order to show that η is surjective, assume that B is an antichain in $\mathcal{W}$. If B is sincere, then it belongs to $\mathcal{S}$ and by definition $\eta(B) = B$. If B is not sincere, let i be the smallest number such that i is not in the support of B. Let A be obtained from B by adding $I(i)$. Then clearly A is sincere and $\eta(A) = B$. $\square$

Remark 15. A similar proof applies to the linearly oriented quiver of type $\mathbb{A}_n$. It yields the formula

$$a(\mathbb{A}_{n-1}) = a_n(\mathbb{A}_n).$$

Also, instead of looking at antichains which do not contain any injective module, we may consider antichains which do not contain any projective module.

Now we are able to determine $u(\mathbb{B}_n)$.

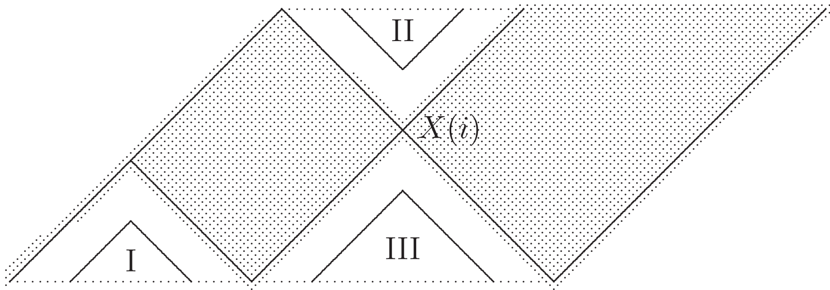
Lemma 16.

$$u(\mathbb{B}_n) = a_{n-1}(\mathbb{B}_n) = \binom{2n-2}{n-1}.$$

Proof. Note that the sincere indecomposable representations of Λ are the modules $X(i) = \tau^{-n+i}P(i)$ with $1 \leq i \leq n$. The dimension vector of $X(n)$ is $(1, \ldots, 1)$, whereas for $1 \leq i < n$, the length of $X(n)$ is $n + i$ and its dimension vector is of the form $(1, \ldots, 1) + (0, \ldots, 0, 1, \ldots, 1)$. It is easy to see that $\mathrm{Hom}(X(i), X(j)) \neq 0$ for $i \geq j$, thus any antichain contains at most one $X(i)$. Let $u_i(\mathbb{B}_n)$ be the antichains which contain $X(i)$, thus

$$u(\mathbb{B}_n) = \sum\nolimits_{i=1}^{n} u_i(\mathbb{B}_n).$$

Let $\mathcal{X}_i$ be the set of indecomposable modules M such that $\operatorname{Hom}(X(i), M) = 0 = \operatorname{Hom}(M, X(i))$. Thus, the antichains which contain $X(i)$ correspond bijectively to the antichains in $\mathcal{X}_i$. In general, the set $\mathcal{X}_i$ consists of three triangles I, II, III:



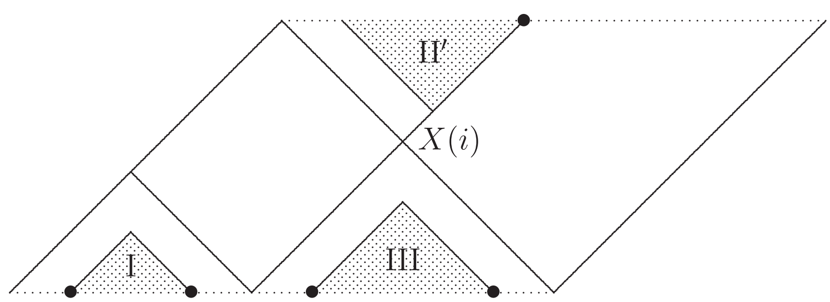
The triangle I is the wing at the vertex $\tau^{-1}P(n-i-1)$, the triangle II is the wing at the vertex $\tau^{-n+i-1}P(i+2)$, and the triangle III is the wing at the vertex $\tau^{-n+i+1}P(i-2)$.

We also are interested in a larger triangle II′ which contains the triangle II as well as $n-i$ additional modules (all being successors of $X(i)$), namely the wing at the vertex $\tau^{-n+i}P(i+1)$.

The full subcategory $\mathcal{X}'$ of all direct sums of indecomposable modules in the wings I, II′, III is the thick subcategory with simple objects

$$S(2), S(3), \dots, S(n-i+1); \quad \tau^{n-i}P(n); \quad S(n-i+3), \dots, S(n-1).$$

The position of these modules is indicated here by bullets:



(A full subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ is called a *thick* subcategory provided it is closed under kernels, cokernels and extensions (see for example [21]). Note that a thick subcategory is an abelian category, and the inclusion functor $\mathcal{A} \to \operatorname{mod}\Lambda$ is exact.)

The category $\mathcal{X}'$ is of type $\mathbb{B}_{n-i} \sqcup \mathbb{A}_{i-2}$ (the $\mathbb{A}_{i-2}$-part is given by the triangle III, whereas the $\mathbb{B}_{n-i}$-part is given by the triangles I and II′). Note that the indecomposables in I and II just correspond to the non-injective indecomposables in the $\mathbb{B}_{n-i}$-part. This shows that

$$u_i(\mathbb{B}_n) = w(\mathbb{B}_{n-i})a(\mathbb{A}_{i-2}) = a_{n-i}(\mathbb{B}_{n-i})a_{i-1}(\mathbb{A}_{i-1}).$$

In the special cases $i = 1, 2, n-1, n$, the same formula holds. Namely, for $i = 1$ and $i = 2$, the triangle III is empty, whereas the triangles I and II′ together yield a category of type $\mathbb{B}_{n-i}$. In the cases $i = n-1$ and $i = n$, the triangles I and II are empty, whereas the triangle III yields a category of type $\mathbb{A}_{i-2}$.

Thus we see

$$u(\mathbb{B}_n) = \sum_{i=1}^{n} u_i(\mathbb{B}_n) = \sum_{i=1}^{n} a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i}).$$

But the latter expression is the recursion formula for $a_{n-1}(\mathbb{B}_n)$, since the number of support-tilting modules T with support $\{1, 2, \ldots, n\} \setminus \{i\}$ is just $a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i})$. $\square$

5.2 The calculation of $v(\mathbb{B}_n)$

Lemma 17.

$$v(\mathbb{B}_n) = a_{n-2}(\mathbb{B}_{n+1}) = \binom{2n-2}{n-2}.$$

Proof. Let $\mathcal{V}$ be the set of sincere antichains of Λ-modules without a sincere element. Let $A = (A_1, \ldots, A_r)$ be in $\mathcal{V}$. Since A is sincere, we may assume that $\operatorname{Hom}(P(1), A_1) \neq 0$. Since A_1 is not sincere, we must have $\operatorname{Hom}(P(n), A_1) = 0$, thus A_1 is a representation of a Dynkin algebra of type $\mathbb{A}_{n-1}$ and actually an indecomposable projective representation (also as a Λ-module), thus $A_1 = P(i)$ for some i with $1 \leq i < n$. Since an antichain can contain only one indecomposable projective module, we see that A_1 is uniquely determined.

We let $\mathcal{V}_i$ denote the sincere antichains A such that $A_1 = P(i)$. For $2 \leq j \leq r$, we have $\operatorname{Hom}(P(i), A_j) = \operatorname{Hom}(A_1, A_j) = 0$. It follows that $(A_2, \ldots, A_r)$ is an antichain with support in $[1, i-1] \cup [i+1, n]$. Altogether, we see that any element of A has support either in $[1, i]$ or in $[i+1, n]$. The elements of A with support in $[1, i]$ but different from A_1 form an arbitrary antichain with support in $[2, i-1]$, thus the number of elements is $a(\mathbb{A}_{i-2})$, at least if $i \geq 2$. Note that $a(\mathbb{A}_{i-2}) = a_{i-1}(\mathbb{A}_{i-1})$.

The elements of A with support in $[i+1, n]$ form a sincere antichain for $\mathbb{B}_{n-i}$. Thus the number of such antichains is $a_{n-i}(\mathbb{B}_{n-i})$. This shows that for $i \geq 2$, the set $\mathcal{V}_i$ has cardinality $a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i})$. This formula holds true also for $i = 1$, since the number of elements of $\mathcal{V}_1$ is $a_{n-1}(\mathbb{B}_{n-1})$ and $a_0(\mathbb{A}_0) = 1$. Thus we see that

$$\begin{aligned}
v(\mathbb{B}_n) &= \sum_{i=1}^{n-1} a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i}) \\
&= -a_{n-1}(\mathbb{A}_{n-1}) + \sum_{i=1}^{n} a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i}) \\
&= -\frac{1}{n}\binom{2n-2}{n-1} + \binom{2n-2}{n-1} = \binom{2n-2}{n-2}.
\end{aligned}$$

Altogether we see

$$u(\mathbb{B}_n) + v(\mathbb{B}_n) = \binom{2n-2}{n-1} + \binom{2n-2}{n-2} = \binom{2n-1}{n-1}.$$

$\square$

Remark 18. The calculation of $v(\mathbb{B}_n)$ shows the following relationship between the cases $\mathbb{A}$ and $\mathbb{B}$:

$$a_{n-1}(\mathbb{B}_n) = a_{n-2}(\mathbb{B}_{n+1}) + a_{n-1}(\mathbb{A}_{n-1}).$$

6 Support-tilting modules: the hook formula

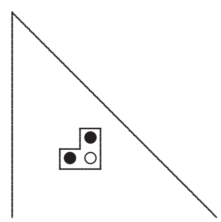
6.1 The hook formula

Proposition 19. *Let $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{E}$. Then*

$$a_s(\Delta_n) = a_s(\Delta_{n-1}) + a_{s-1}(\Delta_n)$$

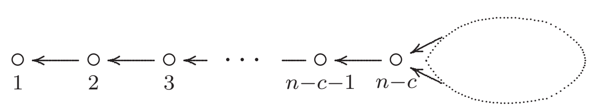
for all $n \geq m$ and $1 \leq s \leq n-c$, where $m = 1, 2, 3, 4$ and $c = 0, 1, 2, 3$ for $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{E}$, respectively.

Here we use the convention that $\mathbb{B}_1 = \mathbb{A}_1, \mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1, \mathbb{E}_3 = \mathbb{A}_2 \sqcup \mathbb{A}_1, \mathbb{E}_4 = \mathbb{A}_4, \mathbb{E}_5 = \mathbb{D}_5$. In the triangles 2.1, 2.2, 2.3, as well as in Section 3, this equality concerns the following kind of hooks:

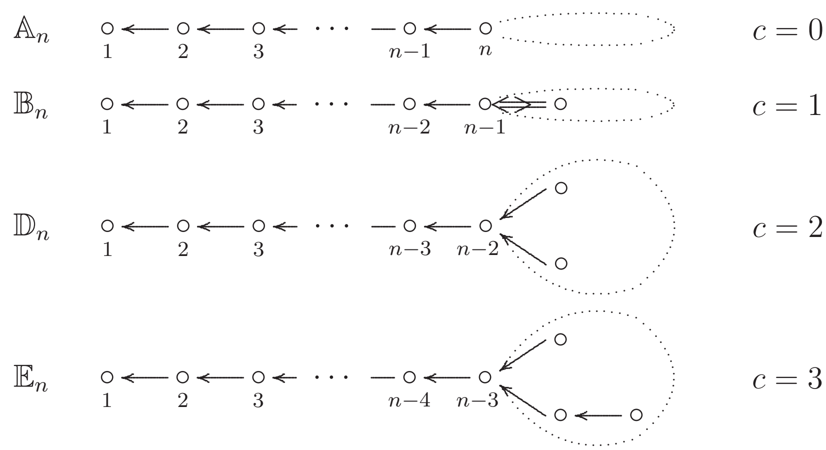


The hook formula asserts that the sum of the values at the positions marked by bullets is the value at the position marked by the circle.

The various assertions concern the following general situation: up to the choice of an orientation, we deal with an artin algebra Λ with the following valued quiver with n vertices:



on the left, we have a quiver of type $\mathbb{A}_{n-c}$ with arrows $i \leftarrow i+1$. The remaining c vertices are in the dotted "cloud" to the right. All arrows between the cloud and the $\mathbb{A}_{n-c}$–quiver end in the vertex $n-c$. We let Q' denote the valued quiver obtained by deleting the vertex 1 and the arrow ending in 1; let Λ' be the corresponding factor algebra of Λ. Here are the cases we are interested in.



Lemma 20. *Let* $1 \le s \le n - c$. *Then*

$$a_s(\Lambda) = a_s(\Lambda') + a_{s-1}(\Lambda).$$

Proof. The support-tilting modules T for Λ with 1 not in the support are just the support-tilting modules for Λ'. Let $\mathcal{S}_s(\Lambda; 1)$ be the set of the basic support-tilting Λ-modules T with support-rank s and $\mathrm{Hom}(P(1), T) \neq 0$. Let $\mathcal{S}_{s-1}(\Lambda)$ be the set of basic support-tilting Λ-modules T with support-rank $s - 1$. We construct a bijection

$$\alpha : \mathcal{S}_s(\Lambda; 1) \longrightarrow \mathcal{S}_{s-1}(\Lambda).$$

This will establish the formula.

Let X be an indecomposable representation with support-rank $s \le n-c$ and $\mathrm{Hom}(P(1), X) \neq 0$. Then the support of X is contained in the $\mathbb{A}_{n-c}$-subquiver, so X is thin and its support is an interval of the form $[1, v]$ with $1 \le v \le n - c$ (a module is said to be *thin* provided the composition factors are pairwise non-isomorphic; in our setting thin indecomposable modules are uniquely determined by the support, thus we may just write $X = [1, v]$).

Let T be a module in $\mathcal{S}_s(\Lambda; 1)$. At least one of the indecomposable direct summand of T, say X, satisfies $\mathrm{Hom}(P(1), X) \neq 0$ and we choose $X = [1, v]$ of largest possible length. We claim that $\mathrm{Hom}(P(w), T) = 0$ for any arrow $v \leftarrow w$. Assume, to the contrary, that there is an indecomposable direct summand Y of T with $\mathrm{Hom}(P(w), Y) \neq 0$. The maximality of X shows that $\mathrm{Hom}(P(1), Y) = 0$. But then $\mathrm{Ext}(Y, X) \neq 0$ contradicts the fact that T has no self-extensions. (Namely, if the support of X and Y is disjoint, then the arrow $v \leftarrow w$ yields directly a non-trivial extension of X by Y; if the support of X and Y is not disjoint, then there is a proper non-zero factor module of X which is a proper submodule of Y, thus there is a non-zero map $X \to Y$ which is neither injective nor surjective — again we obtain a non-trivial extension of X by Y.) Thus the support of T is the disjoint union of the set $\{1, 2, \ldots, v\}$ and a set S'' which does not contain a vertex w with an arrow $v \leftarrow w$.

The indecomposable direct summands of T with support in $\{1, 2, \ldots, v\}$ yield a tilting module for this $\mathbb{A}_v$-quiver, and X is the indecomposable projective-injective representation of this $\mathbb{A}_v$-quiver. Deleting X from this tilting module, we obtain a support-tilting representation of $\mathbb{A}_v$ with support-rank $v-1$.

Thus if we write $T = X \oplus T'$, then T' is a support-tilting Λ-module with support-rank $s-1$ (namely, it is the direct sum of a support-tilting module with support properly contained in $\{1, 2, \ldots, v\}$ and a support-tilting module with support S''). We define $\alpha(T) = T'$; this yields the map

$$\alpha : \mathcal{S}_s(\Lambda; 1) \longrightarrow \mathcal{S}_{s-1}(\Lambda)$$

we are looking for. It remains to be shown that α is surjective and that we can recover T from $\alpha(T)$.

Thus, let T' be in $\mathcal{S}_{s-1}(\Lambda)$. Then there are at least $c+1$ vertices outside of the support of T'.

Case 1: These are the vertices in the cloud and precisely one additional vertex, say i (with $1 \leq i \leq n-c$). Note that in this case $s = n-c$. Let $T = T' \oplus [1, n-c]$. Since T' is a support-tilting module of $\mathbb{A}_{n-c}$ with support-rank $n-c-1$ and $[1, n-c]$ is the indecomposable projective-injective representation of $\mathbb{A}_{n-c}$, we see that $T = T' \oplus [1, n-c]$ is a tilting module for $\mathbb{A}_{n-c}$.

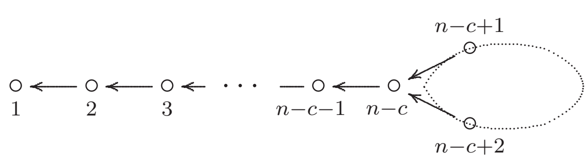
Case 2: At least two vertices between 1 and $n-c$ do not belong to $\operatorname{Supp} T'$, say let $i < j$ be the smallest such numbers. Then let $T = T' \oplus [1, j-1]$. $\square$

6.2 The modified hook formula

Proposition 21.

$$\begin{aligned}
a_{n-1}(\mathbb{D}_n) &= a_{n-1}(\mathbb{D}_{n-1}) + a_{n-2}(\mathbb{D}_n) + a_{n-2}(\mathbb{A}_{n-2}), \\
a_{n-2}(\mathbb{E}_n) &= a_{n-2}(\mathbb{E}_{n-1}) + a_{n-3}(\mathbb{E}_n) + a_{n-3}(\mathbb{A}_{n-3}).
\end{aligned}$$

Again, we consider a general setting, namely we consider an artin algebra Λ with the following valued quiver with n vertices and we assume that $c \geq 2$:



On the left, we have a quiver of type $\mathbb{A}_{n-c}$ with arrows $i \leftarrow i+1$, and the remaining c vertices are in the dotted "cloud" to the right. There are precisely two vertices in the cloud, namely $n-c+1$ and $n-c+2$ with arrows $n-c \leftarrow n-c+1$ and $n-c \leftarrow n-c+2$ and there is no other arrows between the cloud and the $\mathbb{A}_{n-c}$ quiver. Again, we let Q' denote the valued quiver obtained by deleting the vertex 1 and the arrow ending in 1 and by Λ' the corresponding factor algebra of Λ and we show the following:

Lemma 22.

$$a_{n-c+1}(\Lambda) = a_{n-c+1}(\Lambda') + a_{n-c}(\Lambda) + a_{n-c}(\mathbb{A}_{n-c}).$$

Proof. The proof follows closely the proof of Proposition 19. The support-tilting modules T for Λ with 1 not in the support are just the support-tilting modules for Λ'. We construct a surjection α from the set $\mathcal{S}_{n-c+1}(\Lambda;1)$ of the support-tilting Λ-modules T with support-rank $n-c+1$ and $\mathrm{Hom}(P(1),T)\neq 0$ onto the set $\mathcal{S}_{n-c}(\Lambda)$ of support-tilting Λ-modules T with support-rank $n-c$. In the present setting, α will not be injective, but will be a double cover: pairs in $\mathcal{S}(\Lambda;1)$ are identified by α; the number of such pairs will be just $a_{n-c}(\mathbb{A}_{n-c})$.

As above, one shows that any module T in $\mathcal{S}_{n-c+1}(\Lambda;1)$ is of the form $T=X\oplus T'$ where X is indecomposable, $\mathrm{Hom}(P(1),X)\neq 0$ and X is of maximal possible length. Note that the support of X is contained either in $\{1,2,\dots,n-c+1\}$ or in $\{1,2,\dots,n-c,n-c+2\}$. In particular, X is uniquely determined (since the support of T cannot contain all the vertices $1,2,\dots,n-c+2$). As above, the mapping α will be the deletion of the summand X.

Let Z be the indecomposable module with support $\{1,2,\dots,n-c+1\}$ and Z' the indecomposable module with support $\{1,2,\dots,n-c,n-c+2\}$. Starting with a tilting module T' for $\mathbb{A}_{n-c}$, we may form the direct sums $Z\oplus T'$ and $Z'\oplus T'$. Then these are elements of $\mathcal{S}_{n-c+1}(\Lambda;1)$, both of which are mapped under α to the same module T'. These are the $a_{n-c}(\mathbb{A}_{n-c})$ pairs of elements of $\mathcal{S}(\Lambda;1)$ which are identified by α.

It follows that $\mathcal{S}(\Lambda;1)$ has cardinality $a_{n-c}(\Lambda)+a_{n-c}(\mathbb{A}_{n-c})$. ∎

Proof of Proposition 21. The two assertions of Proposition 21 are special cases of Lemma 22. For the first assertion, Λ is of type $\mathbb{D}_n$, Λ' of type $\mathbb{D}_{n-1}$, and $c=2$. For the second assertion, Λ is of type $\mathbb{E}_n$, Λ' of type $\mathbb{E}_{n-1}$ and $c=3$. ∎

Remark 23. For another proof of the modified hook formula, see Hubery [17].

Corollary 24.

$$a_{n-1}(\mathbb{D}_n)=\begin{bmatrix}2n-3\\ n-1\end{bmatrix}.$$

Proof. We start with the previous observation

$$\begin{aligned}a_{n-1}(\mathbb{D}_n) &= a_{n-1}(\mathbb{D}_{n-1})+a_{n-2}(\mathbb{D}_n)+a_{n-2}(\mathbb{A}_{n-2})\\ &= \frac{3n-7}{2n-4}\binom{2n-4}{n-3}+\frac{3n-6}{2n-4}\binom{2n-4}{n-2}+\frac{1}{n-1}\binom{2n-4}{n-2}.\end{aligned}$$

Write

$$\begin{aligned}\binom{2n-4}{n-3} &= \frac{n-2}{2n-3}\binom{2n-3}{n-1},\\ \binom{2n-4}{n-2} &= \frac{n-1}{2n-3}\binom{2n-3}{n-1}.\end{aligned}$$

One easily shows that

$$\frac{3n-7}{2n-4}\cdot\frac{n-2}{2n-3}+\frac{3n-6}{2n-4}\cdot\frac{n-1}{2n-3}+\frac{1}{n-1}\cdot\frac{n-1}{2n-3} = \frac{3n-4}{2n-3}.$$

As a consequence, we get

$$\begin{aligned} a_{n-1}(\mathbb{D}_n) &= \frac{3n-4}{2n-3}\binom{2n-3}{n-1} \\ &= \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix}. \end{aligned}$$

□

7 Summation formulas

An immediate consequence of the previous section is the following assertion:

Proposition 25. *Let* $\Delta = \mathbb{A}$, *or* $\mathbb{B}$ *and* $n \geq 0$, *or* $\Delta = \mathbb{D}$ *and* $n \geq 2$. *If* $1 \leq s \leq n-1$, *then*

$$\sum_{i=0}^{s} a_i(\Delta_n) = a_s(\Delta_{n+1}).$$

Proof. We use induction. For $s = 0$ both sides are equal to 1. For $s \geq 1$ we have

$$\begin{aligned} \sum_{i=0}^{s} a_i(\Delta_n) &= a_s(\Delta_n) + \sum_{i=0}^{s-1} a_i(\Delta_n) \\ &= a_s(\Delta_n) + a_{s-1}(\Delta_{n+1}) \\ &= a_s(\Delta_{n+1}), \end{aligned}$$

the last equality being the hook formula. □

Corollary 26. *Let* $\Delta = \mathbb{A}, \mathbb{B}$ *or* $\mathbb{D}$. *Then*

$$a(\Delta_n) = a_n(\Delta_n) + a_{n-1}(\Delta_{n+1})$$

Case $\mathbb{A}_n$

$$a(\mathbb{A}_n) = \frac{1}{n+1}\binom{2n}{n} + \frac{3}{n+2}\binom{2n}{n-1} = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

Case $\mathbb{B}_n$

$$a(\mathbb{B}_n) = \binom{2n-2}{n-1} + \binom{2n-1}{n} = \binom{2n}{n}.$$

Case $\mathbb{D}_n$

$$a(\mathbb{D}_n) = \begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix} + \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix} = \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}.$$

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