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Carlitz's Identity for the Bernoulli Numbers and Zeon Algebra

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Abstract

In this work we provide a new short proof of Carlitz's identity for the Bernoulli numbers. Our approach is based on the ordinary generating function for the Bernoulli numbers and a Grassmann-Berezin integral representation of the Bernoulli numbers in the context of the Zeon algebra, which comprises an associative and commutative algebra with nilpotent generators.

1 Introduction

In this work we will give a new, simple and short proof of Carlitz's identity for the Bernoulli numbers [6]

$$\sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^{m+n} \sum_{j=0}^{n} \binom{n}{j} B_{m+j}, \tag{1}$$

using the Zeon algebra [16, 17]. The identity in Eq. (1) has been re-obtained many times [7, 8, 12, 23, 25] and also very recently [13, 18, 24]. The proof given here is of independent interest, because of the simplicity of the arguments involved and, as it occurred in other contexts [1, 2, 5, 10, 11, 15, 16, 17, 20, 21, 22], the proof comprises another example of

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the usefulness of using the Zeon algebra and/or the Grassmann algebra towards obtaining combinatorial identities.

Before we continue, we establish the basic underlying algebraic setup needed to give the proof of Eq. (1). Throughout this work we let $\mathbb{Q}$ and $\mathbb{R}$ denote the rational and real numbers, respectively.

2 Basic definitions: Zeon algebra and the Grassmann-Berezin integral

Definition 1. The *Zeon algebra* $\mathcal{Z}_n \supset \mathbb{R}$ is defined as the associative algebra generated by the collection $\{\varepsilon_i\}_{i=1}^n$ $(n < \infty)$ and the scalar $1 \in \mathbb{R}$, such that $1\varepsilon_i = \varepsilon_i = \varepsilon_i 1$, $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$ $\forall$ i, j and $\varepsilon_i^2 = 0$ $\forall$ i.

Note that only linear elements in $\mathcal{Z}_n$ contribute to the calculations.

For $\{i, j, \ldots, k\} \subset \{1, 2, \ldots, n\}$ and $\varepsilon_{ij\cdots k} \equiv \varepsilon_i\varepsilon_j \cdots \varepsilon_k$ the most general element with n generators ε_i can be written as (with the convention of sum over repeated indices implicit)

$$\phi_n = a + a_i\varepsilon_i + a_{ij}\varepsilon_{ij} + \cdots + a_{12\cdots n}\varepsilon_{12\cdots n} = \sum_{\mathbf{i}\in 2^{[n]}} a_{\mathbf{i}}\varepsilon_{\mathbf{i}}, \tag{2}$$

with a, a_i, a_{ij}, ..., $a_{12\cdots n} \in \mathbb{R}$, $2^{[n]}$ being the power set of $[n] := \{1, 2, \ldots, n\}$, and $1 \leq i < j < \cdots \leq n$. We refer to a as the body of ϕ_n and write $b(\phi_n) = a$ and to $\phi_n - a$ as the soul such that $s(\phi_n) = \phi_n - a$.

Definition 2. The *Grassmann-Berezin integral* on $\mathcal{Z}_n$, denoted by $\int$, is the linear functional $\int : \mathcal{Z}_n \to \mathbb{R}$ such that (we use throughout this work the compact notation $d\mu_n := d\varepsilon_n \cdots d\varepsilon_1$)

$$d\varepsilon_i d\varepsilon_j = d\varepsilon_j d\varepsilon_i, \ \int \phi_n\big(\hat{\varepsilon}_i\big) d\varepsilon_i = 0 \text{ and} \int \phi_n\big(\hat{\varepsilon}_i\big)\varepsilon_i d\varepsilon_i = \phi_n\big(\hat{\varepsilon}_i\big),$$

where $\phi_n\big(\hat{\varepsilon}_i\big)$ means any element of $\mathcal{Z}_n$ with no dependence on ε_i. Multiple integrals are iterated integrals, i.e.,

$$\int f(\phi_n) d\mu_n = \int \cdots \left(\int \left(\int f(\phi_n) d\varepsilon_n\right) d\varepsilon_{n-1}\right) \cdots d\varepsilon_1.$$

For example, if we define $\varphi_n := \varepsilon_1 + \cdots + \varepsilon_n$ it follows directly from Definition 2 and the multinomial theorem that

$$\int \varphi_n^i d\mu_n = i!\delta_{i,n}, \tag{3}$$

where $\delta_{i,n}$ is the Kronecker delta. For more details on Grassmann-Berezin integration, we refer the reader to the books of Berezin [3, Chapter 1] and [4, Chapter 2] or the books of DeWitt [9, Chapter 1] and Rogers [19, Chapter 11].

We will now recall some basic facts about the Zeon algebra. First, $a+\phi_n$ with $s(a) = 0 = b(\phi_n)$ is invertible iff $b(a)\neq 0$. More precisely, we have

$$\frac{1}{a+\phi_n} = \frac{1}{a}\left(1-\frac{\phi_n}{a}+\frac{\phi_n^2}{a^2}+\cdots+(-1)^n\frac{\phi_n^n}{a^n}\right). \tag{4}$$

Second, the following expression holds

$$e^{\varphi_n} := \sum_{i=0}^{\infty}\frac{\varphi_n^i}{i!} = \sum_{i=0}^{n}\frac{\varphi_n^i}{i!} = 1+\sum_{1\leq i\leq n}\varepsilon_i+\sum_{1\leq i<j\leq n}\varepsilon_{ij}+\cdots+\varepsilon_{12\cdots n}. \tag{5}$$

To obtain Eq. (5) we have used the multinomial theorem and $\varphi_n^{n+1}=0\ \forall\ n\geq 1$. Third, let $\phi_n\left(\hat{\varepsilon}_i,\hat{\varepsilon}_j,\ldots,\hat{\varepsilon}_k\right)$ and $d\mu_n\left(\hat{\varepsilon}_i,\hat{\varepsilon}_j,\ldots,\hat{\varepsilon}_k\right)$ mean ϕ_n with $\varepsilon_i=\varepsilon_j=\cdots=\varepsilon_k=0$ and $d\mu_n$ with $d\varepsilon_i$, $d\varepsilon_j$, $\ldots$, $d\varepsilon_k$ omitted, respectively. We have

$$\int\phi_n\varepsilon_{ij\cdots k}d\mu_n = \int\phi_n\left(\hat{\varepsilon}_i,\hat{\varepsilon}_j,\ldots,\hat{\varepsilon}_k\right)\varepsilon_{ij\cdots k}d\mu_n = \int\phi_n\left(\hat{\varepsilon}_i,\hat{\varepsilon}_j,\ldots,\hat{\varepsilon}_k\right)d\mu_n\left(\hat{\varepsilon}_i,\hat{\varepsilon}_j,\ldots,\hat{\varepsilon}_k\right). \tag{6}$$

Eq. (6) follows directly from the general expression in Eq. (2) and Definition 2. Finally, from Definition 2, we conclude that the order of integration is irrelevant, i.e., a Fubini-like theorem holds in the setting of Grassmann-Berezin integration.

We are now ready to prove Eq. (1).

3 Proof of Eq. (1)

Let us write $\mathbb{Q}[[z]]$ for the ring of formal power series in the variable z over $\mathbb{Q}$. We recall the generating function for the Bernoulli numbers B_j in $\mathbb{Q}[[z]]$ [26], i.e.,

$$\frac{1}{\sum_{i=0}^{\infty}\frac{z^i}{(i+1)!}} = \frac{z}{e^z-1} = \sum_{j=0}^{\infty}B_j\frac{z^j}{j!} \tag{7}$$

and, making the change $z\to -z$ in Eq. (7), we get

$$\frac{e^z}{\sum_{i=0}^{\infty}\frac{z^i}{(i+1)!}} = \frac{ze^z}{e^z-1} = \sum_{j=0}^{\infty}B_j\frac{(-z)^j}{j!}. \tag{8}$$

Following the strategy of our previous work [16, 17], we consider Eqs. (7) and (8) in the context of the Zeon algebra with the replacement $z\to\phi_k\equiv\varphi_k$. Therefore, we get

$$\frac{1}{\sum_{i=0}^{k}\frac{\varphi_k^i}{(i+1)!}} = \sum_{j=0}^{k}B_j\frac{\varphi_k^j}{j!} \tag{9}$$

and

$$\frac{e^{\varphi_k}}{\sum_{i=0}^{k} \frac{\varphi_k^i}{(i+1)!}} = \sum_{j=0}^{k} B_j \frac{(-\varphi_k)^j}{j!}, \tag{10}$$

using that $\varphi_k^{k+1} = 0 \; \forall \; k \geq 1$. We observe that $b(\sum_{i=0}^{k} \frac{\varphi_k^i}{(i+1)!}) = 1 \neq 0$ and, hence, $\sum_{i=0}^{k} \frac{\varphi_k^i}{(i+1)!}$ is invertible in $\mathcal{Z}_k$.

Now, integrating Eq. (9) in the Zeon algebra and using Eq. (3) we get

$$\int \frac{1}{\sum_{i=0}^{j} \frac{\varphi_j^i}{(i+1)!}} d\mu_j = \sum_{k=0}^{j} \frac{B_k}{k!} \int \varphi_j^k d\mu_j = B_j \tag{11}$$

$\forall \; j \geq 1$. It is straightforward to verify that the representation in Eq. (11) is equivalent to a well-known representation of the Bernoulli numbers [14, Theorem 3.1], i.e.,

$$B_n = n! \sum_{i=1}^{n} (-1)^i \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \cdots + i_n = i \\ i_1 + 2i_2 + \cdots + ni_n = n}} \frac{i!}{i_1! i_2! \cdots i_n!} \frac{1}{2!^{i_1} 3!^{i_2} \cdots (n+1)!^{i_n}}.$$

Indeed, we have

$$\begin{aligned}
B_n &= \sum_{i=1}^{n} (-1)^i \int \left(\frac{\varphi_n}{2!} + \frac{\varphi_n^2}{3!} + \cdots + \frac{\varphi_n^n}{(n+1)!} \right)^i d\mu_n \\
&= \sum_{i=1}^{n} (-1)^i \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \cdots + i_n = i}} \frac{i!}{i_1! i_2! \cdots i_n!} \int \frac{\varphi_n^{i_1} \varphi_n^{2i_2} \cdots \varphi_n^{ni_n}}{2!^{i_1} 3!^{i_2} \cdots (n+1)!^{i_n}} d\mu_n \\
&= n! \sum_{i=1}^{n} (-1)^i \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \cdots + i_n = i}} \frac{i!}{i_1! i_2! \cdots i_n!} \frac{\delta_{n, i_1 + 2i_2 + \cdots + ni_n}}{2!^{i_1} 3!^{i_2} \cdots (n+1)!^{i_n}} \\
&= n! \sum_{i=1}^{n} (-1)^i \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \cdots + i_n = i}} \frac{i!}{i_1! i_2! \cdots i_n!} \frac{\delta_{n, i_1 + 2i_2 + \cdots + ni_n}}{2!^{i_1} 3!^{i_2} \cdots (n+1)!^{i_n}},
\end{aligned}$$

using Eqs. (3), (4) and the multinomial theorem.

By considering Eq. (10), we take $k = m + n$ and write $\varphi_{m+n} = \varphi_m + \phi_n$ with $\varphi_m := \varepsilon_1 + \cdots + \varepsilon_m$, $\phi_n := \epsilon_1 + \cdots + \epsilon_n$, and $\epsilon_i := \varepsilon_{i+m} \; \forall \; 1 \leq i \leq n$. Next, we multiply both sides of Eq. (10) by $e^{-\phi_n}$. Finally, integrating the resulting equation with $d\mu_m := d\varepsilon_m \cdots d\varepsilon_1$ and $d\nu_n := d\epsilon_n \cdots d\epsilon_1$ we get

$$\int \left(\int \frac{e^{\varphi_m}}{\sum_{i=0}^{m+n} \frac{(\varphi_m + \phi_n)^i}{(i+1)!}} d\mu_m \right) d\nu_n = \sum_{j=0}^{m+n} \frac{B_j}{j!} \int \left(\int (-\varphi_m - \phi_n)^j e^{-\phi_n} d\nu_n \right) d\mu_m. \tag{12}$$

In Eq. (12) we have used a Fubini-like argument to perform the integrations. We first consider the left-hand side of Eq. (12). By expanding e^{φ_m} as in Eq. (5) and integrating with respect to $d\mu_m$ we will need to analyze terms such as

$$\sum_{1\le i_1<i_2<\cdots<i_j\le m} \int \left(\int \frac{\varepsilon_{i_1 i_2 \cdots i_j}}{\sum_{i=0}^{m+n} \frac{(\varphi_m+\phi_n)^i}{(i+1)!}} d\mu_m \right) d\nu_n$$
$$= \binom{m}{j} \int \left(\int \frac{1}{\sum_{i=0}^{m-j+n} \frac{(\varphi_{m-j}+\phi_n)^i}{(i+1)!}} d\mu_{m-j} \right) d\nu_n = \binom{m}{j} B_{n+m-j}. \tag{13}$$

Therefore, using Eq. (13), we get for the left-hand side of Eq. (12)

$$\sum_{i=0}^{m} \binom{m}{i} B_{n+i}. \tag{14}$$

Similarly, we expand $e^{-\phi_n}$ as in Eq. (5) and integrate with respect to $d\nu_n$ to obtain for the right-hand side of Eq. (12)

$$(-1)^{m+n} \sum_{j=0}^{n} \binom{n}{j} B_{m+j}. \tag{15}$$

By equating the expressions in (14) and (15) we obtain the desired result, i.e., Eq. (1).

Let $B_i^{(j)}$ be the i-th Bernoulli number of order j with generating function in $\mathbb{Q}[[z]]$ given by

$$\left(\frac{z}{e^z - 1} \right)^j = \sum_{i=0}^{\infty} B_i^{(j)} \frac{z^i}{i!}.$$

Note that $B_n^{(1)} \equiv B_n$. Following the procedure just described, it is straightforward to prove an analogous identity for the Bernoulli numbers of higher order, i.e.,

$$\sum_{i=0}^{m} k^i \binom{m}{i} B_{n+i}^{(k)} = (-1)^{m+n} \sum_{j=0}^{n} k^j \binom{n}{j} B_{m+j}^{(k)}.$$

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