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Enumerating Families of Labeled Graphs

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Abstract

In 1966, Rosa introduced, among others, α- and β-labelings as tools to solve the isomorphic decomposition problem of the complete graph. Ten years later, Sheppard calculated the number of α- and β-labeled graphs with n edges. In this paper we use an extended version of the adjacency matrix of a graph to determine the number of gracefully labeled graphs with n edges; furthermore, we also calculate the number of them with m vertices for every suitable value of m. In addition, we use this technique to determine the number of labeled graphs for other types of labelings as the harmonious, felicitous, and elegant.

1 Introduction

By a *graph* we mean a finite undirected graph with no loops or multiple edges (for all undefined graph-theoretical terminology see [3, 4]). In addition, graphs considered here have no isolated vertices. By a *labeling* f of a graph G, of order m and size n, we understand an injective mapping from $V(G)$ into a subset of the non-negative integers. By L_f we denote the set of labels $f(v_i)$ of the vertices v_i of G. The set W_f is formed by all the numbers $|f(v_i) - f(v_j)|$ where v_iv_j is an edge of G.

Now consider a labeling f of a graph G of size n and the following conditions:

(a) $L_f \subseteq \{0, 1, \ldots, n\}$;

(b) $W_f = \{1, 2, \ldots, n\}$;

(c) there exists λ, $\lambda \in \{0, 1, \ldots, n\}$, such that for any edge v_iv_j of the graph G, either $f(v_i) \leq \lambda < f(v_j)$ or $f(v_j) \leq \lambda < f(v_i)$ holds.

A labeling satisfying the conditions (a) and (b) is called a β-*labeling* (or *graceful* labeling). If in addition the condition (c) is satisfied, we have an α-*labeling*.

For every $v \in V(G)$, the number $f(v)$ is called the *label* of v; for every $uv \in E(G)$, the number $|f(u) - f(v)|$ is called the *weight* of uv. When f is a β-labeling we say that G is *graceful* or it is β-*labeled*; if f is an α-labeling we say that G is an α-*graph* or it is α-*labeled*. The number λ in (c) is called the *boundary value* of f.

These labelings have been intensively studied since their introduction by Rosa [9] in the mid-sixties. Rosa called them β- and α-valuations, respectively. The term graceful labeling was introduced by Golomb [5] years later to refer to Rosa's β-valuation, and it is the standard name nowadays. Multiple families of graphs have been proven to be graceful; however, the graceful tree conjecture (that states that every tree is graceful) remains unsolved. For more information about graph labelings, the interested reader is refered to Gallian's survey [4].

Let G be a graph of order m and size n. It is well-known that if G is graceful, then $m - n \leq 1$. Also, if G is an α-graph, then G is bipartite. Suppose that f is a β-labeling of G, then its complementary labeling $\overline{f}$, defined by $\overline{f}(v) = n - f(v)$, for every $v \in V(G)$, is also a β-labeling. This fact can be used to prove that the number of different β-labelings of G is even, except when $G \cong K_2$.

When f is an α-labeling of G with boundary value λ, the α-labeling f^{-1} of G given by,

$$f^{-1}(v) = \lambda - f(v) \pmod{n+1},$$

is called the *inverse* labeling of f. Thus, depending on the symmetry of the structure of G, the number of different α-labelings of G is a multiple of 4.

In 1976, Sheppard [10] counted the number of different α- and β-labeled graphs with n edges. He used a sequence of integers to represent the labelings, showing the existence of an injective correspondence between the size n and these n-element sequences. Sheppard proved that the number of β-labeled graphs of size n is $n!$. He also counted the number of α-labeled graphs of size n to be

$$2\sum_{\lambda=0}^{\frac{n-2}{2}} (\lambda!)^2(\lambda+1)^{n-2\lambda}$$

when n is even, and

$$2\sum_{\lambda=0}^{\frac{n-3}{2}} (\lambda!)^2(\lambda+1)^{n-2\lambda} + \left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!$$

when n is odd.

In Section 2 we introduce another representation of gracefully labeled graphs which simplifies the attainment of the previous formulas. We also use this representation to count the number of labeled graphs where the labeling used is harmonious, felicitous, or elegant. In Section 3 we determine the exact number of gracefully labeled graphs of size n and order m.

2 Graceful Triangles

Let f be a β-labeling of a graph G of order m and size n. The β-labeled graph G can be described by means of matrices. One such matrix is the *graceful matrix* $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if there exists $uv \in E(G)$ such that $f(u) = i$ and $f(v) = j$, and $a_{ij} = 0$ otherwise, for $0 \le i, j \le n$. These matrices were introduced by Zhi-Zeng [14]. The graceful matrix is just an extension of the adjacency matrix and as this one, it is symmetric and all the elements in the main diagonal equal zero. Therefore, all the characteristics of the β-labeled graph G can be seen in the triangular arrangement formed by the cells a_{ij} of $A(G)$ where $i < j$, $0 \le i \le n-1$, and $1 \le j \le n$. We call this arragement the *graceful triangle*. In Figure 1 we show a β-labeled graph of order 6 and size 10 together with its graceful triangle, where the adjacencies have been represented by dots.

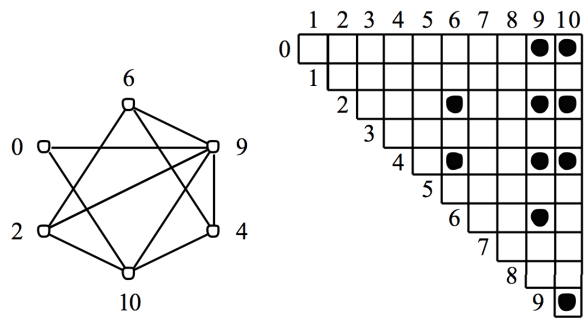

Figure 1: Graceful Triangle

The graceful matrix tool has been used by other authors. Barrientos [1], Zhi-Zeng [14], and Shiue and Lu [11], use it to prove the existence of α- and β-labelings for certain families of graphs. In addition, Haviar and Ivaška [7] use, in their book, a similar tool to count these types of labeled graphs, providing an alternative proof to Sheppard's result.

In general, for a β-labeled graph G of order m and size n, its graceful triangle contains n diagonals $d_1, d_2, \ldots, d_n$, where d_k is formed by the cells a_{ij} where $j-i = n+1-k$, $1 \le k \le n$. Thus d_k has exactly k cells. Moreover, each diagonal contains exactly one adjencency or dot.

If in a triangular arrangement with n diagonals, every diagonal has one dot, then clearly the arrangement is the graceful triangle of a β-labeled graph of size n. Thus, we can use them to count the number of β-labeled graphs with n edges.

Let $d_1, d_2, \ldots, d_n$ be the diagonals of a triangular arrangement. Since for every $1 \leq k \leq n$, the diagonal d_k has k cells and only one of them can be present, that is, contains a dot, in a graceful triangle, it is possible to construct $n!$ of these triangles, i.e., there are $n!$ β-labeled graphs of size n. Within a graceful triangle, when all the adjacencies are rotated around the axis formed by the cells a_{ij} with $i + j = n$, the resulting graceful triangle corresponds to the complementary labeling of the original. Furthermore, when an α-labeling f, with boundary value λ, is represented in a triangular arrangement, all the adjacencies lie in a rectangle with corners $a_{0n}, a_{0(\lambda+1)} a_{\lambda(\lambda+1)}, a_{\lambda n}$. We refer to this rectangle as the *rectangle determined by* λ. An example of this fact is shown in Figure 2 for an α-graph of size 7 and boundary value 3.

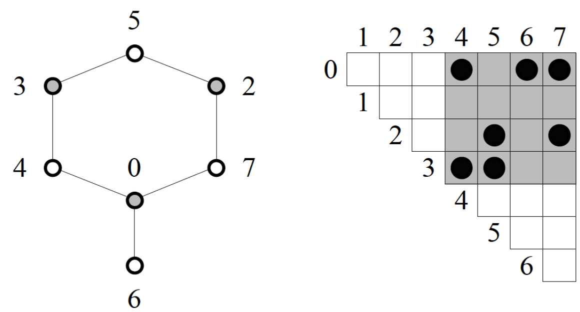

Figure 2: Triangular representation of an α-labeling

Notice that the inverse labeling of f is obtained rotating this rectangle by 180 degrees.

This representation of α-labelings is extremely useful to count the number of α-labeled graphs of size n. In fact, we just need to count the number of possible rectangles and within each rectangle, the number of possible distributions of dots. Using the symmetry we can consider only half of the possible values for λ and multiply by 2. When n is even, the rectangle determined by λ for $0 \leq \lambda \leq \frac{n-2}{2}$, is symmetric to the one determined by $n - \lambda - 1$.

Let f be an α-labeling of G with boundary value λ, where $0 \leq \lambda \leq \frac{n-2}{2}$. For $k \in \{1, 2, \ldots, \lambda\}$, the diagonals d_k and d_{n+1-k} have k cells that can be used to place a dot; thus there are $(\lambda!)^2$ possible distributions of the dots. For $k \in \{\lambda + 1, \lambda + 2, \ldots, n - \lambda\}$, the diagonals d_k have $\lambda + 1$ cells where a dot can be placed, thus there are $(\lambda + 1)^{n-2\lambda}$ possible distributions of the dots. Hence, there are $(\lambda!)^2(\lambda + 1)^{n-2\lambda}$ graphs of even size n that admit an α-labeling with boundary value λ. Taking the sum over all possible values of λ and multiplying by 2, we obtain the number of α-labeled graphs of even size n.

When n is odd, the rectangle determined by λ for $0 \leq \lambda \leq \frac{n-3}{2}$, is symmetric to the one determined by $n - \lambda - 1$. The only difference with the even case is when $\lambda = \frac{n-1}{2}$. In this case the diagonals d_k and d_{n+1-k}, for $1 \leq k \leq \lambda$, have k cells that can be used to place a dot, i.e., there are $\left(\frac{n-1}{2}!\right)^2$ possible distributions of the dots. The remaining diagonal, $d_{\frac{n+1}{2}}$, has $\frac{n+1}{2}$ cells where a dot can be placed. Since

$$\left(\left(\frac{n-1}{2}\right)!\right)^2 \frac{n+1}{2} = \left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!,$$

we have that there are $(\lambda!)^2(\lambda+1)^{n-2\lambda}$ graphs of size n that admit an α-labeing with boundary value λ, $0 \leq \lambda \leq \frac{n-3}{2}$ and $\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!$ with boundary value $\lambda = \frac{n-1}{2}$. Once again, taking the sum over all possible values of λ and multiplying by 2, except when $\lambda = \frac{n-1}{2}$, we get the number of α-labeled graphs of odd size n. Thus, we have given a different and more intuitive proof to Sheppard's result [10].

In Figure 3 we show the facts about cells per diagonal, where $n = 7$ and $\lambda = 2$.

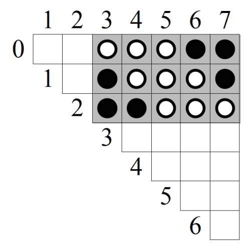

Figure 3: Triangular atrrangements for $n = 7$ and $\lambda = 2$

Proposition 1. *The number of α-labeled graphs of size n is*

$$2\sum_{\lambda=0}^{\frac{n-2}{2}} (\lambda!)^2(\lambda+1)^{n-2\lambda}$$

when n is even, and

$$2\sum_{\lambda=0}^{\frac{n-3}{2}} (\lambda!)^2(\lambda+1)^{n-2\lambda} + \left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!$$

when n is odd.

In Table 1, we show the number of α-labeled graphs of size n for the first values of n. This is the sequence A005193 in Sloane's *Online Encyclopedia of Integer Sequences* [12].

2.1 Enumerating Other Types of Labeled Graphs

Triangular arrangements can be also used to count other types of labeled graphs. Suppose G is a graph of size n. Let $f : V(G) \to L$ be a labeling of the vertices of G such that the weight of each edge uv of G is now defined as $(f(u) + f(v)) \pmod z$.

n	$\alpha(n)$	n	$\alpha(n)$
1	1	8	1930
2	2	9	9690
3	4	10	53578
4	10	11	322650
5	30	12	2106250
6	106	13	14790810
7	426	14	111327178

Table 1: Number of α-labeled graphs of size n

(a) f is *harmonious* when $L = \{0, 1, \ldots, n-1\}$, $z = n$, and
$W_f = \{0, 1, \ldots, n-1\}$. (See [6].)

(b) f is *felicitous* when $L = \{0, 1, \ldots, n\}$, $z = n$, and
$W_f = \{0, 1, \ldots, n-1\}$. (See [4, 8].)

(c) f is *elegant* when $L = \{0, 1, \ldots, n\}$, $z = n+1$, and
$W_f = \{0, 1, \ldots, n\}$. (See [2].)

Even when the nature of the weights is different now, all these labelings can be analyzed using these triangular arrangements. Now, the cell a_{ij} corresponds to $(i+j) \pmod{z}$. We focus our attention on the harmonious labelings.

The definition of harmonious labelings can be extended to include graphs as trees, i.e., graphs of size n and order $n+1$, by allowing the repetition of one label on two vertices. As a consequence of this repetition, the enumeration technique presented below gives the exact number of harmoniously labeled graphs of size n and order at most n; and a lower bound for the number of harmoniously labeled graphs of size n.

Because we want to determine $h(n)$, that is, the number of harmoniously labeled graphs of size n and order at most n, we have $n \geq 3$.

The weight $w = n-1$ appears exactly once in each of the diagonals d_k where k is odd. Thus, when n is odd, $w = n-1$ appears $\frac{n-1}{2}$ times and $\frac{n}{2}$ when n is even.

For every $0 \leq w \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, both weights, w and $n-2-w$, appear in the diagonals d_k for every $k \in A = \{w+2, w+4, \ldots, n-x\}$ and every $k \in B = \{n-w, n-w+2, \ldots, n-y\}$ where

$$x = \begin{cases} 1, & \text{if } n \text{ is odd and } w \text{ is even;} \\ 2, & \text{if } n \text{ is odd and } w \text{ is odd;} \\ 2, & \text{if } n \text{ is even and } w \text{ is even;} \\ 1, & \text{if } n \text{ is even and } w \text{ is odd.} \end{cases}$$

and

$$y = \begin{cases} 1, & \text{if } w \text{ is even;} \\ 2, & \text{if } w \text{ is odd.} \end{cases}$$

regardless the parity of n.

Notice that $B = \varnothing$ when $w = 0$ and $w = n - 2$.

When n is even, $|A| = \frac{n-x-w}{2}$ and $|B| = \frac{w+2-x}{2}$ because $x = y$. Then the weight w appears in $\frac{n}{2} + 1 - x$ diagonals, that is, $\frac{n}{2} - 1$ times when it is even, and $\frac{n}{2}$ times when it is odd. Therefore, the number of harmoniously labeled graphs of even size n, $h(n)$, is given by

$$h(n) = \left(\frac{n}{2} - 1\right)^{\frac{n}{2}} \left(\frac{n}{2}\right)^{\frac{n}{2}}.$$

When n is odd, $|A| = \frac{n-x-w}{2}$ and $|B| = \frac{w+2-y}{2}$. Then the weight w appears in $\frac{n+2-x-y}{2}$ diagonals, that is, $\frac{n-1}{2}$ times. Hence,

$$h(n) = \left(\frac{n-1}{2}\right)^{n}.$$

Thus, we have proved the following proposition.

Proposition 2. *The number of harmoniously labeled graphs of size n and order at most n is equal to:*

- $\left(\frac{n-1}{2}\right)^n$ *when $n \geq 3$ is odd,*
- $\left(\frac{n}{2} - 1\right)^{\frac{n}{2}} \left(\frac{n}{2}\right)^{\frac{n}{2}}$ *when $n \geq 4$ is even.*

Using similar arguments we determined the number of felicitously and elegantly labeled graphs of size n.

Proposition 3. *The number of felicitously labeled graphs of size n is*

- $\left(\frac{n+1}{2}\right)^n$ *when n is odd,*
- $\left(\frac{n}{2}\right)^{\frac{n}{2}} \left(\frac{n+2}{2}\right)^{\frac{n}{2}}$ *when n is even.*

Proposition 4. *The number of elegantly labeled graphs of size n is*

- $\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}} \left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}$ *when n is odd,*
- $\left(\frac{n}{2}\right)^n$ *when n is even.*

3 The Number of Graceful Graphs

Let $\mathcal{G}_{m,n}$ be the family of all β-labeled graphs of order m and size n. We want to determine its cardinality. To do this, we use graceful triangles.

Recall that every graph G in $\mathcal{G}_{m,n}$ must have the integers 0 and n as labels. Since $m \leq n+1$, the remaining $m-2$ labels of G form a subset of $\{1, 2, \ldots, n-1\}$. Thus, any β-labeling of G does not assign $t = n+1-m$ integers from $\{1, 2, \ldots, n-1\}$ as labels of G. Let $A = \{x_1, x_2, \ldots, x_t\}$ be the set formed by these numbers, where $x_i < x_{i+1}$ for every $1 \leq i \leq t-1$.

Let $d(j, x_i)$ denote the number of deleted (or forbidden) cells on the diagonal d_j of the graceful triangle when x_i is not a label of G. Suppose $t = 1$; we can see that:

- For $1 \leq x_1 \leq \lfloor \frac{n}{2} \rfloor$

$$d(j, x_1) = \begin{cases} 0, & \text{if } 1 \leq j \leq x_1; \\ 1, & \text{if } x_1 + 1 \leq j \leq n - x_1; \\ 2, & \text{if } n - x_1 + 1 \leq j \leq n. \end{cases}$$

- For $\lceil \frac{n+1}{2} \rceil \leq x_1 \leq n-1$

$$d(j, x_1) = \begin{cases} 0, & \text{if } 1 \leq j \leq n - x_1; \\ 1, & \text{if } n - x_1 + 1 \leq j \leq x_1; \\ 2, & \text{if } x_1 + 1 \leq j \leq n. \end{cases}$$

In Figure 4 we show an example for $n = 10$, where $x_1 = 4$. All the cells in the first 4 diagonals are available, all the cells except 1 can be used in the diagonals d_5 and d_6, while in the remaining diagonals two cells cannot be used in each of them.

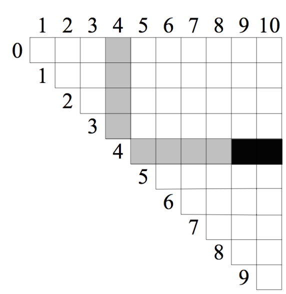

Figure 4: Counting of deleted cells

Thus, $j - d(j, x_1)$ is the number of cells in d_j where a dot can be placed. Therefore, using the product rule, the number $g(n, x_1)$ of β-labeled graphs of size n that do not have the integer x_1 as a label is given by

$$g(n, x_1) = \prod_{j=1}^{n} (j - d(j, x_1))$$

In Table 2 we show the number $g(n, i)$ of β-labeled graphs of size $n \geq 2$ that do not use the integer $i \in \{1, 2, \ldots, n-1\}$ as a label. This is the sequence A241094 in Sloane's *Online Encyclopedia of Integer Sequences* [12].

$n \backslash i$	1	2	3	4	5	6	7
2	0						
3	1	1					
4	4	4	4				
5	18	24	24	18			
6	96	144	144	144	96		
7	600	960	1080	1080	960	600	
8	4320	7200	8640	8640	8640	7200	4320

Table 2: β-labeled graphs where i is not a label

Suppose now that the integers x_1 and x_2 are not labels of G. The number of available cells on d_j is

$$j - d(j, x_1) - d(j, x_2),$$

except when $j = n + 1 - (x_2 - x_1)$, where this number is

$$j - d(j, x_1) - d(j, x_2) + 1,$$

because the cell $a_{x_1 x_2}$ was eliminated twice (see Figure 5a). Then the number $g(n, x_1, x_2)$ of β-labeled graphs of size n that do not have the integers x_1 and x_2 as labels, is given by

$$g(n, x_1, x_2) = \prod_{j=1}^{n} (j - d(j, x_1) - d(j, x_2) + \delta_{x_1 x_2}),$$

where

$$\delta_{x_1 x_2} = \begin{cases} 1, & \text{if } j = n + 1 - (x_2 - x_1); \\ 0, & \text{otherwise.} \end{cases}$$

Now that a pattern has been established, we proceed to find the general formula that allows us to count the number of β-labeled graphs of size n that do not have as labels, t numbers from $\{1, 2, \ldots, n-1\}$.

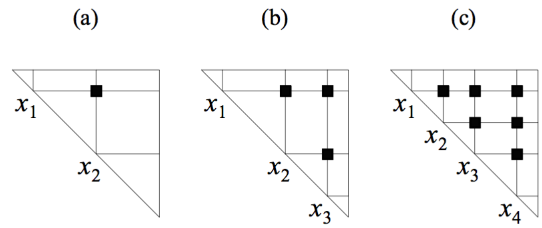

Figure 5: Overcounting of deleted cells

Notice that when t numbers from $\{1, 2, \ldots, n-1\}$ are not used, the number of cells, in the triangular arrangement, that have been eliminated twice is given by $\frac{t(t-1)}{2}$ (see Figure 4). Thus,

$$\delta = \sum_{\forall x_i, x_j \in A} \delta_{x_i x_j},$$

and the number of β-labeled graphs of size n that do not have, as labels, the integers in $A = \{x_1, x_2, \ldots, x_t\}$ is given by

$$g(n, A) = \prod_{j=1}^{n} \left(j - \sum_{i=1}^{t} d(j; x_i) + \delta \right).$$

Therefore, the number $g(n, t)$ of β-labeled graphs of size n that do not use t numbers from $\{1, 2, \ldots, n-1\}$ as labels, is obtained by adding the numbers $g(n, A)$ for all possible t-element subsets A of $\{1, 2, \ldots, n-1\}$, that is,

$$\begin{aligned} g(n, t) &= \sum_{A \subseteq \{1,2,\ldots,n-1\}} g(n, A) \\ &= \sum_{A \subseteq \{1,2,\ldots,n-1\}} \left(\prod_{j=1}^{n} \left(j - \sum_{i=1}^{t} d(j, x_i) + \delta \right) \right) \end{aligned}$$

In the next theorem we use the principle of inclusion and exclusion and the numbers $g(n, t)$ to count the number of β-labeled graphs of size n and order $n+1$. Notice that these kind of graphs include all β-labeled trees.

Theorem 5. *The number of graceful graphs of order $n+1$ and size n is given by*

$$g(n, 0) = n! + \sum_{k=1}^{n-1} (-1)^k g(n, k).$$

Proof. Let $g(n, 0)$ represent the number of β-labeled graphs of size n and order $n+1$. For any $1 \leq k < n-1$, the number $g(n, t)$ includes the number $g(n, t+1)$. Since there are $n!$ graceful-graphs of size n and the numbers $g(n, k)$ satisfy the conditions of the principle of inclusion and exclusion, we have proven our result. $\square$

The first values of $g(n, 0)$ can be seen in the lower diagonal of Table 3. For example, 7008 is the number of β-labeled graphs of size 8 and order 9. We can also use $g(n, 0)$ as an upper bound for the number of β-labeled trees of size n.

Now we can focus on our final goal, that is, to know the value of $|\mathcal{G}_{m,n}|$. Since $g(n, t)$ counts the number of β-labeled graphs of size n that do not use the t numbers in $A = \{x_1, x_2, \ldots, x_t\}$ as labels, $g(n, t)$ includes $g(n, t+1)$. Suppose that k is the largest integer in $\{1, 2, \ldots, n-1\}$ such that $g(n, k) \neq 0$. Then

$$|\mathcal{G}_{n+1-k,n}| = g(n, k).$$

Let $\overline{g}(n, t)$ denote the number of β-labeled graphs of size n that do not use exactly t numbers from $\{1, 2, \ldots, n-1\}$ as labels. Thus $\overline{g}(n, k) = g(n, k)$. Moreover, to determine $\overline{g}(n, k-1)$ we need to eliminate from $g(n, k-1)$ all those labeled graphs counted by $\overline{g}(n, k)$. Since every set with k elements contains $\binom{k}{k-1} = k$ subsets of cardinality $k-1$, we know that $\overline{g}(n, k-1) = g(n, k-1) - k\overline{g}(n, k)$. This equation is used within the proof of our main result.

Theorem 6. *The number $\overline{g}(n, t)$ of β-labeled graphs of size n and order $m = n+1-t$ is given by*

$$\overline{g}(n, t) = g(n, t) - \sum_{i=1}^{k-t} \binom{t+i}{t} \overline{g}(n, t+i)$$

where k is the largest integer in $\{1, 2, \ldots, n-1\}$ such that $\overline{g}(n, k) \neq 0$.

Proof. Suppose that the value of $\overline{g}(n, t+i)$ is known for every $1 \leq i \leq k-t$. Thus

$$\binom{t+i}{t} \overline{g}(n, t+i)$$

counts the number of times a β-labeled graph of size n and order $n+1-t-i$ has been counted inside $g(n, t)$. Since $1 \leq i \leq k$,

$$\overline{g}(n, t) = g(n, t) - \sum_{i=1}^{k-t} \binom{t+i}{t} \overline{g}(n, t+i).$$

This concludes the proof. $\square$

Observe that this last formula can be written in terms of $g(n, t+i)$ instead of $\overline{g}(n, t+i)$. Doing back substitution we arrive to the following equivalent formula.

Theorem 7. *The number $\overline{g}(n,t)$ of β-labeled graphs of size n and order $n+1-t$ is given by*

$$\overline{g}(n,t) = g(n,t) + \sum_{i=1}^{k-t} (-1)^i \binom{t+i}{t} g(n,t+i).$$

In Table 3 we show $|\mathcal{G}_{m,n}|$ for the first values of n and m.

$m \backslash n$	1	2	3	4	5	6	7	8	9	10
2	1									
3		2	2							
4			4	12	8	2				
5				12	68	106	88	32	8	
6					44	406	1186	1728	1696	964
7						206	2644	12096	29536	45496
8							1122	19456	126304	448512
9								7008	155960	1365992
10									49376	1380476
11										387360

Table 3: First values of $|\mathcal{G}_{m,n}|$

4 Conclusions

Triangular arrangements have shown to be extremely useful to count the number of β-labeled graphs of size n and order m. Now, that these numbers have been determined, we can use them for further investigations of this type of labeled graphs. In particular, we can take the subfamily of size n and order $n+1$ and study how many of them correspond to connected graphs, that is, trees. If this is possible, we would have more information toward the proof of the graceful tree conjecture.

5 Note

During the preperation of the final version of this paper we became aware of the work of Whitty [13]. In this paper, Whitty determined the number of β-labeled trees of order n to be the leading coefficient of the rook polymonial associated with the Klein bottle.

6 Acknowledgments

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