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Reciprocal Series of Squares of Fibonacci Related Sequences with Subscripts in Arithmetic Progression

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Abstract

In this paper, we derive closed forms for reciprocal series, both finite and infinite, that involve Fibonacci numbers. The term that defines the denominator of each summand generates squares of Fibonacci related numbers with subscripts in arithmetic progression. Our method employs certain algebraic identities that we believe are new. These identities exhibit the telescoping effect when summed.

1 Introduction

As usual, the Fibonacci and Lucas numbers are defined, respectively, for all integers n, by

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1.$$

With $\alpha = (1+\sqrt{5})/2$, the Binet (closed) forms for F_n and L_n are

$$F_n = \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{5},$$
$$L_n = \alpha^n + (-1)^n\alpha^{-n},$$

and these closed forms are valid for all integers n.

In this paper, we present closed forms for Fibonacci related reciprocal sums, both finite and infinite, in which the denominator of the summand is a perfect square. For instance, two summands that we consider are

$$\frac{L_{2ai+b}}{(F_{2ai+b}+c)^2}, \text{ and } \frac{F_{2ai+b}}{(L_{2ai+b}+c)^2},$$

for certain values of the parameters a, b, and c. In each of these two cases, the term in the numerator of the summand, together with its counterpart in the denominator, have subscripts in arithmetic progression.

Before proceeding, we give some instances of infinite sums that we have discovered. These infinite sums arise from families of such sums that occur in the sequel.

With $s = 2$, (25) becomes

$$\sum_{i=0}^{\infty} \frac{F_{2i}}{(L_{2i}+3)^2} = \frac{9}{100}. \tag{1}$$

With $s = 1$, (26) and (28) yield, respectively,

$$\sum_{i=0}^{\infty} \frac{F_{2i}}{\left(L_{2i}+\sqrt{5}\right)^2} = \frac{5-2\sqrt{5}}{5}, \tag{2}$$

and

$$\sum_{i=0}^{\infty} \frac{L_{2i+1}}{(F_{2i+1}+1)^2} = \frac{5}{4}. \tag{3}$$

We also have

$$\sum_{i=0}^{\infty} \frac{L_{2i+3}}{(F_{2i+3}+1)^2} = \sum_{i=0}^{\infty} \frac{L_{2i+5}}{(F_{2i+5}-1)^2} = 1. \tag{4}$$

In (4), the first sum comes from (19), with $(a, b, s) = (1, 3, 1)$, while the second sum comes from (35), with $(a, b, s) = (1, 5, 1)$. We give several more results of a similar nature after we present our main results on infinite sums.

All the results that we present flow from elementary algebraic identities that we believe are new. In Sections 2 and 3, we present our results on finite sums, together with the infinite sums that are readily derived from them. In Sections 4 and 5, we present our results on infinite sums.

2 Finite sums I

The lemma that follows is a statement of our first algebraic identity. All the results in this section flow from this lemma.

Lemma 1. *Let $t > 1$ be a real number, and let n, a, and b be integers. Then*

$$\frac{(t^a - t^{-a})\left(t^{2an+b} - t^{-2an-b}\right)}{(t^{2an+b} + t^{-2an-b} + t^a + t^{-a})^2} = \frac{t^{2an+b-a}}{(1+t^{2an+b-a})^2} - \frac{t^{2an+b+a}}{(1+t^{2an+b+a})^2}. \tag{5}$$

Proof. Expressing the right side of (5) on a single denominator, we obtain

$$\frac{t^{2an+b-a}\left(1+t^{2an+b+a}\right)^2 - t^{2an+b+a}\left(1+t^{2an+b-a}\right)^2}{\left(1+t^{2an+b-a}+t^{2an+b+a}+t^{4an+2b}\right)^2}. \tag{6}$$

Upon dividing both the numerator and denominator of (6) by t^{4an+2b}, we obtain the left side of (5). $\square$

In this paper, n, a, and b are integers, and henceforth we do not restate this. Many of the results that follow remain valid if a and b are allowed to be negative. Our preference, however, has been to opt for simplicity by demanding that a and b be non-negative . Readers wishing to take any of our results and choose parameters that yield negative subscripts should remember that, for $m \geq 0$, $F_{-m} = (-1)^{m-1}F_m$, and $L_{-m} = (-1)^m L_m$. With this in mind, we define, for $a \geq 1$ and $b \geq 0$,

$$\begin{aligned}
S_1(n,a,b) &= L_a \sum_{i=0}^{n} \frac{L_{2ai+b}}{\left(F_{2ai+b}+F_a\right)^2},\\
S_2(n,a,b) &= 5F_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{\left(L_{2ai+b}+L_a\right)^2},\\
S_3(n,a,b) &= \sqrt{5}L_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{\left(L_{2ai+b}+\sqrt{5}F_a\right)^2},\\
S_4(n,a,b) &= \frac{F_a}{\sqrt{5}} \sum_{i=0}^{n} \frac{L_{2ai+b}}{\left(F_{2ai+b}+L_a/\sqrt{5}\right)^2}.
\end{aligned}$$

As we soon see, under certain restrictions on a and b, the closed forms for each of the S_i above follow from Lemma 1. Before proceeding, we note that, for odd integers a and b, $(a+b)/2$ is even (odd) if and only if $(b-a)/2$ is odd (even). However, if a and b are both even, then $(a+b)/2$ is even (odd) if and only if $(b-a)/2$ is even (odd). Our first theorem gives the closed form for S_1.

Theorem 2. *Let $n \geq 0$. Let $a \geq 1$, and $b \geq 1$ both be odd. Then*

$$S_1(n,a,b) = \begin{cases}
\frac{5}{L^2_{(b-a)/2}} - \frac{1}{F^2_{an+(a+b)/2}}, & \text{if } n \text{ is even and } (a+b)/2 \text{ is odd;}\\
\frac{1}{F^2_{(b-a)/2}} - \frac{5}{L^2_{an+(a+b)/2}}, & \text{if } n \text{ is even and } (a+b)/2 \text{ is even;}\\
5\left(\frac{1}{L^2_{(b-a)/2}} - \frac{1}{L^2_{an+(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is odd;}\\
\frac{1}{F^2_{(b-a)/2}} - \frac{1}{F^2_{an+(a+b)/2}}, & \text{if } n \text{ is odd and } (a+b)/2 \text{ is even.}
\end{cases}$$

Proof. From Lemma 1 we have, due to the telescoping effect,

$$\begin{aligned}
&\sum_{i=0}^{n} \frac{\left(t^a-t^{-a}\right)\left(t^{2ai+b}-t^{-2ai-b}\right)}{\left(t^{2ai+b}+t^{-2ai-b}+t^a+t^{-a}\right)^2} = \frac{t^{b-a}}{\left(1+t^{b-a}\right)^2} - \frac{t^{2an+a+b}}{\left(1+t^{2an+a+b}\right)^2}\\
&= \frac{1}{\left(t^{(b-a)/2}+t^{-(b-a)/2}\right)^2} - \frac{1}{\left(t^{an+(a+b)/2}+t^{-an-(a+b)/2}\right)^2}.
\end{aligned} \tag{7}$$

In (7), with $t = \alpha$, and taking into account each of the four cases imposed upon n, a, and b in the statement of Theorem 2, we use the Binet forms to transform the summand on the left side of (7), and the two fractions on the right side of (7), into Fibonacci/Lucas numbers. This proves Theorem 2. $\square$

As an immediate corollary of Theorem 2, we have

Corollary 3. *Let $a \geq 1$, and $b \geq 1$ both be odd. Then*

$$L_a \sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b} + F_a\right)^2} = \begin{cases} \frac{5}{L^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is odd;} \\ \frac{1}{F^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is even.} \end{cases}$$

Our next theorem gives the closed form for S_2 which, unlike the closed form for S_1, is independent of the parity of n. The proof begins with (7), and proceeds in precisely the same manner as the proof of Theorem 2. We leave the details to the reader.

Theorem 4. *Let $n \geq 0$. Let $a \geq 2$, and $b \geq 0$ both be even. Then*

$$S_2(n,a,b) = \begin{cases} \frac{1}{L^2_{(b-a)/2}} - \frac{1}{L^2_{an+(a+b)/2}}, & \text{if } (a+b)/2 \text{ is even;} \\ \frac{1}{5}\left(\frac{1}{F^2_{(b-a)/2}} - \frac{1}{F^2_{an+(a+b)/2}}\right), & \text{if } (a+b)/2 \text{ is odd.} \end{cases}$$

As a corollary of Theorem 4, we have

Corollary 5. *Let $a \geq 2$, and $b \geq 0$ both be even. Then*

$$5F_a \sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b} + L_a\right)^2} = \begin{cases} \frac{1}{L^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is even;} \\ \frac{1}{5F^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is odd.} \end{cases}$$

Observe that, for n an odd integer,

$$\begin{aligned} \frac{\alpha^n}{\left(\alpha^n+1\right)^2} &= \frac{\alpha^n\left(\alpha^n-1\right)^2}{\left(\alpha^n+1\right)^2\left(\alpha^n-1\right)^2} = \frac{\alpha^n\left(\alpha^n-1\right)^2}{\left(\alpha^{2n}-\alpha^n\alpha^{-n}\right)^2} = \frac{\alpha^n+\alpha^{-n}-2}{\left(\alpha^n-\alpha^{-n}\right)^2} \\ &= \frac{\sqrt{5}F_n-2}{L_n^2}. \end{aligned} \tag{8}$$

For n an odd integer, we also have

$$\frac{\alpha^n}{\left(\alpha^n-1\right)^2} = \frac{\sqrt{5}F_n+2}{L_n^2}, \tag{9}$$

which we require in the sequel.

Our next theorem gives the closed forms for S_3 and S_4, where the assumption is that a and b have different parities. We follow this theorem with a corollary that gives the corresponding infinite sums.

Theorem 6. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$S_3(n,a,b) = \frac{\sqrt{5}F_{b-a} - 2}{L_{b-a}^2} - \frac{\sqrt{5}F_{2an+a+b} - 2}{L_{2an+a+b}^2}, \text{ if } a \text{ is odd and } b \text{ is even,}$$

$$S_4(n,a,b) = \frac{\sqrt{5}F_{b-a} - 2}{L_{b-a}^2} - \frac{\sqrt{5}F_{2an+a+b} - 2}{L_{2an+a+b}^2}, \text{ if } a \text{ is even and } b \text{ is odd.}$$

Proof. With $t = \alpha$, and taking into account the parities of a and b, we make use of the Binet forms, together with (8), to transform the first equality in (7) into Fibonacci/Lucas numbers. □

As a corollary of Theorem 6, we have

Corollary 7. *Let $a \geq 1$, $b \geq 0$. Then*

$$\sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b} + \sqrt{5}F_a\right)^2} = \frac{\sqrt{5}F_{b-a} - 2}{\sqrt{5}L_a L_{b-a}^2}, \text{ if } a \text{ is odd and } b \text{ is even,}$$

$$\sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b} + L_a/\sqrt{5}\right)^2} = \frac{5F_{b-a} - 2\sqrt{5}}{F_a L_{b-a}^2}, \text{ if } a \text{ is even and } b \text{ is odd.}$$

3 Finite sums II

In this section, we give results that are parallel to those presented in the previous section. For $a \geq 1$, and $b \geq 0$, the finite sums that we consider are

$$S_5(n,a,b) = L_a \sum_{i=0}^{n} \frac{L_{2ai+b}}{\left(F_{2ai+b} - F_a\right)^2},$$

$$S_6(n,a,b) = 5F_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{\left(L_{2ai+b} - L_a\right)^2},$$

$$S_7(n,a,b) = \sqrt{5}L_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{\left(L_{2ai+b} - \sqrt{5}F_a\right)^2},$$

$$S_8(n,a,b) = \frac{F_a}{\sqrt{5}} \sum_{i=0}^{n} \frac{L_{2ai+b}}{\left(F_{2ai+b} - L_a/\sqrt{5}\right)^2}.$$

Under certain restrictions on a and b, the closed forms for each of the finite sums defined in the previous paragraph are derived from Lemma 8, which follows. Since the proof of Lemma 8 is similar to the proof of Lemma 1, we state it without proof. Note that, in Lemma 8, the condition $a \neq b$ ensures that denominators do not vanish when $n = 0$. Actually, in this section, the possibility of vanishing denominators would not have arisen had we chosen to present all finite sums with lower limit $i = 1$. However, to maintain consistency with the results of Section 2, we have chosen to present all sums in this paper with lower limit $i = 0$.

Lemma 8. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, with $a \neq b$. Then*

$$\frac{(t^a - t^{-a})\left(t^{2an+b} - t^{-2an-b}\right)}{\left(t^{2an+b} + t^{-2an-b} - t^a - t^{-a}\right)^2} = \frac{t^{2an+b-a}}{\left(1 - t^{2an+b-a}\right)^2} - \frac{t^{2an+b+a}}{\left(1 - t^{2an+b+a}\right)^2}. \tag{10}$$

Our first theorem in this section gives the closed form for S_5.

Theorem 9. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 1$ be odd integers with $a \neq b$. Then*

$$S_5(n,a,b) = \begin{cases} \frac{5}{L^2_{(b-a)/2}} - \frac{1}{F^2_{an+(a+b)/2}}, & \text{if } n \text{ is even and } (a+b)/2 \text{ is even;} \\ \frac{1}{F^2_{(b-a)/2}} - \frac{5}{L^2_{an+(a+b)/2}}, & \text{if } n \text{ is even and } (a+b)/2 \text{ is odd;} \\ 5\left(\frac{1}{L^2_{(b-a)/2}} - \frac{1}{L^2_{an+(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is even;} \\ \frac{1}{F^2_{(b-a)/2}} - \frac{1}{F^2_{an+(a+b)/2}}, & \text{if } n \text{ is odd and } (a+b)/2 \text{ is odd.} \end{cases}$$

Proof. From Lemma 8 we have, due to the telescoping effect,

$$\sum_{i=0}^{n} \frac{(t^a - t^{-a})\left(t^{2ai+b} - t^{-2ai-b}\right)}{\left(t^{2ai+b} + t^{-2ai-b} - t^a - t^{-a}\right)^2} = \frac{t^{b-a}}{\left(1 - t^{b-a}\right)^2} - \frac{t^{2an+a+b}}{\left(1 - t^{2an+a+b}\right)^2}$$
$$= \frac{1}{\left(t^{(b-a)/2} - t^{-(b-a)/2}\right)^2} - \frac{1}{\left(t^{an+(a+b)/2} - t^{-an-(a+b)/2}\right)^2}. \tag{11}$$

In (11), with $t = \alpha$, we proceed as in the proof of Theorem 2. □

The following is a corollary of Theorem 9.

Corollary 10. *Let $a \geq 1$, $b \geq 1$ be odd integers with $a \neq b$. Then*

$$L_a \sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b} - F_a\right)^2} = \begin{cases} \frac{5}{L^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is even;} \\ \frac{1}{F^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is odd.} \end{cases}$$

In the next theorem, we give the closed form for S_6, which, like the closed form for S_5, is a consequence of (11).

Theorem 11. *Let $n \geq 0$. Let $a \geq 2$, $b \geq 0$ be even integers with $a \neq b$. Then*

$$S_6(n,a,b) = \begin{cases} \frac{1}{L^2_{(b-a)/2}} - \frac{1}{L^2_{an+(a+b)/2}}, & \text{if } (a+b)/2 \text{ is odd;} \\ \frac{1}{5}\left(\frac{1}{F^2_{(b-a)/2}} - \frac{1}{F^2_{an+(a+b)/2}}\right), & \text{if } (a+b)/2 \text{ is even.} \end{cases}$$

As a corollary of Theorem 11, we have

Corollary 12. *Let $n \geq 0$. Let $a \geq 2$, $b \geq 0$ be even integers with $a \neq b$. Then*

$$5F_a \sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b} - L_a\right)^2} = \begin{cases} \frac{1}{L^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is odd;} \\ \frac{1}{5F^2_{(b-a)/2}}, & \text{if } (a+b)/2 \text{ is even.} \end{cases}$$

Our next theorem gives the closed forms for S_7 and S_8, where the assumption is that a and b have different parities. We follow this with a corollary that gives the corresponding infinite sums.

Theorem 13. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$S_7(n,a,b) = \frac{\sqrt{5}F_{b-a}+2}{L_{b-a}^2} - \frac{\sqrt{5}F_{2an+a+b}+2}{L_{2an+a+b}^2}, \text{ if } a \text{ is odd and } b \text{ is even,}$$
$$S_8(n,a,b) = \frac{\sqrt{5}F_{b-a}+2}{L_{b-a}^2} - \frac{\sqrt{5}F_{2an+a+b}+2}{L_{2an+a+b}^2}, \text{ if } a \text{ is even and } b \text{ is odd.}$$

Proof. With $t = \alpha$, and taking into account the parities of a and b, we make use of the Binet forms, together with (9), to transform the first equality in (11) into Fibonacci/Lucas numbers. ☐

The following is a corollary of Theorem 13.

Corollary 14. *Let $a \geq 1$, $b \geq 0$. Then*

$$\sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b} - \sqrt{5}F_a\right)^2} = \frac{\sqrt{5}F_{b-a}+2}{\sqrt{5}L_a L_{b-a}^2}, \text{ if } a \text{ is odd and } b \text{ is even,}$$
$$\sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b} - L_a/\sqrt{5}\right)^2} = \frac{5F_{b-a}+2\sqrt{5}}{F_a L_{b-a}^2}, \text{ if } a \text{ is even and } b \text{ is odd.}$$

4 Infinite sums I

In this section, we present closed forms for infinite sums that are more general than those in Corollaries 3, 5, and 7. We accomplish this via an algebraic identity that contains an additional parameter, s. Here, and for the remainder of this paper, s is taken to be a positive integer. The algebraic identity in question is given in Lemma 15. We state Lemma 15 without proof, since its proof is analogous to the proof of Lemma 1.

Lemma 15. *Let $t > 1$ be a real number. Let $n \geq 0$, $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$\frac{\left(t^{as}-t^{-as}\right)\left(t^{2an+b}-t^{-2an-b}\right)}{\left(t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}\right)^2} = \frac{t^{2an+b-as}}{\left(1+t^{2an+b-as}\right)^2} - \frac{t^{2an+b+as}}{\left(1+t^{2an+b+as}\right)^2}. \tag{12}$$

As a consequence of Lemma 15, we have

Lemma 16. *Let $t > 1$ be a real number. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$\sum_{i=0}^{\infty} \frac{\left(t^{as}-t^{-as}\right)\left(t^{2ai+b}-t^{-2ai-b}\right)}{\left(t^{2ai+b}+t^{-2ai-b}+t^{as}+t^{-as}\right)^2} = \sum_{i=0}^{s-1} \frac{t^{2ai+b-as}}{\left(1+t^{2ai+b-as}\right)^2}. \tag{13}$$

Proof. Consider (12) for $n > s$. Then, by the telescoping effect,

$$\sum_{i=0}^{n} \frac{\left(t^{as} - t^{-as}\right)\left(t^{2ai+b} - t^{-2ai-b}\right)}{\left(t^{2ai+b} + t^{-2ai-b} + t^{as} + t^{-as}\right)^2}$$
$$= \sum_{i=0}^{s-1} \frac{t^{2ai+b-as}}{\left(1 + t^{2ai+b-as}\right)^2} - \sum_{i=n-s+1}^{n} \frac{t^{2ai+b+as}}{\left(1 + t^{2ai+b+as}\right)^2}. \tag{14}$$

Upon letting $n \to \infty$ in (14), we obtain (13). $\square$

Motivated by (13), we define, for $a \geq 1$, $b \geq 0$, and $s \geq 1$,

$$T_s(a, b) = \sum_{i=0}^{s-1} \frac{\alpha^{2ai+b-as}}{\left(1 + \alpha^{2ai+b-as}\right)^2}.$$

We then have

$$\sum_{i=0}^{\infty} \frac{\left(\alpha^{as} - \alpha^{-as}\right)\left(\alpha^{2ai+b} - \alpha^{-2ai-b}\right)}{\left(\alpha^{2ai+b} + \alpha^{-2ai-b} + \alpha^{as} + \alpha^{-as}\right)^2} = T_s(a, b). \tag{15}$$

Taking into account the parities of a, b, and s, we make use of the Binet forms to transform the left side of (15) into Fibonacci/Lucas numbers. The various outcomes are recorded in the theorem that follows, which is the main result in this section.

Theorem 17. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$\sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b} + L_{as}\right)^2} = \frac{1}{5F_{as}} T_s(a, b), \text{ if } b \text{ is even and } a \text{ or } s \text{ is even}, \tag{16}$$

$$\sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b} + \sqrt{5}F_{as}\right)^2} = \frac{1}{\sqrt{5}L_{as}} T_s(a, b), \text{ if } b \text{ is even and } a \text{ and } s \text{ are odd}, \tag{17}$$

$$\sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b} + L_{as}/\sqrt{5}\right)^2} = \frac{\sqrt{5}}{F_{as}} T_s(a, b), \text{ if } b \text{ is odd and } a \text{ or } s \text{ is even}, \tag{18}$$

$$\sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b} + F_{as}\right)^2} = \frac{5}{L_{as}} T_s(a, b), \text{ if } b \text{ is odd and } a \text{ and } s \text{ are odd}. \tag{19}$$

It is a simple matter to check that, with $s = 1$, the sums (16)-(19) produce all the infinite sums in Corollaries 3, 5, and 7. Of course (16)-(19) yield an infinitude of infinite sums. With a little effort, we can also write down pairs of infinite sums that converge to the same limit. For instance, with $(b, a, s) = (2, 3, 2)$, and $(b, a, s) = (4, 3, 2)$, (16) produces

$$\sum_{i=0}^{\infty} \frac{F_{6i+2}}{\left(L_{6i+2} + 18\right)^2} = \sum_{i=0}^{\infty} \frac{F_{6i+4}}{\left(L_{6i+4} + 18\right)^2} = \frac{7}{900}.$$

Again, with $(b, a, s) = (1, 2, 2)$, and $(b, a, s) = (3, 2, 2)$, (18) produces

$$\sum_{i=0}^{\infty} \frac{L_{4i+1}}{\left(F_{4i+1} + 7/\sqrt{5}\right)^2} = \sum_{i=0}^{\infty} \frac{L_{4i+3}}{\left(F_{4i+3} + 7/\sqrt{5}\right)^2} = \frac{45 - 17\sqrt{5}}{24}.$$

The expectation is that $T_s(a,b)$ has a Fibonacci connection. We demonstrate that is the case for the special cases $(a,b)=(1,0)$, and $(a,b)=(1,1)$, by writing down explicit expressions for $T_s(1,0)$ and $T_s(1,1)$ in terms of their rational and irrational parts. We begin with an analysis of $T_s(1,0)$ under the assumption that s is even.

$$\begin{aligned}
T_{2s}(1,0) &= \sum_{i=0}^{2s-1} \frac{\alpha^{2i-2s}}{(\alpha^{2i-2s}+1)^2} \\
&= \sum_{i=0}^{2s-1} \frac{1}{(\alpha^{i-s}+\alpha^{-i+s})^2} \\
&= \sum_{i=0}^{s-1} \left(\frac{1}{(\alpha^{2i-s}+\alpha^{-2i+s})^2} + \frac{1}{(\alpha^{2i+1-s}+\alpha^{-2i-1+s})^2} \right),
\end{aligned} \tag{20}$$

so that

$$T_{2s}(1,0) = \begin{cases} \sum_{i=0}^{s-1} \left(\frac{1}{L_{2i-s}^2} + \frac{1}{5F_{2i-s+1}^2} \right), & \text{if } s \geq 2 \text{ is even;} \\ \sum_{i=0}^{s-1} \left(\frac{1}{5F_{2i-s}^2} + \frac{1}{L_{2i-s+1}^2} \right), & \text{if } s \geq 1 \text{ is odd.} \end{cases} \tag{21}$$

Furthermore, it follows from (8) that

$$T_s(1,0) = \sum_{i=0}^{s-1} \frac{\sqrt{5}F_{2i-s}-2}{L_{2i-s}^2}, \text{if } s \geq 1 \text{ is odd.} \tag{22}$$

Together, (21) and (22) can be used to evaluate $T_s(1,0)$, in terms of its rational and irrational parts, for any integer $s \geq 1$. Since the evaluation of $T_s(1,1)$ follows from similar reasoning, we simply write down the results.

$$T_s(1,1) = \sum_{i=0}^{s-1} \frac{\sqrt{5}F_{2i-s+1}-2}{L_{2i-s+1}^2}, \text{if } s \geq 2 \text{ is even.} \tag{23}$$

$$T_{2s+1}(1,1) = \begin{cases} \frac{1}{4}, \ s=0; \\ \frac{1}{L_s^2} + \sum_{i=0}^{s-1} \left(\frac{1}{L_{2i-s}^2} + \frac{1}{5F_{2i-s+1}^2} \right), & \text{if } s \geq 2 \text{ is even;} \\ \frac{1}{5F_s^2} + \sum_{i=0}^{s-1} \left(\frac{1}{5F_{2i-s}^2} + \frac{1}{L_{2i-s+1}^2} \right), & \text{if } s \geq 1 \text{ is odd.} \end{cases} \tag{24}$$

With (21)-(24) in mind, we state the following corollary of Theorem 17.

Corollary 18. *If $s \geq 1$ then*

$$\sum_{i=0}^{\infty} \frac{F_{2i}}{(L_{2i}+L_s)^2} = \frac{1}{5F_s} T_s(1,0), \textit{ if } s \textit{ is even}, \tag{25}$$

$$\sum_{i=0}^{\infty} \frac{F_{2i}}{\left(L_{2i}+\sqrt{5}F_s\right)^2} = \frac{1}{\sqrt{5}L_s} T_s(1,0), \textit{ if } s \textit{ is odd}, \tag{26}$$

$$\sum_{i=0}^{\infty} \frac{L_{2i+1}}{\left(F_{2i+1}+L_s/\sqrt{5}\right)^2} = \frac{\sqrt{5}}{F_s} T_s(1,1), \textit{ if } s \textit{ is even}, \tag{27}$$

$$\sum_{i=0}^{\infty} \frac{L_{2i+1}}{(F_{2i+1}+F_s)^2} = \frac{5}{L_s} T_s(1,1), \textit{ if } s \textit{ is odd}. \tag{28}$$

Proof. Let $(a,b) = (1,0)$. Then (25) and (26) follow from (16) and (17), respectively. Next let $(a,b) = (1,1)$. Then (27) and (28) follow from (18) and (19), respectively. □

5 Infinite sums II

The main theorem in this section, which gives infinite sums that generalize those in Corollaries 10, 12, and 14, is derived from our final algebraic identity. We first specify a condition that excludes the possibility of vanishing denominators in this algebraic identity.

Condition 19. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ be integers. Then we say that a, b, and s satisfy Condition 19 if $2a|(as-b)$ implies that $as-b<0$.*

The results that follow are counterparts of results in Section 4, and can be proved similarly. We therefore present what follows without proof.

Lemma 20. *Let $t > 1$ be a real number. Let $n \geq 0$, $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 19. Then*

$$\frac{\left(t^{as}-t^{-as}\right)\left(t^{2an+b}-t^{-2an-b}\right)}{\left(t^{2an+b}+t^{-2an-b}-t^{as}-t^{-as}\right)^2} = \frac{t^{2an+b-as}}{\left(1-t^{2an+b-as}\right)^2} - \frac{t^{2an+b+as}}{\left(1-t^{2an+b+as}\right)^2}. \tag{29}$$

From Lemma 20 we obtain, by the telescoping effect,

Lemma 21. *Let $t > 1$ be a real number. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 19. Then*

$$\sum_{i=0}^{\infty} \frac{\left(t^{as}-t^{-as}\right)\left(t^{2ai+b}-t^{-2ai-b}\right)}{\left(t^{2ai+b}+t^{-2ai-b}-t^{as}-t^{-as}\right)^2} = \sum_{i=0}^{s-1} \frac{t^{2ai+b-as}}{\left(1-t^{2ai+b-as}\right)^2}. \tag{30}$$

Motivated by (30), we define

$$T_s^1(a,b) = \sum_{i=0}^{s-1} \frac{\alpha^{2ai+b-as}}{\left(1-\alpha^{2ai+b-as}\right)^2},$$

where the parameters have the same restrictions as for Lemma 21. We then have

$$\sum_{i=0}^{\infty} \frac{\left(\alpha^{as}-\alpha^{-as}\right)\left(\alpha^{2ai+b}-\alpha^{-2ai-b}\right)}{\left(\alpha^{2ai+b}+\alpha^{-2ai-b}-\alpha^{as}-\alpha^{-as}\right)^2} = T_s^1(a,b). \tag{31}$$

The main theorem in this section, which we now state, follows from (31).

Theorem 22. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 19. Then*

$$\sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b}-L_{as}\right)^2} = \frac{1}{5F_{as}} T_s^1(a,b), \text{ if } b \text{ is even and } a \text{ or } s \text{ is even,} \tag{32}$$

$$\sum_{i=0}^{\infty} \frac{F_{2ai+b}}{\left(L_{2ai+b}-\sqrt{5}F_{as}\right)^2} = \frac{1}{\sqrt{5}L_{as}} T_s^1(a,b), \text{ if } b \text{ is even and } a \text{ and } s \text{ are odd,} \tag{33}$$

$$\sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b}-L_{as}/\sqrt{5}\right)^2} = \frac{\sqrt{5}}{F_{as}} T_s^1(a,b), \text{ if } b \text{ is odd and } a \text{ or } s \text{ is even,} \tag{34}$$

$$\sum_{i=0}^{\infty} \frac{L_{2ai+b}}{\left(F_{2ai+b}-F_{as}\right)^2} = \frac{5}{L_{as}} T_s^1(a,b), \text{ if } b \text{ is odd and } a \text{ and } s \text{ are odd.} \tag{35}$$

Before proceeding, we give two interesting equalities produced by Theorem 22. We have found several others. With $(b,a,s)=(2,3,2)$, and $(b,a,s)=(4,3,2)$, (32) produces

$$\sum_{i=0}^{\infty} \frac{F_{6i+2}}{\left(L_{6i+2}-18\right)^2} = \sum_{i=0}^{\infty} \frac{F_{6i+4}}{\left(L_{6i+4}-18\right)^2} = \frac{3}{100}.$$

Again, with $(b,a,s)=(1,3,1)$, and $(b,a,s)=(5,3,1)$, (35) produces

$$\sum_{i=0}^{\infty} \frac{L_{6i+1}}{\left(F_{6i+1}-2\right)^2} = \sum_{i=0}^{\infty} \frac{L_{6i+5}}{\left(F_{6i+5}-2\right)^2} = \frac{5}{4}.$$

Also recall (3) in the introduction to this paper.

Next, for the special cases where $(a,b)=(1,0)$, and $(a,b)=(1,1)$, we write down expressions for $T_s^1(a,b)$ in terms of its rational and irrational parts. The results that we present here, however, are less complicated than their counterparts in Section 4. The reason is that, because of the possibility of vanishing denominators, $T_s^1(1,0)$ exists only for s odd, while $T_s^1(1,1)$ exists only for s even. It follows from (9) that

$$T_s^1(1,0) = \sum_{i=0}^{s-1} \frac{\sqrt{5}F_{2i-s}+2}{L_{2i-s}^2}, \text{if } s \geq 1 \text{ is odd.} \tag{36}$$

We also have

$$T_s^1(1,1) = 2\sum_{i=0}^{(s-2)/2} \frac{\sqrt{5}F_{2i+1}+2}{L_{2i+1}^2}, \text{if } s \geq 2 \text{ is even.} \tag{37}$$

With (36) and (37) in mind, we state the following corollary of Theorem 22. Note that, due to the occurrence of vanishing denominators, (32) and (35) do not make a contribution to Corollary 23.

Corollary 23. *If $s \geq 1$ then*

$$\sum_{i=0}^{\infty} \frac{F_{2i}}{\left(L_{2i} - \sqrt{5}F_s\right)^2} = \frac{1}{5L_s} \sum_{i=0}^{s-1} \frac{5F_{2i-s} + 2\sqrt{5}}{L_{2i-s}^2}, \text{ if } s \text{ is odd,} \tag{38}$$

$$\sum_{i=0}^{\infty} \frac{L_{2i+1}}{\left(F_{2i+1} - L_s/\sqrt{5}\right)^2} = \frac{2}{F_s} \sum_{i=0}^{(s-2)/2} \frac{5F_{2i+1} + 2\sqrt{5}}{L_{2i+1}^2}, \text{ if } s \text{ is even.} \tag{39}$$

For instance, with $s = 1$ and $s = 2$, (38) and (39) yield, respectively,

$$\sum_{i=0}^{\infty} \frac{F_{2i}}{\left(L_{2i} - \sqrt{5}\right)^2} = \frac{5 + 2\sqrt{5}}{5},$$

$$\sum_{i=0}^{\infty} \frac{L_{2i+1}}{\left(F_{2i+1} - 3/\sqrt{5}\right)^2} = 10 + 4\sqrt{5}.$$

6 Concluding comments

In this paper, we do not make use of Fibonacci/Lucas identities to prove our results. Rather, we establish certain algebraic identities that translate to the Fibonacci and Lucas numbers via the Binet forms. Here, the influence of the insightful work of Almkvist [1] is clear. Almkvist's paper was in response to the seminal paper of Backstrom [3], who initiates study into finite and infinite sums with summands of the form

$$\frac{1}{F_{2ai+b} + c}, \text{ and } \frac{1}{L_{2ai+b} + c},$$

for certain constants a, b, and c. Soon after, Popov [4, 5] offers alternative proofs of Backstrom's results, and produces further results similar in nature. Popov's work relies upon well known Fibonacci identities. Later, André-Jeannin [2], with his characteristic elegance, offers more insight. This is followed by the contributions of Zhao [6, 7].

Finally, it is natural to ask if the results in this paper can be translated to the pair of sequences defined, for all integers n, by

$$U_n = pU_{n-1} - qU_{n-2}, U_0 = 0, U_1 = 1,$$
$$V_n = pV_{n-1} - qV_{n-2}, V_0 = 2, V_1 = p.$$

Here, p and q are real numbers with $p^2 - 4q \neq 0$. In the case of $q = -1$, one takes each of our results and simply replaces F_n by U_n, L_n by V_n, and 5 by $p^2 + 4$. The case where $q \neq -1$ is more involved. The interested reader wishing to proceed in this direction can obtain guidance from the work of Zhao [7].

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