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Finite Sequences Dominated by the Squares

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Abstract

We evaluate the number $u(n)$ of length-n finite sequences $(a_k)_{1\leq k\leq n}$ of natural numbers that satisfy the inequality $a_k \leq k^2$ for all k. We thus determine two recurrence relations for $u(n)$ by two different methods, and we give an explicit expression in closed form for it.

1 Introduction

The general Erdős-Turán conjecture([2, 1, 5]) states that if $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ is an infinite subset of the set $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ of natural numbers satisfying $a_n \leq cn^2$ for all integers $n \geq 1$, with any real constant $c > 0$, then the number of representations function associated with A, defined by $r(A, n) = |\{(a_i, a_j) \in A \times A : a_i + a_j = n\}|$, for $n \in \mathbb{N}$, is unbounded. In a previous paper ([3]), we established that it is enough to prove the conjecture for sequences satisfying $a_n \leq n^2$ for all $n \geq 1$. More recently, we undertook a quantitative exploration of the conjecture ([4]) by studying a function $\psi(n)$ defined on the set $\mathcal{A}(n)$ of all sequences $A = \{a_1 < a_2 < \cdots < a_n\}$ of n elements in $\mathbb{N}$ dominated by the squares, i.e., satisfying $a_k \le k^2$ for $1 \le k \le n$. In the process, a natural question that came up was the determination of the number $u(n) = |\mathcal{A}(n)|$ of such sequences. It proved to be a non-trivial combinatorial problem of intrinsic interest. We here present two solutions to this problem by two different methods which agree numerically.

We first introduce two auxiliary sequences of numbers $u\,(n,t) = |\mathcal{A}(n,t)|$ and $v\,(n,t) = |\mathcal{A}(n|t)|$, depending on two integer parameters $n, t \in \mathbb{N}$, in terms of which $u(n)$ can be simply expressed, where $\mathcal{A}(n,t)$ (resp. $\mathcal{A}(n|t)$) is the set of all subsets of $\mathcal{A}(n)$ contained in the interval $[0,t]$ (resp. whose largest element $a_n = t$). We thus prove ((9) and (11)) that

$$u(n) = \sum_{t=n-1}^{n^2} v(n,t) = u(n, n^2-k) + k \cdot u(n-1), \quad \text{for} \quad n \ge 1,\ 1 \le k \le 2n-1.$$

These double sequences $(u(n,t))_{n,t}$ and $(v\,(n,t))_{n,t}$ have properties of intrinsic interest. For instance, they satisfy the relations ((13) and (14))

$$u(n,t) = u(n,t-1) + u(n-1,t-1), \quad v(n,t) = v(n,t-1) + v(n-1,t-1), \quad \text{for } 1 \le t \le n^2.$$

These relations are similar to the ones satisfied by the binomial coefficients, but they are subject to the restriction $t \le n^2$. It is thus possible to construct for $u(n,t)$ and $v(n,t)$ analogues of Pascal triangle.

As for the main sequence $u(n)$, we obtain two different recurrence formulas for it by two different methods, namely ((22) and (32)), for $n \ge 2$,

$$u(n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{(n-k+1)^2 + k - 1}{k} u(n-k);$$

and

$$u(n) = \binom{n^2}{n-1} + \binom{n^2-1}{n-1} - \sum_{k=2}^{n-1} \binom{n^2-k^2}{n-k+1} u\,(k-1)\,.$$

We also establish an explicit expression in "closed form" for $u(n)$, namely ((37))

$$u(k-1) = \frac{\Delta_n(k)}{\Delta_n}, \quad \text{for } 2 \le k \le n-1,$$

where Δ_n is the $(n-2) \times (n-2)$ determinant

$$\Delta_n = \det\left((a_n(h,k))_{\substack{0 \le h \le n-3 \\ 2 \le k \le n-1}} \right),$$

and $\Delta_n(k)$ is the determinant obtained by replacing in Δ_n the k-th column by the column $(b_n(h))_{0 \le h \le n-3}$, with

$$a_n(h,k) = \binom{n^2 - k^2 + h - 1}{n-k-1}, \quad \text{and} \quad b_n(h) = \binom{n^2+h-1}{n-3} + \binom{n^2+h-2}{n-3},$$

for $0 \leq h \leq n-3$ and $2 \leq k \leq n-1$. However this expression is not practical, since the size of the determinant grows with n.

The sequence $(u(n))_{n\in\mathbb{N}}$ appears in N. J. A. Sloane's database, *The On-line Encyclopedia of Integer Sequences* (OEIS) [6], as A242105.

2 The auxiliary double sequences

For any $n \in \mathbb{N}$, let $\mathcal{A}(n)$ be the set of all subsets $A = \{a_1 < a_2 < \cdots < a_n\}$ of $\mathbb{N}$ of cardinality $|A| = n$, satisfying $a_k \leq k^2$ for $1 \leq k \leq n$. Also, for any $n, t \in \mathbb{N}$, let

$$\mathcal{A}(n,t) = \{A \in \mathcal{A}(n) : A \subset [0,t]\},$$

and

$$\mathcal{A}(n|t) = \{A \in \mathcal{A}(n) : \max(A) = t\}.$$

Further, let $u(n) = |\mathcal{A}(n)|$, $u(n,t) = |\mathcal{A}(n,t)|$, and $v(n,t) = |\mathcal{A}(n|t)|$.

Obviously,

$$u(0) = 1, \qquad u(1) = 2. \tag{1}$$

It follows from the definitions that if $n > 0$ and $A = \{a_1 < \cdots < a_n\} \in \mathcal{A}(n)$, then

$$k - 1 \leq a_k \leq k^2, \quad \text{for } 1 \leq k \leq n. \tag{2}$$

It is not difficult to see that

$$u(n,t) = \begin{cases} 0, & \text{if } 0 \leq t < n-1; \\ 1, & \text{if } n = 0 \text{ or } (n = 1, t = 0); \\ u(n), & \text{if } t \geq n^2 \geq 1. \end{cases} \tag{3}$$

Also,

$$v(n,t) = \begin{cases} 0, & \text{if } n = 0 \text{ or } (t = 0, n \neq 1) \text{ or } t < n-1 \text{ or } t > n^2; \\ 1, & \text{if } n = 1 \text{ and } t = 0, 1. \end{cases} \tag{4}$$

Moreover,

$$u(n, n-1) = v(n, n-1) = 1, \quad \text{for } n \geq 1. \tag{5}$$

Lemma 1. *For $n, t \in \mathbb{N}$, we have*

$$v(n,t) = \begin{cases} u(n-1, t-1), & \text{if } t \leq n^2; \\ 0, & \text{if } t > n^2. \end{cases} \tag{6}$$

Proof. If $t \leq n^2$, then $\mathcal{A}(n|t) = \{A \cup \{t\} : A \in \mathcal{A}(n-1, t-1)\}$, and therefore $v(n,t) = u(n-1, t-1)$. For $t > n^2$, the result follows from (4). $\square$

Remark 2. It follows from (6) and (3) that

$$v(n,t) = u\left(n-1, t-1\right) = u\left(n-1\right), \quad \text{for } \left(n-1\right)^2 < t \leq n^2. \tag{7}$$

Therefore

$$u(n) = v\left(n+1, t\right), \quad \text{for } n^2 < t \leq (n+1)^2. \tag{8}$$

Lemma 3. *For $n \geq 1$, we have*

$$u(n) = \sum_{t=n-1}^{n^2} v(n,t) = \sum_{t=0}^{\infty} v(n,t). \tag{9}$$

Proof. It follows from (2) that $\mathcal{A}(n) = \bigcup_{t=n-1}^{n^2} \mathcal{A}(n|t)$, a union of pairwise disjoint sets. Hence the first equality. The second equality follows from (4). □

Lemma 4. *For any $n, s, t \in \mathbb{N}$ such that $s < t$, we have*

$$u(n,t) = u(n,s) + \sum_{k=s+1}^{t} v(n,k). \tag{10}$$

Proof. This follows from the simple equality $\mathcal{A}(n,t) = \mathcal{A}(n,s) \bigcup \left(\bigcup_{k=s+1}^{t} \mathcal{A}(n|k) \right)$, with a union of pairwise disjoint sets. □

Lemma 5. *For $n \geq 1$ and $1 \leq k \leq 2n-1$, we have*

$$u(n) = u(n, n^2 - k) + k \cdot u(n-1). \tag{11}$$

Proof. By (3) and (10), we have

$$u(n) = u(n, n^2) = u(n, n^2 - k) + \sum_{j=n^2-k+1}^{n^2} v(n,j).$$

Moreover, for $1 \leq k \leq 2n-1$ and $n^2 - k < j \leq n^2$, we have $\left(n-1\right)^2 < j \leq n^2$, and therefore $v(n,j) = u(n-1)$, by (7). Hence the result. □

Remark 6. It follows from (10) with $s = t-1$, that, for $n \geq 0, t \geq 1$, we have

$$u(n,t) = u\left(n, t-1\right) + v\left(n,t\right). \tag{12}$$

Lemma 7. *For $1 \leq t \leq n^2$, we have*

$$u(n,t) = u(n,t-1) + u(n-1,t-1). \tag{13}$$

Proof. We clearly have

$$\mathcal{A}(n,t) = \mathcal{A}(n|t) \cup \mathcal{A}(n,t-1),$$

and

$$\mathcal{A}(n|t) \cap \mathcal{A}(n,t-1) = \emptyset.$$

Hence

$$u(n,t) = |\mathcal{A}(n,t)| = |\mathcal{A}(n|t)| + |\mathcal{A}(n,t-1)| = v(n,t) + u(n,t-1).$$

Moreover, by (6),

$$v(n,t) = u(n-1,t-1).$$

The result follows immediately. □

Lemma 8. *For $1 \le t \le n^2$, we have*

$$v(n,t) = v(n,t-1) + v(n-1,t-1). \tag{14}$$

Proof. It can be easily verified that the relation holds for $t = 1$. So we may assume that $2 \le t \le n^2$. We have a bijection

$$f : \mathcal{A}(n|t) \to \mathcal{A}(n-1|t-1) \cup \mathcal{A}(n-1,t-2)$$

which, to $A = \{a_1 < a_2 < \cdots < a_n\}$, with $a_n = t$, makes correspond

$$f(A) = A \setminus \{t\} = \{a_1 < a_2 < \cdots < a_{n-1}\}$$

. And we have

$$\mathcal{A}(n-1|t-1) \cap \mathcal{A}(n-1,t-2) = \emptyset.$$

Hence

$$v(n,t) = |\mathcal{A}(n|t)| = |\mathcal{A}(n-1|t-1)| + |\mathcal{A}(n-1,t-2)| = v(n-1,t-1) + u(n-1,t-2).$$

Moreover, by (6), $v(n,t-1) = u(n-1,t-2)$. The result follows immediately. □

Lemma 9. *For $n,t \ge 1$, we have*

$$u(n,t) - u(n,t-1) = \begin{cases} u(n-1,t-1), & \textit{if } t \le n^2; \\ 0, & \textit{if } t > n^2. \end{cases} \tag{15}$$

Proof. By (12), we have $u(n,t) - u\,(n,t-1) = v\,(n,t)$, and the result follows from (6). □

Lemma 10. *For $n,t \ge 1$, we have*

$$v(n,t) - v(n,t-1) - v\,(n-1,t-1) = \begin{cases} -u(n-1), & \textit{if } t = n^2+1; \\ 0 & \textit{if } t \ne n^2+1. \end{cases} \tag{16}$$

Proof. If $t = n^2 + 1$, then, by (4) and (7),

$$v(n, n^2 + 1) - v(n, n^2) - v(n-1, n^2) = -u(n-1).$$

If $t > n^2 + 1$, then, by (4), all terms on the left are $= 0$ and the last case holds.

If $1 \le t \le n^2$, then, by (6), $v(n,t) = u(n-1, t-1)$ and $v(n, t-1) = u\,(n-1, t-2)$; and, by (12), $v\,(n-1, t-1) = u(n-1, t-1) - u(n-1, t-2)$. Therefore

$$v(n,t) - v(n, t-1) - v\,(n-1, t-1) = 0.$$

Hence the result. $\square$

Example 11. Using the relation (14), we can construct an analogue of Pascal's triangle for the numbers $v(n,t)$. We here give the first few rows of such a table.

$v(n,t)$	$n=1$	2	3	4	5	6	7	8	9	10
$t=0$	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	0	2	1	0	0	0	0	0	0	0
3	0	2	3	1	0	0	0	0	0	0
4	0	2	5	4	1	0	0	0	0	0
5	0	0	7	9	5	1	0	0	0	0
6	0	0	7	16	14	6	1	0	0	0
7	0	0	7	23	30	20	7	1	0	0
8	0	0	7	30	53	50	27	8	1	0
9	0	0	7	37	83	103	77	35	9	1
10	0	0	0	44	120	186	180	112	44	10
11	0	0	0	44	164	306	366	292	156	54
12	0	0	0	44	208	470	672	658	448	210
13	0	0	0	44	252	678	1142	1330	1106	658
14	0	0	0	44	296	930	1820	2472	2436	1764
15	0	0	0	44	340	1226	2750	4292	4908	4200
16	0	0	0	44	384	1566	3976	7042	9200	9108
17	0	0	0	0	428	1950	5542	11018	16242	18308
18	0	0	0	0	428	2378	7492	16560	27260	34550
19	0	0	0	0	428	2806	9870	24052	43820	61810
20	0	0	0	0	428	3234	12676	33922	67872	105630

Proposition 12. *For $k, n, t \in \mathbb{N}$ such that $1 \le k \le n$ and $k \le t \le (n-k+1)^2 + k - 1$, we have*

$$u(n,t) = \sum_{i=0}^{k} \binom{k}{i} u(n-i, t-k). \tag{17}$$

Proof. For a given $n \geq 1$, the proof is by induction on k.

For $k = 1$, the equality amounts to $u(n,t) = u(n,t-1) + u(n-1,t-1)$, which is true for $1 \leq t \leq n^2$, by (13).

Assume the equality holds for some $1 \leq k < n$ and all $k \leq t \leq (n-k+1)^2 + k - 1$. Then, for $k+1 \leq t \leq (n-k)^2 + k$, we have, by (13),

$$u(n-i, t-k) = u(n-i, t-k-1) + u(n-i-1, t-k-1),$$

for $0 \leq i \leq k$. Hence

$$\begin{aligned} u(n,t) &= \sum_{i=0}^{k} \binom{k}{i} u(n-i,t-k) = \sum_{i=0}^{k} \binom{k}{i} \left(u(n-i,t-k-1) + u(n-i-1,t-k-1) \right) = \\ &= \sum_{i=0}^{k} \binom{k}{i} u(n-i,t-k-1) + \sum_{j=1}^{k+1} \binom{k}{j-1} u(n-j,t-k-1) = \\ &= \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1} \right) u(n-i,t-k-1) + \\ &+ u(n,t-k-1) + u(n-k-1,t-k-1) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} u(n-i,t-(k+1)), \quad \text{for } k+1 \leq t \leq (n-k)^2 + k. \end{aligned}$$

So the equality holds for $k+1$ and all $k+1 \leq t \leq (n-k)^2 + k$. This completes the induction. □

3 The main sequence $(u(n))_{n \geq 0}$

We now propose to establish a recurrence formula for the numbers $u(n)$. In the course of the work, we will need the following summation formulas for binomial coefficients.

Lemma 13. *For all integers $N \geq m \geq 0$, we have*

$$\sum_{k=m}^{N} \binom{k}{m} = \binom{N+1}{m+1}. \tag{18}$$

Proof. The proof is by a simple induction on N, using the fundamental recurrence for binomial coefficients

$$\binom{N}{m} + \binom{N}{m+1} = \binom{N+1}{m+1}.$$

□

Corollary 14. *For all integers $N \geq M \geq m \geq 0$, we have*

$$\sum_{k=M}^{N} \binom{k}{m} = \sum_{k=m}^{N} \binom{k}{m} - \sum_{k=m}^{M-1} \binom{k}{m} = \binom{N+1}{m+1} - \binom{M}{m+1}. \tag{19}$$

Remark 15. In the remainder of this section, we set $a_0 = 0$.

Lemma 16. *For $n \geq 1$, we have*

$$u\left(n\right) = \sum_{a_1=0}^{1} \sum_{a_2=a_1+1}^{4} \cdots \sum_{a_n=a_{n-1}+1}^{n^2} 1. \tag{20}$$

Proof. By definition, $\mathcal{A}(n)$ consists of all subsets $A = \{a_1 < a_2 < \cdots < a_n\}$ of $\mathbb{N}$ satisfying

$$0 \leq a_1 \leq 1,\ a_1 + 1 \leq a_2 \leq 4,\ a_2 + 1 \leq a_3 \leq 9, \ldots, a_{n-1} + 1 \leq a_n \leq n^2.$$

The stated relation for $u\left(n\right) = \left|\mathcal{A}\left(n\right)\right|$ follows immediately. $\square$

Theorem 17. *For $n \geq 2$ and $1 \leq h \leq n-1$, we have*

$$\begin{aligned} u(n) = {} & \sum_{k=1}^{h} (-1)^{k-1} \binom{(n-k+1)^2 + k - 1}{k} u(n-k) + \\ & + (-1)^h \sum_{a_1=0}^{1} \sum_{a_2=a_1+1}^{4} \cdots \sum_{a_{n-h}=a_{n-h-1}+1}^{(n-h)^2} \binom{a_{n-h} + h - 1}{h}. \end{aligned} \tag{21}$$

Proof. For a fixed $n \geq 2$, the proof is by induction on h.

First, note that the truth of (21) for $h = 1$ follows immediately from (20)

Assume now, by induction, that (21) holds for some $1 \leq h < n - 1$. By (19), we have

$$\sum_{a_{n-h}=a_{n-h-1}+1}^{(n-h)^2} \binom{a_{n-h} + h - 1}{h} = \sum_{k=a_{n-h-1}+h}^{(n-h)^2+h-1} \binom{k}{h} = \binom{(n-h)^2 + h}{h+1} - \binom{a_{n-h-1} + h}{h+1},$$

and substituting this into (21) yields

$$
\begin{aligned}
u(n) = &\sum_{k=1}^{h}(-1)^{k-1}\binom{(n-k+1)^2+k-1}{k}u(n-k)+\\
&+(-1)^h\sum_{a_1=0}^{1}\sum_{a_2=a_1+1}^{4}\cdots\sum_{a_{n-h-1}=a_{n-h-2}+1}^{(n-h-1)^2}\left(\binom{(n-h)^2+h}{h+1}-\binom{a_{n-h-1}+h}{h+1}\right)=\\
&=\sum_{k=1}^{h}(-1)^{k-1}\binom{(n-k+1)^2+k-1}{k}u(n-k)+\\
&+(-1)^h\binom{(n-h)^2+h}{h+1}\sum_{a_1=0}^{1}\sum_{a_2=a_1+1}^{4}\cdots\sum_{a_{n-h-1}=a_{n-h-2}+1}^{(n-h-1)^2}1+\\
&+(-1)^{h+1}\sum_{a_1=0}^{1}\sum_{a_2=a_1+1}^{4}\cdots\sum_{a_{n-h-1}=a_{n-h-2}+1}^{(n-h-1)^2}\binom{a_{n-h-1}+h}{h+1}.
\end{aligned}
$$

Moreover, by (20),

$$
\sum_{a_1=0}^{1}\sum_{a_2=a_1+1}^{4}\cdots\sum_{a_{n-h-1}=a_{n-h-2}+1}^{(n-h-1)^2}1=u(n-h-1).
$$

Hence

$$
\begin{aligned}
u(n) = &\sum_{k=1}^{h}(-1)^{k-1}\binom{(n-k+1)^2+k-1}{k}u(n-k)+(-1)^h\binom{(n-h)^2+h}{h+1}u(n-h-1)+\\
&+(-1)^{h+1}\sum_{a_1=0}^{1}\sum_{a_2=a_1+1}^{4}\cdots\sum_{a_{n-h-1}=a_{n-h-2}+1}^{(n-h-1)^2}\binom{a_{n-h-1}+h}{h+1}=\\
&=\sum_{k=1}^{h+1}(-1)^{k-1}\binom{(n-k+1)^2+k-1}{k}u(n-k)+\\
&(-1)^{h+1}\sum_{a_1=0}^{1}\sum_{a_2=a_1+1}^{4}\cdots\sum_{a_{n-h-1}=a_{n-h-2}+1}^{(n-h-1)^2}\binom{a_{n-h-1}+h}{h+1},
\end{aligned}
$$

which is just the expression (21) for $h+1$ replacing h. This completes the proof by induction of (21). □

Corollary 18. *For $n \geq 2$, we have*

$$
u(n)=\sum_{k=1}^{n}(-1)^{k-1}\binom{(n-k+1)^2+k-1}{k}u(n-k). \tag{22}
$$

n	$u(n)$
0	1
1	2
2	7
3	44
4	428
5	5802
6	102322
7	2239844
8	58849332
9	1810039960
10	63930543419

Table 1: First few values of $u(n)$

Proof. Taking $h = n-1$, the multiple sum in (21) reduces to

$$(-1)^{n-1}\sum_{a_1=0}^{1}\binom{a_1+n-2}{n-1} = (-1)^{n-1} = (-1)^{n-1}u(0).$$

Substituting this into (21) yields the desired expression for $u(n)$. ∎

Example 19. Using (22) and (1), we can compute $u(n)$ recursively. We thus obtain the data in Table 1.

More values of $u(n)$ can be found in A242105 of the OEIS ([6]).

4 An alternative approach

Definition 20. For $t \in \mathbb{N}$, let $t^* = \lceil\sqrt{t}\rceil$ be the least integer $\geq \sqrt{t}$, so that $t^* = m$ if and only if $(m-1)^2 < t \leq m^2$ for some $m \in \mathbb{N}$, and let

$$p_t(x) = \sum_{n=t^*}^{t+1} v(n,t)x^{n-t^*} = x^{t+1-t^*} + \sum_{n=t^*}^{t} v(n,t)x^{n-t^*}. \tag{23}$$

Proposition 21. *For $t \geq 1$, the polynomials p_t satisfy the following recurrence formula.*

$$p_{t+1}(x) = \begin{cases} \dfrac{1}{x}\left(\left(x+1\right)p_t(x) - u\left(m-1\right)\right), & \text{if } t = m^2, \text{ with } m \in \mathbb{N}; \\ \left(x+1\right)p_t(x), & \text{otherwise.} \end{cases} \tag{24}$$

Proof. Note first that if $n \geq (t+1)^*$ then $n^2 > t$, and that by (14), $v\,(n,t+1) = v(n,t) + v(n-1,t)$, for $0 \leq t < n^2$. Hence

$$p_{t+1}(x) = \sum_{n=(t+1)^*}^{t+2} v(n,t+1)x^{n-(t+1)^*} = \sum_{n=(t+1)^*}^{t+2} \left(v(n,t) + v(n-1,t)\right) x^{n-(t+1)^*} =$$

$$= \sum_{n=(t+1)^*}^{t+2} v(n,t)x^{n-(t+1)^*} + \sum_{n=(t+1)^*-1}^{t+1} v(n,t)x^{n+1-(t+1)^*}.$$

If $t = m^2$, then $t^* = m$ and $(t+1)^* = m+1$, so that

$$\sum_{n=(t+1)^*}^{t+2} v(n,t)x^{n-(t+1)^*} = \sum_{n=m+1}^{m^2+2} v(n,m^2)x^{n-m-1} = \frac{1}{x}\sum_{n=t^*+1}^{t+2} v(n,t)x^{n-t^*} =$$

$$= \frac{1}{x}\left(p_t(x) - v\left(m,m^2\right) + v\left(m^2+2,m^2\right)x^{m^2+2-m}\right) = \frac{1}{x}\left(p_t(x) - u\left(m-1\right)\right),$$

since $v\,(m,m^2) = u(m-1)$ by (8), and $v\,(m^2+2,m^2) = 0$ by (4), and

$$\sum_{n=(t+1)^*-1}^{t+1} v(n,t)x^{n+1-(t+1)^*} = \sum_{n=m}^{m^2+1} v(n,m^2)x^{n-m} = \sum_{n=t^*}^{t+1} v(n,t)x^{n-t^*} = p_t(x).$$

Therefore, in the case $t = m^2$, we have

$$p_{t+1}(x) = \frac{1}{x}\left(p_t(x) - u\left(m-1\right)\right) + p_t(x) = \frac{1}{x}\left(\left(x+1\right)p_t(x) - u\left(m-1\right)\right).$$

Otherwise, if t is not a square, i.e., $m^2 < t < (m+1)^2$, with $m \in \mathbb{N}$, then $t^* = (t+1)^* = m+1$, so that

$$\sum_{n=(t+1)^*}^{t+2} v(n,t)x^{n-(t+1)^*} = \sum_{n=t^*}^{t+2} v(n,t)x^{n-t^*} = p_t(x) + v\left(t+2,t\right)x^{t+2-t^*} = p_t(x),$$

since $v\,(t+2,t) = 0$ by (4); and

$$\sum_{n=(t+1)^*-1}^{t+1} v(n,t)x^{n+1-(t+1)^*} = \sum_{n=t^*-1}^{t+1} v(n,t)x^{n+1-t^*} = x\sum_{n=t^*-1}^{t+1} v(n,t)x^{n-t^*} =$$

$$= x\left(p_t(x) + v\left(t^*-1,t\right)x^{-1}\right) = xp_t\left(x\right),$$

since $v\,(t^*-1,t) = v\,(m,t) = 0$, by (4). Therefore, in this case, we have

$$p_{t+1}(x) = p_t(x) + xp_t(x) = \left(x+1\right)p_t(x).$$

$\square$

Corollary 22. *For $n \geq 1$ and $1 \leq h \leq 2n+1$, we have*

$$p_{n^2+h}(x) = \frac{1}{x}\left((x+1)^h p_{n^2}(x) - u(n-1)(x+1)^{h-1}\right). \tag{25}$$

Proof. The proof is by induction on h. By (24), $p_{n^2+1}(x) = \frac{1}{x}((x+1)p_{n^2}(x) - u(n-1))$, so that the property holds for $h=1$. Assume, by induction that it holds for some $1 \leq h \leq 2n$. Then, by (24) and the induction assumption,

$$\begin{aligned} p_{n^2+h+1}(x) &= (x+1)p_{n^2+h}(x) = \frac{x+1}{x}\left((x+1)^h p_{n^2}(x) - u(n-1)(x+1)^{h-1}\right) = \\ &= \frac{1}{x}\left((x+1)^{h+1} p_{n^2}(x) - u(n-1)(x+1)^h\right), \end{aligned}$$

which shows that the result holds for $h+1$ and completes the induction. □

Example 23. We have

$$p_0(x) = x, \qquad p_1(x) = x+1, \qquad p_2(x) = x+2,$$

$$p_3(x) = x^2+3x+2, \qquad p_4(x) = x^3+4x^2+5x+2, \qquad p_5(x) = x^3+5x^2+9x+7.$$

Definition 24. For any integer $n \geq 2$, let

$$T_n(x) = \sum_{k=2}^{n-1} u(k-1)\, x^{k-2}(x+1)^{n^2-k^2-1}.$$

Example 25. We have

$$T_2(x) = 0, \quad T_3(x) = 2(x+1)^4,$$

$$T_4(x) = 2x^{11}+22x^{10}+110x^9+330x^8+667x^7+966x^6+1029x^5+800x^4+435x^3+152x^2+29x+2.$$

Lemma 26. *For $n \geq 2$, we have*

$$T_{n+1}(x) = (x+1)^{2n+1} T_n(x) + u(n-1)\, x^{n-2}(x+1)^{2n}. \tag{26}$$

Proof. By definition,

$$\begin{aligned} T_{n+1}(x) &= \sum_{k=2}^{n} u(k-1)\, x^{k-2}(x+1)^{(n+1)^2-k^2-1} = \sum_{k=2}^{n} u(k-1)\, x^{k-2}(x+1)^{n^2+2n-k^2} \\ &= (x+1)^{2n+1}\sum_{k=2}^{n-1} u(k-1)\, x^{k-2}(x+1)^{n^2-k^2-1} + u(n-1)x^{n-2}(x+1)^{2n} = \\ &= (x+1)^{2n+1} T_n(x) + u(n-1)\, x^{n-2}(x+1)^{2n}. \end{aligned}$$

□

Lemma 27. *For $n \geq 2$, we have*

$$p_{n^2}(x) = \frac{(x+1)^{n^2-1} + (x+1)^{n^2-2} - T_n(x)}{x^{n-2}}. \tag{27}$$

Proof. The proof is by induction on n. Using the Examples given above, it is easily verified that $p_4(x) = (x+1)^3 + (x+1)^2 - T_2(x)$, so that the relation holds for $n = 2$.

Assume inductively that it holds for n. Then, by (25), the induction assumption and (26), we have

$$\begin{aligned} xp_{(n+1)^2}(x) &= xp_{n^2+2n+1}(x) = (x+1)^{2n+1} p_{n^2}(x) - u(n-1)(x+1)^{2n} = \\ &= (x+1)^{2n+1} \frac{(x+1)^{n^2-1} + (x+1)^{n^2-2} - T_n(x)}{x^{n-2}} - u(n-1)(x+1)^{2n}. \end{aligned}$$

Therefore

$$\begin{aligned} x^{n-1}p_{(n+1)^2}(x) &= (x+1)^{n^2+2n} + (x+1)^{n^2+2n-1} - (x+1)^{2n+1} T_n(x) \\ &\quad - u(n-1)x^{n-2}(x+1)^{2n} = (x+1)^{(n+1)^2-1} + (x+1)^{(n+1)^2-2} - T_{n+1}(x). \end{aligned}$$

Thus the relation holds for $n+1$. $\square$

Corollary 28. *For $n \geq 2$ and $1 \leq h \leq 2n+1$, we have*

$$\begin{aligned} p_{n^2+h}(x) &= \\ &= \frac{(x+1)^{n^2+h-1} + (x+1)^{n^2+h-2} - (x+1)^h T_n(x) - u(n-1)x^{n-2}(x+1)^{h-1}}{x^{n-1}}. \end{aligned} \tag{28}$$

Proof. By (25), we have $p_{n^2+h}(x) = \frac{1}{x}\left((x+1)^h p_{n^2}(x) - u(n-1)(x+1)^{h-1}\right)$. Substituting in this equality the expression in (27) for $p_{n^2}(x)$ yields the result. $\square$

Lemma 29. *For $n \geq 2$, we have*

$$T_n(x) = \sum_{m=0}^{n^2-5} \left(\sum_{k=2}^{n-1} \binom{n^2-k^2-1}{m-k+2} u(k-1) \right) x^m. \tag{29}$$

Proof. From the definition and the binomial expansion, and taking into account that $\binom{N}{j} = 0$

if $j > N$ $(j, N \in \mathbb{N})$, we have

$$\begin{aligned}
T_n(x) &= \sum_{k=2}^{n-1} u\,(k-1)\,x^{k-2}\,(x+1)^{n^2-k^2-1} \\
&= \sum_{k=2}^{n-1} u\,(k-1)\,x^{k-2} \sum_{j=0}^{n^2-k^2-1} \binom{n^2-k^2-1}{j} x^j \\
&= \sum_{k=2}^{n-1} u\,(k-1)\,x^{k-2} \sum_{j\geq 0} \binom{n^2-k^2-1}{j} x^j = \sum_{j\geq 0}\sum_{k=2}^{n-1} \binom{n^2-k^2-1}{j} u(k-1)x^{k+j-2} \\
&= \sum_{m\geq 0}\sum_{k=2}^{n-1} \binom{n^2-k^2-1}{m-k+2} u(k-1)x^m = \sum_{m=0}^{n^2-5}\sum_{k=2}^{n-1} \binom{n^2-k^2-1}{m-k+2} u(k-1)x^m.
\end{aligned}$$

Here, we have set $m = k + j - 2$, and we note that, for $2 \leq k \leq n-1$, the condition $0 \leq m - k + 2 \leq n^2 - k^2 - 1$ amounts to $0 \leq m \leq n^2 - k(k-1) - 3 \leq n^2 - 5$. ☐

Theorem 30. *For $1 \leq h \leq 2n+1$ and $n \geq 2$, we have*

$$\begin{aligned}
u(n) =& \binom{n^2+h-1}{n-1} + \binom{n^2+h-2}{n-1} - \sum_{k=2}^{n-1} \binom{n^2-k^2+h-1}{n-k+1} u\,(k-1) \\
&- (h-1)\,u\,(n-1)\,.
\end{aligned} \tag{30}$$

Proof. The constant coefficient of $p_{n^2+h}(x) = \sum\limits_{k=n+1}^{n^2+h+1} v(k, n^2+h)x^{k-n-1}$ is

$$p_{n^2+h}\,(0) = v\left(n+1, n^2+h\right) = u(n),$$

by (7). So $u(n)$ is the coefficient of x^{n-1} in $x^{n-1}p_{n^2+h}\,(x)$. Moreover by (28), we have

$x^{n-1}p_{n^2+h}\,(x) = (x+1)^{n^2+h-1} + (x+1)^{n^2+h-2} - (x+1)^h\,T_n\,(x) - u\,(n-1)\,x^{n-2}\,(x+1)^{h-1}$;

and, by binomial expansions, we get

$$\begin{aligned}
(x+1)^h T_n(x) &= \sum_{k=2}^{n-1} u\,(k-1)\,x^{k-2}\,(x+1)^{n^2-k^2+h-1} = \\
&= \sum_{k=2}^{n-1} u(k-1) \sum_{j\geq 0} \binom{n^2-k^2+h-1}{j} x^{j+k-2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
x^{n-1}p_{n^2+h}\,(x) =& \sum_{k=0}^{n^2+h-1} \binom{n^2+h-1}{k} x^k + \sum_{k=0}^{n^2+h-2} \binom{n^2+h-2}{k} x^k \\
&- \sum_{j\geq 0}\sum_{k=2}^{n-1} \binom{n^2-k^2+h-1}{j} u(k-1)x^{j+k-2} - u\,(n-1) \sum_{k=0}^{h-1} \binom{h-1}{k} x^{k+n-2}.
\end{aligned}$$

So the coefficient of x^{n-1} in $x^{n-1}p_{n^2+h}(x)$ is the sum of the coefficients of x^{n-1} in each of the last four sums, namely

$$\binom{n^2+h-1}{n-1}+\binom{n^2+h-2}{n-1}-\sum_{k=2}^{n-1}\binom{n^2-k^2+h-1}{n-k+1}u\left(k-1\right)-\left(h-1\right)u\left(n-1\right).$$

The result follows by identifying the two expressions for the coefficient of x^{n-1}. ☐

Corollary 31. *For $n \geq 2$, we have*

$$u(n)=\binom{n^2}{n-1}+\binom{n^2-1}{n-1}-\sum_{k=2}^{n-1}\binom{n^2-k^2}{n-k+1}u\left(k-1\right). \tag{31}$$

Proof. This is the special case $h=1$ of (30). ☐

Corollary 32. *For $n \geq 2$ and $0 \leq h \leq 2n$, we have*

$$u(n-1)=\binom{n^2+h-1}{n-2}+\binom{n^2+h-2}{n-2}-\sum_{k=2}^{n-1}\binom{n^2-k^2+h-1}{n-k}u(k-1). \tag{32}$$

Proof. Replacing h by $h+1$ in (30), we get

$$u(n)=\binom{n^2+h}{n-1}+\binom{n^2+h-1}{n-1}-\sum_{k=2}^{n-1}\binom{n^2-k^2+h}{n-k+1}u\left(k-1\right)-hu\left(n-1\right). \tag{33}$$

Then, substracting (30) from (33), and using the fundamental relation for binomial coefficients,

$$\binom{n}{k}-\binom{n-1}{k}=\binom{n-1}{k-1}, \tag{34}$$

we get the desired result. ☐

Corollary 33. *For $n \geq 3$ and $0 \leq h \leq 2n-1$, we have*

$$\sum_{k=2}^{n-1}\binom{n^2-k^2+h-1}{n-k-1}u(k-1)=\binom{n^2+h-1}{n-3}+\binom{n^2+h-2}{n-3}. \tag{35}$$

Proof. It follows from (32) that the expression

$$F(n,h)=\binom{n^2+h-1}{n-2}+\binom{n^2+h-2}{n-2}-\sum_{k=2}^{n-1}\binom{n^2-k^2+h-1}{n-k}u(k-1)$$

is independent of $0 \leq h \leq 2n$. Therefore $F(n,h)-F(n,h+1)=0$ for $0 \leq h \leq 2n-1$, which, in view of (34), yields the desired result. ☐

Corollary 34. *For $n \geq 3$ and $2 \leq k \leq n-1$, we have the expression in "closed form"*

$$u(k-1) = \frac{\Delta_n(k)}{\Delta_n}, \tag{36}$$

where Δ_n is the $(n-2) \times (n-2)$ determinant

$$\Delta_n = \det\left(\left(\binom{n^2 - k^2 + h - 1}{n-k-1}\right)_{\substack{0 \leq h \leq n-3 \\ 2 \leq k \leq n-1}}\right)$$

and $\Delta_n(k)$ is the determinant obtained by replacing in Δ_n the k-th column by the column

$$\left(\binom{n^2 + h - 1}{n-3} + \binom{n^2 + h - 2}{n-3}\right)_{0 \leq h \leq n-3}.$$

Proof. The relations (35), restricted to $0 \leq h \leq n-3$, give a system of $n-2$ linear equations into the $n-2$ unknowns $u(1), u(2), \ldots, u(n-2)$, which can be written

$$\sum_{k=2}^{n-1} a_n(k,h)\, u(k-1) = b_n(h), \quad 0 \leq h \leq n-3, \tag{37}$$

with

$$a_n(k,h) = \binom{n^2 - k^2 + h - 1}{n-k-1} \quad \text{and} \quad b_n(h) = \binom{n^2 + h - 1}{n-3} + \binom{n^2 + h - 2}{n-3}.$$

So, by Cramer's rule, we get the stated expressions for $u(k-1)$. $\square$

Remark 35. Computations of a number of values of Δ_n suggested that $\Delta_n = \pm 1$. We have since proved that $\Delta_n = (-1)^{(n-2)(n-3)/2}$.

Remark 36. Computations of $f(n) = \log_2 u(n)$ and of $g(n) = \frac{f(n)}{n}$ for $1 \leq n \leq 105$ seem to indicate that there exist constants $0 < c_1 < c_2$ such that $c_1 n \leq g(n) \leq c_2 n$, i.e. $c_1 n^2 \leq f(n) \leq c_2 n^2$, so that $2^{c_1 n^2} \leq u(n) \leq 2^{c_2 n^2}$, for large enough n.

Remark 37. The expression (37) gives another method for computing the values of $u(n)$ quite different from the one given in (22). The two methods "fortunately" give identical numerical results. Our attempts to relate the two methods, by deducing one from the other, have so far not been fruitful. Thus the relation between the two approaches described in sections 3 and 4 remains for now an open problem.

Remark 38. Here are some of the steps that led us to the polynomials p_t and T_n. We used *generating functions* and their ability to hold data *quite tight*. Set $I = \mathbb{N} \times \mathbb{N}$ and introduce the following formal series:

$$U(x,y) = \sum_{(n,t) \in I} u(n,t) x^n y^t, \quad V(x,y) = \sum_{(n,t) \in I} v(n,t) x^n y^t, \quad G(n,y) = \sum_{t \geq 0} u(n,t) y^t,$$

and

$$P(x,y)=\sum_{n\geq 1}u(n-1)x^n y^{n^2+1}.$$

Thus

$$U(x,y)=\sum_{n\geq 0}G(n,y)x^n.$$

Also, by (3)

$$G(n,y)=\sum_{0\leq t<n^2}u(n,t)y^t+u(n)\frac{y^{n^2}}{1-y},\quad \text{for } n\geq 0.$$

In particular,

$$G(0,y)=\sum_{t\geq 0}y^t=\frac{1}{1-y}.$$

Now, using some formulas established in section 2, we obtain relations among these generating functions. Thus, by (4) and (9), we have

$$V(x,1)=\sum_{n\geq 0,t\geq 0}v(n,t)x^n=\sum_{t\geq 0}v(0,t)+\sum_{n\geq 1}\sum_{t=0}^{\infty}v(n,t)x^n=\sum_{n\geq 1}u(n)x^n.$$

Similarly using (12), we get

$$(1-y)U(x,y)=V(x,y)+1,$$

and using (16), we get

$$(1-y-xy)V=x-P,\quad (1-y)(1-y-xy)U=1+x-y-xy-P.$$

Then, introducing $F(x,y)=\sum_{t\geq n-1\geq 0}\binom{t}{n-1}x^n y^t$, we get $(1-y-xy)F=x$. Hence

$$(1-y-xy)(F-V)=P.$$

Those relations were meant to deal with the data contained in the pseudo *pascalian* tableau for the $v(n,t)$'s. To extract information from those relations, some heavy analytical machinery is probably needed.

The *truncated pascalian* triangle of the $v(n,t)$'s, given in Table 2, starts as follows (where the points represent 0's):

The original binomial generating polynomial for the lines of Pascal's triangle, $(x+y)^n$, had to be emulated. A glance at the tableau made it clear that only lines having a square order were needed to deal with the $u(n)$'s. This gave birth to the polynomials p_t. Starting with $p_0(x)=v(1,0)x=x$, the first few polynomials are easy to obtain, by induction:

$$\begin{aligned}
p_1(x)&=((1+x)x-0)/x=1+x,\\
p_2(x)&=((1+x)(1+x)-1)/x=2+x,\\
p_3(x)&=(2+x)(1+x)=2+3x+x^2,\\
p_4(x)&=(2+x)(1+x)^2=2+5x+4x^2+x^3,\\
p_5(x)&=((2+x)(1+x)^3-2)/x=7+9x+5x^2+x^4.
\end{aligned}$$

t/n	1	2	3	4	5	6	7
0	1						
1	1	1					
2	·	2	1				
3	·	2	3	1			
4	·	2	5	4	1		
5	·	·	7	9	5	1	
6	·	·	7	16	14	6	1
7	·	·	7	23	30	20	7
8	·	·	7	30	53	50	27
9	·	·	7	37	83	103	77
10	·	·	·	44	120		
11	·	·	·	44	164		
12	·	·	·	44	208		
13	·	·	·	44	252		
14	·	·	·	44	296		
15	·	·	·	44	340		
16	·	·	·	44	384		
17	·	·	·	·	428		
18	·	·	·	·	428		

Table 2: Truncated pascalian table

A look at the list strongly suggested the introduction of the T_n's. That is about everything.

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