Journal of Integer Sequences, Vol. 18 (2015),
Article 15.11.8

# On a Matrix Arising from a Family of Iterated Self-Compositions

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**Abstract**

We obtain here a number of results associated with an infinite matrix arising from a family of iterated self-compositions. This matrix exhibits a rich structure, and our results include an intricate property of its rows, a characterization of its entries in terms of their Zeckendorf representations, and a link between its columns and a mathematical object known as the Fibonacci word.

## 1 Introduction and initial definitions

Fraenkel [5] defines the notion of an iterated floor function, and derives identities involving sums of such functions. Applications of these iterated floor functions to discrete dynamical systems and to other other areas of mathematics are then exhibited. The purpose of our paper is to use the iterated floor function in order to generate an infinite matrix $\mathcal{M}$, as defined below, and then to study certain aspects of its rich structure.

In order to construct $\mathcal{M}$, we utilize both the *floor function* and the *golden ratio*. The former, written $\lfloor x \rfloor$, gives the largest integer not exceeding $x$, while the latter is given by

$$\alpha = \frac{1+\sqrt{5}}{2}.$$

| $k$ | $g_1(k)$ | $g_2(k)$ | $g_3(k)$ | $g_4(k)$ | $g_5(k)$ | $g_6(k)$ | $g_7(k)$ | $g_8(k)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 3 | 4 | 6 | 9 | 14 | 22 | 35 | 56 |
| 3 | 4 | 6 | 9 | 14 | 22 | 35 | 56 | 90 |
| 4 | 6 | 9 | 14 | 22 | 35 | 56 | 90 | 145 |
| 5 | 8 | 12 | 19 | 30 | 48 | 77 | 124 | 200 |
| 6 | 9 | 14 | 22 | 35 | 56 | 90 | 145 | 234 |
| 7 | 11 | 17 | 27 | 43 | 69 | 111 | 179 | 289 |
| 8 | 12 | 19 | 30 | 48 | 77 | 124 | 200 | 323 |

Table 1: The top left-hand corner $8 \times 8$ sub-matrix of the infinite matrix $\mathcal{M}$.

Let $\mathbb{N}$ denote the set of positive integers. For each $k \in \mathbb{N}$, $g_j(k)$ is defined by way of $g_1(k) = \lfloor k\alpha \rfloor$ and, for $j \geq 2$, $g_j(k) = g_1(g_{j-1}(k))$. Then $\mathcal{M}$ is defined to be the infinite matrix having entries $g_j(k)$, where $k$ and $j$ denote the row and column numbers, respectively.

The $8 \times 8$ matrix comprising the top left-hand corner sub-matrix of $\mathcal{M}$ is shown in Table 1. The first column of $\mathcal{M}$ is a particular *Beatty sequence* [2, 3] known as the *lower Wythoff sequence*, and given by $\mathcal{B}(\alpha) = (\lfloor n\alpha \rfloor)_{n\geq 1}$. This appears as sequence A000201 in the *On-line Encyclopedia of Integer Sequences* [16]. The second, third and fourth columns of $\mathcal{M}$ arise as sequences A003622, A134859, and A151915, respectively, but none of the further columns appear in the OEIS. Shifted versions of some of the rows of $\mathcal{M}$ are also to be found in the OEIS. For example, A001588, A001611, and A001612 are related to rows seven, two and five, respectively, by way of shifts. Some of the rows have interesting combinatorial interpretations, such as the 13th row, the terms of which, via an appropriate shift, enumerate the binary strings of length $n$ with no substrings equal to 0001, 1000, or 1001 (A164485).

Although the *upper Wythoff sequence* (A001950), $\mathcal{B}(\alpha^2) = (\lfloor n\alpha^2 \rfloor)_{n\geq 1}$, does not appear as a row or a column in $\mathcal{M}$, it does play a role here in the proof of some of the lemmas and theorems. We will make use of the fact that, as a pair of complementary sequences, $\mathcal{B}(\alpha)$ and $\mathcal{B}(\alpha^2)$ satisfy both $\mathcal{B}(\alpha) \cap \mathcal{B}(\alpha^2) = \emptyset$ and $\mathcal{B}(\alpha) \cup \mathcal{B}(\alpha^2) = \mathbb{N}$. The upper Wythoff sequence does, however, appear as the second column of a matrix known as the *Wythoff array* [9]. There are similarities between $\mathcal{M}$ and the Wythoff array, although there is also a major difference; the former contains repeated entries but not every positive integer, while the latter contains every positive integer precisely once.

Both the Fibonacci and Lucas numbers feature in some of our proofs and results. The *Fibonacci sequence* $(F_n)_{n\geq 0}$ is defined by setting $F_0 = 0$ and $F_1 = 1$ and then $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The *Lucas sequence* $(L_n)_{n\geq 0}$ is defined by setting $L_0 = 2$ and $L_1 = 1$ and then $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

In the following sections we prove a number of results concerning the structure of $\mathcal{M}$. In Section 2 some introductory results are given. Then, in Section 3, we demonstrate a characterization of the entries of $\mathcal{M}$ in terms of their Zeckendorf representations. A link between the columns of $\mathcal{M}$ and a mathematical object known as the Fibonacci word is shown in Section 4. Our final theorems, involving a property of the rows of $\mathcal{M}$, highlight very clearly the close relationship between the Fibonacci and Lucas numbers.

# 2 Some initial results associated with $\mathcal{M}$

First, we obtain a formula for the entry $g_j(k)$ of $\mathcal{M}$ and a straightforward corollary of this result. Note that use will be made of the equality $\alpha^2 = \alpha + 1$ and its many rearrangements throughout this paper.

**Theorem 1.** *The entry in the $k$th row and $j$th column of $\mathcal{M}$ is given by*

$$g_j(k) = F_j \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+1} + 1.$$

*Proof.* Once more, we proceed by induction on $j$. First, when $j = 1$, we have

$$\begin{aligned} F_1 \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_2 + 1 &= \left\lfloor \frac{k}{\alpha} \right\rfloor + k \\ &= \left\lfloor \frac{k(1+\alpha)}{\alpha} \right\rfloor \\ &= \lfloor k\alpha \rfloor \\ &= g_1(k). \end{aligned}$$

The statement of the theorem is thus true for all $k \in \mathbb{N}$ when $j = 1$.

Let us now assume that it is true for all $k \in \mathbb{N}$ for some $j \in \mathbb{N}$. We then have

$$g_{j+1}(k) = \left\lfloor \alpha \left( F_j \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+1} + 1 \right) \right\rfloor \tag{1}$$

for each $k \in \mathbb{N}$. In order to complete the proof of the theorem, it would suffice to show that (1) is equal to

$$F_{j+1} \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+2} + 1 \tag{2}$$

for all $k \in \mathbb{N}$. To this end, let us consider $d(j,k)$, the difference between (1) and (2). Using Binet's formula [4, 8, 10], it is a simple matter to show that

$$\alpha F_{j+1} - F_{j+2} = \frac{(-1)^j}{\alpha^{j+1}}. \tag{3}$$

We may then obtain

$$\begin{aligned}
d(j,k) &= \left\lfloor \alpha \left( F_j \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+1} + 1 \right) \right\rfloor - \left( F_{j+1} \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+2} + 1 \right) \\
&= \left\lfloor \left\lfloor \frac{k}{\alpha} \right\rfloor (\alpha F_j - F_{j+1}) + (k-1)(\alpha F_{j+1} - F_{j+2}) + (\alpha - 1) \right\rfloor \\
&= \left\lfloor \frac{(-1)^{j-1}}{\alpha^j} \left( \left\lfloor \frac{k}{\alpha} \right\rfloor - \frac{k-1}{\alpha} \right) + \frac{1}{\alpha} \right\rfloor, \qquad (4)
\end{aligned}$$

where (3) has been utilized in going from the second to the third line above.

Now note that for all $k \in \mathbb{N}$,

$$\left| \left\lfloor \frac{k}{\alpha} \right\rfloor - \frac{k-1}{\alpha} \right| = \left| \frac{1}{\alpha} - \left( \frac{k}{\alpha} - \left\lfloor \frac{k}{\alpha} \right\rfloor \right) \right| = \left| \frac{1}{\alpha} - \left\{ \frac{k}{\alpha} \right\} \right| < \frac{1}{\alpha}, \qquad (5)$$

where $\{x\} = x - \lfloor x \rfloor$, the *fractional part* of $x$. From (5) it follows that

$$-\frac{1}{\alpha^{j+1}} + \frac{1}{\alpha} < \frac{(-1)^{j-1}}{\alpha^j} \left( \left\lfloor \frac{k}{\alpha} \right\rfloor - \frac{k-1}{\alpha} \right) + \frac{1}{\alpha} < \frac{1}{\alpha^{j+1}} + \frac{1}{\alpha},$$

which, since

$$0 < -\frac{1}{\alpha^{j+1}} + \frac{1}{\alpha} \quad \text{and} \quad \frac{1}{\alpha^{j+1}} + \frac{1}{\alpha} < 1,$$

implies via (4) that $d(j,k) = 0$, as required. □

**Corollary 2.** *For all $j, k \in \mathbb{N}$ we have*

$$g_j(k) + g_{j+1}(k) = g_{j+2}(k) + 1.$$

*Proof.* From Theorem 1 we have

$$\begin{aligned}
g_j(k) + g_{j+1}(k) &= F_j \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+1} + 1 + F_{j+1} \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+2} + 1 \\
&= F_{j+2} \left\lfloor \frac{k}{\alpha} \right\rfloor + (k-1)F_{j+3} + 2 \\
&= g_{j+2}(k) + 1.
\end{aligned}$$

□

By way of Corollary 2 we have $g_1(k) + g_2(k) = g_3(k) + 1$ and $g_2(k) + g_3(k) = g_4(k) + 1$, from which it follows that $g_1(k) - 2g_3(k) + g_4(k) = 0$. Since this is valid for any $k \in \mathbb{N}$, it is the case that for any $n \geq 4$, the columns of the top left-hand $n \times n$ sub-matrix of $\mathcal{M}$ are not linearly independent. This in turn implies that the top left-hand $n \times n$ sub-matrix of $\mathcal{M}$ has determinant 0 when $n \geq 4$.

On looking at Table 1 it becomes apparent that, for each pair $(j,k)$ such that $1 \leq j \leq 7$ and $1 \leq k \leq 7$, the expression $g_{j+1}(k) - g_j(k+1)$ is equal to $F_m$ or $-F_m$ for some non-negative integer $m$ (noting that $F_0 = 0$). However, on extending the table it may be seen that this property does not hold in general. For example, $g_2(9) - g_1(10) = 6$.

## 3 A characterization via Zeckendorf representations

Silber [15] obtains various results concerning the lower Wythoff sequence, including a property of the Zeckendorf representations of the terms in this sequence. As will be explained, it follows from this property that the entries in $\mathcal{M}$ may be characterized in a very straightforward manner by way of their Zeckendorf representations.

To this end, let $\mathcal{F} = \{F_2, F_3, F_4, \ldots\}$ be the set of Fibonacci numbers with subscripts at least 2. It is well-known that any positive integer $n$ can be expressed as a sum of distinct elements from $\mathcal{M}$ in at least one way. We term

$$F_{c_1} + F_{c_2} + \cdots + F_{c_k}$$

an *F-representation* for $n$ if $n = F_{c_1} + F_{c_2} + \cdots + F_{c_k}$, where $(c_1, c_2, \ldots, c_k)$ is an increasing sequence such that $c_1 \geq 2$.

Zeckendorf's theorem provides conditions under which each positive integer may be represented in a unique way as a sum of distinct Fibonacci numbers. Specifically, it states that, for any $n \in \mathbb{N}$, there is exactly one way in which $n$ can be expressed as a sum of distinct Fibonacci numbers such that the sum does not include any two consecutive Fibonacci numbers. This gives the *Zeckendorf representation* of $n$. A more formal statement of Zeckendorf's theorem is as follows:

**Theorem 3.** *For any $n \in \mathbb{N}$ there exists a unique increasing sequence of positive integers, $(c_1, c_2, \ldots, c_k)$ say, such that $c_1 \geq 2$, $c_i \geq c_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and*

$$n = \sum_{i=1}^{k} F_{c_i}.$$

Lekkerkerker [11] and Zeckendorf [17] both supply proofs of this result. We note here that Zeckendorf representation is a special case of Ostrowski representation [1, 14], the latter of which gives a general method of integer representation utilizing continued fraction representations of irrational numbers.

Silber [15] shows that the smallest term in the Zeckendorf representation of $n$ is of the form $F_{2m}$ if, and only if, $n = \lfloor q\alpha \rfloor$ for some $q \in \mathbb{N}$. By definition, the first column of $\mathcal{M}$ contains every integer of the form $\lfloor q\alpha \rfloor$. Furthermore, by the construction of $\mathcal{M}$, every entry will be of this form. It follows from this that the integer $n$ appears as an entry in $\mathcal{M}$ if, and only if, the smallest term in its Zeckendorf representation is of the form $F_{2m}$ for some $m \in \mathbb{N}$, thereby providing a way of characterizing the entries of $\mathcal{M}$ via a property of their Zeckendorf representations.

## 4 A result associated with the Fibonacci word

Let $W$ be some infinite word over a binary alphabet, where, without loss of generality, the alphabet is given by $\{0, 1\}$. Then $W$ is a *Sturmian word* if it possesses exactly $n + 1$ factors of length $n$ for each $n \geq 0$. Equivalently, a word $W$ over the alphabet $\{0, 1\}$ is a Sturmian word if, and only if, there exist $a, b \in \mathbb{R}$, with $a$ irrational, such that the $n$th letter of $W$ corresponds to the integer $\lfloor a(n+1) + b \rfloor - \lfloor an + b \rfloor - \lfloor a \rfloor$ for each $n \in \mathbb{N}$ [1, 12].

The special case that results on setting $a = \alpha$ and $b = 0$ is known as the *Fibonacci word* [1, 12, 13]. This mathematical object $S_\infty = 0100101001001\ldots$ is defined [10] to be the infinite string of 0s and 1s constructed recursively as follows. Let $S_1 = 1$ and $S_2 = 0$, and then, for $k \geq 3$, $S_k = S_{k-1} \cdot S_{k-2}$, the *concatenation* of the strings $S_{k-1}$ and $S_{k-2}$. Thus

$$\begin{aligned} S_3 &= S_2 \cdot S_1 = 0 \cdot 1 = 01, \\ S_4 &= S_3 \cdot S_2 = 01 \cdot 0 = 010, \\ S_5 &= S_4 \cdot S_3 = 010 \cdot 01 = 01001, \end{aligned}$$

and so on. The Fibonacci word is the unique infinite string $S_\infty$ such that for all $k \geq 2$, $S_k$ is a prefix of $S_\infty$.

As will be shown in the following theorem, the matrix $\mathcal{M}$ is intimately related to the *Fibonacci word*. In particular, the structure of each column of $\mathcal{M}$ mirrors that of the Fibonacci word.

**Theorem 4.** *Let $l_k$ denote the $k$th digit of the Fibonacci word $S_\infty$, so $l_1 = 0$, $l_2 = 1$, and so on. Then*

$$g_j(k+1) - g_j(k) = \begin{cases} F_{j+1}, & \textit{if } l_k = 1; \\ F_{j+2}, & \textit{if } l_k = 0. \end{cases}$$

*Proof.* From Theorem 1 we have

$$\begin{aligned} g_j(k+1) - g_j(k) &= F_j \left\lfloor \frac{k+1}{\alpha} \right\rfloor + kF_{j+1} + 1 - F_j \left\lfloor \frac{k}{\alpha} \right\rfloor - (k-1)F_{j+1} - 1 \\ &= F_j \left( \lfloor (k+1)(\alpha - 1) \rfloor - \lfloor k(\alpha - 1) \rfloor \right) + F_{j+1} \\ &= F_j \left( \lfloor (k+1)\alpha \rfloor - (k+1) - \lfloor k\alpha \rfloor + k \right) + F_{j+1} \\ &= F_j \left( \lfloor (k+1)\alpha \rfloor - \lfloor k\alpha \rfloor - 1 \right) + F_{j+1}. \end{aligned} \tag{6}$$

If we now use the well-known result

$$\lfloor (k+1)\alpha \rfloor - \lfloor k\alpha \rfloor = \begin{cases} 2, & \text{if } k \in \mathcal{B}(\alpha); \\ 1, & \text{if } k \in \mathcal{B}\left(\alpha^2\right) \end{cases}$$

in conjunction with (6), it follows that

$$g_j(k+1) - g_j(k) = F_j + F_{j+1} = F_{j+2}$$

if, and only if, $k \in \mathcal{B}(\alpha)$, and

$$g_j(k+1) - g_j(k) = F_{j+1}$$

if, and only if, $k \in \mathcal{B}(\alpha^2)$. Finally, Griffiths [6] shows that $l_k = 0$ if, and only if, $k$ is of the form $\lfloor n\alpha \rfloor$ for some $n \in \mathbb{N}$, thereby proving the theorem. ∎

## 5 A result concerning the rows

Finally, by proving the pair of theorems below, we demonstrate some of the intricate structure possessed by the rows of $\mathcal{M}$ whilst simultaneously highlighting an aspect of the association between the Fibonacci and Lucas numbers.

**Theorem 5.** *Let*

$$f(n) = 2\left\lfloor \frac{n}{2} \right\rfloor - 1.$$

*For each $n \geq 2$, the sequences comprising the*

$$(F_n + 1)\text{ th row}, (2F_n + 1)\text{ th row}, (3F_n + 1)\text{ th row}, \ldots, \text{ and } \left(L_{f(n)}F_n + 1\right)\text{ th row}$$

*of $\mathcal{M}$ have $j$th terms given by*

$$F_{j+n} + 1, 2F_{j+n} + 1, 3F_{j+n} + 1, \ldots, \text{ and } L_{f(n)}F_{j+n} + 1,$$

*respectively. Furthermore, this property does not extend to the $\left(\left(L_{f(n)} + 1\right)F_n + 1\right)$ th row.*

*Proof.* From Theorem 1 we know that the $j$th entry in the $(qF_n + 1)$th row is

$$F_j \left\lfloor \frac{qF_n + 1}{\alpha} \right\rfloor + qF_nF_{j+1} + 1.$$

Furthermore, Hansen [7] shows that

$$F_{k-1}F_{m-1} + F_kF_m = F_{k+m-1}$$

for any $n, m \in \mathbb{Z}$. If follows, on setting $k = j + 1$ and $m = n$, that, in order to complete the proof of the theorem, it would suffice to show that the first $L_{f(n)}$ terms of the sequence

$$\left(\left\lfloor \frac{qF_n + 1}{\alpha} \right\rfloor\right)_{q\geq 1}$$

comprise the finite arithmetic progression $F_{n-1}, 2F_{n-1}, 3F_{n-1}, \ldots, L_{f(n)}F_{n-1}$, while the next term is either $\left(L_{f(n)} + 1\right)F_{n-1} + 1$ or $\left(L_{f(n)} + 1\right)F_{n-1} - 1$.

It is straightforward to prove, using Binet's formula, that

$$\frac{F_n}{\alpha} = F_{n-1} + \frac{(-1)^{n-1}}{\alpha^n}.$$

Now consider the sequence

$$\left(\frac{qF_n + 1}{\alpha}\right)_{q\geq 0}.$$

The difference between successive terms is given by

$$\frac{(r + 1)F_n + 1}{\alpha} - \frac{rF_n + 1}{\alpha} = \frac{F_n}{\alpha} = F_{n-1} + \frac{(-1)^{n-1}}{\alpha^n}.$$

Let us suppose first that $n$ is even, in which case this difference is

$$\frac{F_n}{\alpha} = F_{n-1} - \frac{1}{\alpha^n}$$

and we are looking for the largest value of $m \in \mathbb{N}$ such that

$$\frac{1}{\alpha} - \frac{m}{\alpha^n} > 0.$$

This rearranges to $m < \alpha^{n-1}$. Since $L_{n-1} = \alpha^{n-1} + \beta^{n-1}$ [8], where

$$\beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha},$$

we know, remembering that $n-1$ is odd in this case, that $-1 < \beta^{n-1} < 0$. It follows from this that $\alpha^{n-1} - 1 < L_{n-1} < \alpha^{n-1}$, and hence that $m = L_{n-1}$.

On the other hand, if $n$ is odd, then the difference between successive terms is

$$\frac{F_n}{\alpha} = F_{n-1} + \frac{1}{\alpha^n}.$$

In this case we require the largest value of $m \in \mathbb{N}$ subject to

$$\left(1 - \frac{1}{\alpha}\right) - \frac{m}{\alpha^n} > 0.$$

This rearranges to $m < \alpha^{n-2}$. Since $n-2$ is odd, we have $-1 < \beta^{n-2} < 0$. It follows from this that $\alpha^{n-2} - 1 < L_{n-2} < \alpha^{n-2}$, so that $m = L_{n-2}$.

Finally, on noting that the $n$th term of the sequence $(a(n))_{n\geq 2}$ generated by

$$a(n) = \begin{cases} L_{n-1}, & \text{if } n \text{ is even;} \\ L_{n-2}, & \text{of } n \text{ is odd,} \end{cases}$$

is identical to $L_{f(n)}$, where

$$f(n) = 2\left\lfloor \frac{n}{2} \right\rfloor - 1,$$

the proof of the theorem has been completed. $\square$

Although the proof of Theorem 6 very much mirrors that of Theorem 5, we include it here in its entirety for the sake both of completeness and of showing the interplay between the Fibonacci and Lucas numbers.

**Theorem 6.** *Let*

$$g(n) = 2\left\lfloor \frac{n-1}{2} \right\rfloor.$$

*For each $n \geq 3$, the sequences comprising the*

$$(L_n + 1)\text{ th row}, (2L_n + 1)\text{ th row}, (3L_n + 1)\text{ th row}, \ldots, \text{ and } \left(F_{g(n)}L_n + 1\right)\text{ th row}$$

*of $\mathcal{M}$ have $j$th terms given by*

$$L_{j+n} + 1, 2L_{j+n} + 1, 3L_{j+n} + 1, \ldots, \text{ and } F_{g(n)}L_{j+n} + 1,$$

*respectively. The above property does not extend to the $\left(\left(F_{g(n)} + 1\right) L_n + 1\right)$ th row.*

*Proof.* Using Theorem 1 once more, we know that the $j$th entry in the $(qL_n + 1)$th row is

$$F_j \left\lfloor \frac{qL_n + 1}{\alpha} \right\rfloor + qL_nF_{j+1} + 1.$$

This time we just need to show that the first $F_{g(n)}$ terms of the sequence

$$\left(\left\lfloor \frac{qL_n + 1}{\alpha} \right\rfloor\right)_{q \geq 1}$$

comprise the finite arithmetic progression $L_{n-1}, 2L_{n-1}, 3L_{n-1}, \ldots, F_{g(n)}L_{n-1}$, while the next term is either $\left(F_{g(n)} + 1\right) L_{n-1} + 1$ or $\left(F_{g(n)} + 1\right) L_{n-1} - 1$.

From $L_n = \alpha^n + \beta^n$ it follows that

$$\frac{L_n}{\alpha} = L_{n-1} + \frac{(-1)^n\sqrt{5}}{\alpha^n}.$$

The difference between successive terms of the sequence

$$\left(\frac{qL_n + 1}{\alpha}\right)_{q \geq 0}$$

is therefore given by

$$\frac{(r+1)L_n + 1}{\alpha} - \frac{rL_n + 1}{\alpha} = \frac{L_n}{\alpha} = L_{n-1} + \frac{(-1)^n\sqrt{5}}{\alpha^n}.$$

If $n$ is even, this difference is

$$\frac{L_n}{\alpha} = L_{n-1} + \frac{\sqrt{5}}{\alpha^n}.$$

In this case we are looking for the largest value of $m \in \mathbb{N}$ such that

$$\left(1 - \frac{1}{\alpha}\right) - \frac{m\sqrt{5}}{\alpha^n} > 0.$$

This rearranges to

$$m < \frac{\alpha^{n-2}}{\sqrt{5}}.$$

Since $n-2$ is even we have $0 < \beta^{n-2} < 1$. Thus, since

$$F_{n-2} = \frac{1}{\sqrt{5}}\left(\alpha^{n-2} - \beta^{n-2}\right),$$

we know that

$$\frac{\alpha^{n-2}}{\sqrt{5}} - 1 < F_{n-2} < \frac{\alpha^{n-2}}{\sqrt{5}},$$

so that $m = F_{n-2}$.

However, if $n$ is odd, then the difference between successive terms is

$$\frac{L_n}{\alpha} = L_{n-1} - \frac{\sqrt{5}}{\alpha^n}.$$

In this case we require the largest value of $m \in \mathbb{N}$ subject to

$$\frac{1}{\alpha} - \frac{m\sqrt{5}}{\alpha^n} > 0.$$

This rearranges to $m < \alpha^{n-1}\sqrt{5}$. Since $n-1$ is even, we have $0 < \beta^{n-1} < 1$. It follows from this that

$$\frac{\alpha^{n-1}}{\sqrt{5}} - 1 < F_{n-1} < \frac{\alpha^{n-1}}{\sqrt{5}},$$

so that $m = F_{n-1}$.

This completes the proof of the theorem since the $n$th term of the sequence $(b(n))_{n\geq 2}$ generated by

$$b(n) = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-1}, & \text{of } n \text{ is odd,} \end{cases}$$

is identical to $F_{g(n)}$, where

$$g(n) = 2\left\lfloor \frac{n-1}{2} \right\rfloor.$$

□

# 6 Acknowledgments

I would like to thank two anonymous referees and the editor. Their helpful comments have led to significant improvements in all aspects of this article. In particular, I am grateful to them for alerting me to previous results in this area of which I was unaware.

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2010 *Mathematics Subject Classification*: Primary 11B83; Secondary 11B39.
*Keywords:* Beatty sequence, Fibonacci sequence, iteration, Lucas sequence, Zeckendorf representation.

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(Concerned with sequences A000201, A001588, A001611, A001612, A001950, A003622, A134859, A151915, and A164485.)

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Received March 27 2015; revised versions received October 25 2015; November 28 2015; December 11 2015. Published in *Journal of Integer Sequences*, December 16 2015.

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