Journal of Integer Sequences, Vol. 18 (2015),
Article 15.11.5

Partitions with Fixed Number of Sizes

David Christopher
Department of Mathematics
The American College
Madurai
Tamil Nadu
India
davchrame@yahoo.co.in

Abstract

Let $t(n, s)$ and $t(n, k, s)$, respectively, be the number of partitions of n with s different sizes, and the number of partitions of n with exactly k parts and s different sizes. In this article, an asymptotic estimate for $t(n, k, s)$ is presented for the following two cases: (i) $s = k - 1$ and (ii) when k is a prime number with $s = 2$. Further, the enumeration of uniform partitions with exactly 2 sizes is considered and the estimate for its partial sum is derived. Finally, a parity result for $t(n, 2)$ is obtained.

1 Introduction and statement of results

Let n be a positive integer. By a partition of n we mean a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. The λ_i are called the parts of the partition and each element in the set of parts is called a size of λ.

Let $t(n, s)$ and $t(n, k, s)$, respectively, be the number of partitions of n with s different sizes, and the number of partitions of n with exactly k parts and s different sizes.

These two classes of functions are subjects of investigation in the recent past. MacMahon [7] was the first who considered this kind of partitions. In 2005, Deutsch included the number of partitions of n with exactly two odd sizes (see A117955) and the number of partitions of n with exactly two sizes, one odd and one even (see A117956) in the *On-Line Encyclopedia of Integer Sequences* (OEIS). Sloane [9] included the function $t(n, 2)$ in the OEIS (see A002133).

Deutsch presented the following generating function of $t(n,2)$ (see A002133):

$$\sum_{n=1}^{\infty} t(n,2)x^n = \sum_{j=1}^{i-1}\sum_{i=1}^{\infty} \frac{x^{i+j}}{(1-x^i)(1-x^j)}.$$

Andrews [1] gave the following formula for $t(n,2)$ by means of q series manipulations:

$$t(n,2) = \frac{\sum_{k=1}^{n-1} \tau(k)\tau(n-k) + \tau(n) - \sigma(n)}{2},$$

where $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of the positive divisors of n.

In 2011, Tani and Bouroubi [8] found an elegant exact formula for the function $t(n,k,2)$.

The purpose of this article is twofold. First, we give an asymptotic estimate for $t(n,k,s)$ for certain cases.

Theorem 1. *We have*

$$t(n,k+1,k) \sim \frac{C_{k-1}}{(k+1)!^{k-1}} n^{k-1}, \tag{1}$$

where C_{k-1} is given by the recurrence relation:

$$C_{k-1} = (k-2)!kC_{k-2}(k+1)^{k-2} + k\frac{(k+1)!^{k-2}}{(k-1)!} \tag{2}$$

with $C_1 = 3$.

Theorem 2. *For prime $p \geq 3$, we have*

$$t(n,p,2) \sim \frac{\sum_{j=1}^{p-1} \frac{(p-1)!}{j}}{p!} n.$$

The method adopted to arrive at the above estimates is similar to the one used in a recent paper by David Christopher et al. [5]. In addition to the estimates above, we consider the uniform partitions with 2 sizes and we derive the estimate for the partial sums of its enumeration function.

Definition 3. A partition λ is said to be an *uniform partition* (see [3]) if, the number of occurrences of a size in λ is identical with that of all the other sizes in λ. The enumeration function $U(n,2)$ is defined to be the number of uniform partitions of n with exactly 2 sizes.

Theorem 4. *We have*

$$\sum_{m=1}^{n} U(m,2) \sim \frac{\pi^2}{24} n^2.$$

The next result in this article is a parity expression for $t(n,2)$.

Theorem 5. *Let n be a positive integer.*

1. *If $n \equiv 1 \pmod 2$, then we have*

$$t(n,2) \equiv \begin{cases} \frac{\tau(n)-1}{2} \pmod 2, & \text{if } n \text{ is a square;} \\ \frac{\tau(n)}{2} \pmod 2, & \text{if } n \text{ is not a square.} \end{cases}$$

2. *If $n \equiv 0 \pmod 2$ and β is the highest power of 2 that divides n, then we have*

$$t(n,2) \equiv \begin{cases} \frac{(\beta-1)\tau(\frac{n}{2^\beta})-1}{2} \pmod 2, & \text{if } n \text{ is a square;} \\ \frac{(\beta-1)\tau(\frac{n}{2^\beta})}{2} \pmod 2, & \text{if } n \text{ is not a square.} \end{cases}$$

In the final section, we provide two different proofs for this theorem.

2 Proof of the Estimates

2.1 Prerequisites

To begin with, we need the following recurrence relation satisfied by $t(n,k,s)$, which has $k-s+2$ preceding terms. Note that, the following recurrence relation is identical to the one given by Tani and Bouroubi [8], but it is presented here in a linear form. The method of proof below is bijective whereas the former proof is based upon a simple variation in the system of equations that admits partitions with fixed number of sizes as its solution.

Lemma 6 (Tani and Bouroubi [8])**.** *We have*

$$t(n,k,s) = t(n-k,k,s) + \sum_{r=1}^{k-s+1} t(n-k,k-r,s-1).$$

Proof. Let $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n with exactly k parts and s different sizes.

Case (i): Assume that $\lambda_k > 1$. Consider the mapping

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) \to (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1);$$

this mapping establishes an one-to-one correspondence between the following sets:

- The set of all partitions of n with exactly k parts, s different sizes and least part $a_k > 1$.
- The set of all partitions of $n-k$ with exactly k parts and s different sizes.

We observe that the cardinality of the latter set is $t(n-k,k,s)$.

Case (ii): Assume that $\lambda_k = \lambda_{k-1} = \cdots = \lambda_{k-(r-1)} = 1$ and $\lambda_{k-r} > 1$ for some positive integer $r \geq 1$.

We see that the mapping

$$(\lambda_1, \lambda_2, \ldots, \lambda_k) \to (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{k-r} - 1)$$

establishes an one-to-one correspondence between the following sets:

- The set of all partitions of n with exactly k parts, s different sizes and $\lambda_k = \lambda_{k-1} = \cdots = \lambda_{k-(r-1)} = 1$ and $\lambda_{k-r} > 1$.
- The set of all partitions of $n-k$ with exactly $k-r$ parts and $s-1$ different sizes.

Notice that the cardinality of the latter set is $t(n-k, k-r, s-1)$. Since r varies from 1 to $k-s+1$, the result follows. $\square$

Definition 7. Let n and k be two positive integers with $n \geq \frac{k(k+1)}{2}$. The enumeration function $q(n,k)$ is defined to be the number of partitions of n with k distinct parts.

The following result is required in the proof of Theorem 1.

Lemma 8 (David Christopher and Davamani Christober [4])**.** *Let $k \geq 2$ be a positive integer. For each $r \in \{0, 1, \ldots, k!-1\}$, the function $q(k!l + r, k)$ is a polynomial in l of degree $k-1$ with the leading coefficient $\frac{k!^{k-2}}{(k-1)!}$.*

2.2 Proof of Theorem 1

Wielding Lemma 6 and Lemma 8 we will prove the following statements:

1. The function $t((k+1)!l + r, k+1, k)$ is a polynomial in l of degree $k-1$ for each $r \in \{0, 1, \ldots, (k+1)! - 1\}$.
2. The leading coefficient of $t((k+1)!l + r, k+1, k)$, denoted C_{k-1}, satisfies the recurrence relation (2).

From these statements one can get the following limit:

$$\lim_{l \to \infty} \frac{t((k+1)!l + r, k+1, k)}{((k+1)!l + r)^{k-1}} = \frac{C_{k-1}}{(k+1)!^{k-1}}$$

for each $r \in \{0, 1, \ldots, (k+1)! - 1\}$; which is equivalent to the estimate (1).

Now we prove the statements above simultaneously by induction over k. We observe that: $t(n,k,k)=q(n,k)$. Put $k=3$ and $s=2$ in Lemma 6 to get

$$\begin{aligned} t(n,3,2)-t(n-3,3,2) &= t(n-3,2,1)+q(n-3,1) \\ &= \begin{cases} 2, & \text{if } n\equiv 3 \pmod 2; \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Put $n=3!l+r$ with $0\le r\le 3!-1$, where l is a non-negative integer. Then we have

$$t(3!l+r,3,2)-t(3!l+r-3,3,2)=\begin{cases} 2, & \text{if } r\equiv 1 \pmod 2; \\ 1, & \text{if } r\equiv 0 \pmod 2, \end{cases}$$

and

$$t(3!l+r-3,3,2)-t(3!l+r-6,3,2)=\begin{cases} 2, & \text{if } r\equiv 0 \pmod 2; \\ 1, & \text{if } r\equiv 1 \pmod 2. \end{cases}$$

The sum of the above two expressions can be put as $t(3!l+r,3,2)-t(3!(l-1)+r,3,2)=3$ for each $r\in\{0,1,\dots,3!-1\}$. Then substituting $2,3,\dots,l$ in place of l gives $t(3!l+r,3,2)=3(l-1)+t(r,3,2)$ for each $r\in\{0,1,\dots,3!-1\}$. Thus, the statements above hold for $k=2$.

Assume that the assertion is true up to some $k\ge 2$. Set $k+1$ as the number of parts, k as the number of sizes and $n=(k+1)!l+r$, where l is a non-negative integer and $r\in\{0,1,\dots,(k+1)!-1\}$. Then applying Lemma 6 $k!$ times, we get

$$\begin{aligned} t((k+1)!l+r,k+1,k) &- t((k+1)!(l-1)+r,k+1,k) \\ &= \sum_{i=1}^{k!} t((k+1)!l+r-i(k+1),k,k-1) \\ &+ \sum_{i=1}^{k!} q((k+1)!l+r-i(k+1),k-1). \end{aligned}$$

In view of the division algorithm one can obtain unique pair of integers say (r_i,q_i) for each $i\in\{1,2,\dots,k!\}$ satisfying the equality $k!q_i+r_i=r-i(k+1)$ with $0\le r_i\le k!-1$. Similarly one can get unique pair of integers say (r_i',q_i') for each $i\in\{1,2,\dots,k!\}$ satisfying the equality $(k-1)!q_i'+r_i'=r-i(k+1)$ with $0\le r_i'\le(k-1)!-1$. Consequently, the right side of the above equality can be written as

$$\sum_{i=1}^{k!} t(k!((k+1)l+q_i)+r_i,k,k-1)+\sum_{i=`1}^{k!} q\left((k-1)!\left(\frac{(k+1)!}{(k-1)!}l+q_i'\right)+r_i',k-1\right). \quad (3)$$

Put $l_i=(k+1)l+q_i$. By induction assumption, the function $t(k!l_i+r_i,k,k-1)$ is a polynomial in l_i of degree $k-2$ with the leading coefficient C_{k-2} as given in the recurrence relation (2) for each $r_i\in\{0,1,\dots,k!-1\}$ and $i\in\{1,2,\dots,k!\}$. Just assuming this polynomial representation, we see that the first summation in (3) is a sum of $k!$ polynomials in l, each of which is of degree $k-2$ with the leading coefficient $C_{k-2}(k+1)^{k-2}$. (At this juncture, we note that $(k+1)l+q_i$ may be negative for some initial values of l. In such instances, we assume the extrapolation of the function $t(k!l_i+r_i,k,k-1)$ as a polynomial in l_i and not as an enumeration function.) Thus, the first summation in (3) is itself a polynomial in l of degree $k-2$ with the leading coefficient $k!C_{k-2}(k+1)^{k-2}$. Put $l_i' = \frac{(k+1)!}{(k-1)!}l + q_i'$. From Lemma 8 it follows that $q((k-1)!l_i'+r_i',k-1)$ is a polynomial in l_i' of degree $k-2$ for each $r_i' \in \{0,1,\ldots,(k-1)!-1\}$ and $i \in \{1,2,\ldots,k!\}$. Consequently, the second summation in (3) is itself a polynomial in l of degree $k-2$ with the leading coefficient $k!\frac{(k-1)!^{k-3}}{(k-2)!}\frac{(k+1)!^{k-2}}{(k-1)!^{k-2}}$. Thus, the summation term in (3) is a polynomial in l of degree $k-2$ with the leading coefficient $k!C_{k-2}(k+1)^{k-2} + k!\frac{(k-1)!^{k-3}}{(k-2)!}\frac{(k+1)!^{k-2}}{(k-1)!^{k-2}}$. Accordingly, $t((k+1)!l+r,k+1,k) - t((k+1)!(l-1)+r,k+1,k)$ is a polynomial in l of degree $k-2$. Put $f(l) = t((k+1)!l + r,k+1,k) - t((k+1)!(l-1)+r,k+1,k)$. Substituting the values $1,2,\ldots,l$ in place of l and adding we get $t((k+1)!l+r,k+1,k) = \sum_{i=1}^{l} f(i) - t(r,k+1,k)$. This equality implies that $t((k+1)!l+r,k+1,k)$ is a polynomial in l of degree $k-1$.

If one denotes the leading coefficient of $t((k+1)!l+r,k+1,k)$ by C_{k-1}, then from the conclusions above we get

$$(k-1)C_{k-1} = k!C_{k-2}(k+1)^{k-2} + k!\frac{(k-1)!^{k-3}}{(k-2)!}\frac{(k+1)!^{k-2}}{(k-1)!^{k-2}}.$$

This simplifies to the recurrence relation (2). The proof is now completed.

Corollary 9. *From the recurrence relation of C_n given in the Theorem 1 we get*

$$C_1 = 3, C_2 = 72, C_3 = 24000, C_4 = 233280000, \ldots$$

From these values of C_i, we have the following estimates:

$$\begin{aligned} t(n,3,2) &\sim \frac{1}{2}n; \\ t(n,4,3) &\sim \frac{1}{8}n^2; \\ t(n,5,4) &\sim \frac{1}{72}n^3; \\ t(n,6,5) &\sim \frac{1}{1152}n^4. \end{aligned}$$

2.3 Proof of Theorem 2

Let p be a prime number. Then from Lemma 6 it follows that

$$t(n,p,2) - t(n-p,p,2) = \sum_{j=1}^{p-1} t(n-p,j,1).$$

Put $n = p!l + r$ (where l is a positive integer and $0 \leq r \leq p! - 1$) and apply Lemma 6 $(p-1)!$ times in the above relation to get

$$t(p!l + r, p, 2) - t(p!(l-1) + r, p, 2) = \sum_{j=1}^{p-1} \sum_{i=1}^{(p-1)!} t(p!l + r - pi, j, 1). \tag{4}$$

Notice that

$$t(p!l + r - pi, j, 1) = \begin{cases} 1, & \text{if } pi \equiv r \pmod{j}; \\ 0, & \text{otherwise,} \end{cases}$$

and the congruence equation

$$px \equiv r \pmod{j}$$

has unique solution modulo j for each $j \in \{1, 2, \ldots, p-1\}$. Consequently, we get the right side of (4) as $\sum_{j=1}^{p-1} \frac{(p-1)!}{j}$. Then replacing l by $1, 2, \ldots, l$ in (4) and adding gives

$$t(p!l + r, p, 2) = t(r, p, 2) + \left(\sum_{j=1}^{p-1} \frac{(p-1)!}{j} \right) l$$

for each $r \in \{0, 1, \ldots, p! - 1\}$. This implies that

$$\lim_{l \to \infty} \frac{t(p!l + r, p, 2)}{p!l + r} = \frac{\sum_{j=1}^{p-1} \frac{(p-1)!}{j}}{p!}$$

for each $r \in \{0, 1, \ldots, p! - 1\}$. Equivalently,

$$t(n, p, 2) \sim \frac{\sum_{j=1}^{p-1} \frac{(p-1)!}{j}}{p!} n;$$

this is the contention of the Theorem 2.

Remark 10. At this juncture, we remark that, the method adopted so far to derive asymptotic estimates may turn futile for some other cases. For instance, from the exact expression for the function $t(n, 4, 2)$ given in [8] one can get the following values:

1. $t(12l + 4, 4, 2) = 7l$ for every non-negative integer l.
2. $t(12l + 5, 4, 2) = 4l + 1$ for every non-negative integer l.

This leads to the conclusion that

$$t(n, 4, 2) \nsim Kn$$

for every real number K.

2.4 Proof of Theorem 4

Recall that the function $U(n,2)$ is the number of uniform partitions of n with exactly 2 sizes. We have

$$\begin{aligned} U(n,2) &= \sum_{d|n} q(d,2) \\ &= \sum_{d|n} \left\lfloor \frac{d-1}{2} \right\rfloor \\ &\leq \sum_{d|n} \frac{d-1}{2} \\ &= \frac{1}{2}\left(\sigma(n) - \tau(n)\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} U(n,2) &= \sum_{d|n} q(d,2) \\ &= \sum_{d|n} \left\lfloor \frac{d-1}{2} \right\rfloor \\ &\geq \sum_{d|n} \frac{d-2}{2} \\ &= \frac{1}{2}\left(\sigma(n) - 2\tau(n)\right). \end{aligned}$$

Consequently, we have

$$\frac{1}{2}\left(\sum_{m=1}^{n} \sigma(m) - 2\sum_{m=1}^{n} \tau(m)\right) \leq \sum_{m=1}^{n} U(m,2) \leq \frac{1}{2}\left(\sum_{m=1}^{n} \sigma(m) - \sum_{m=1}^{n} \tau(m)\right). \tag{5}$$

Since

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \sigma(m)}{\frac{\pi^2}{12} n^2} = 1 \tag{6}$$

and

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \tau(m)}{n \log n} = 1, \tag{7}$$

dividing the inequality (5) by $\frac{\pi^2}{12} n^2$ and letting $n \to \infty$ gives

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} U(m,2)}{\frac{\pi^2}{12} n^2} = \frac{1}{2},$$

which is equivalent to the estimate given in Theorem 4. (The validity of the limits (6) and (7) can be seen in [2, p. 57–60].)

3 Two proofs of Theorem 5

In this section, we adopt the following notation: if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n and $\{a_1, a_2, \ldots\}$ is the set of all sizes of λ with $a_1 > a_2 > \cdots$, then we denote $\lambda = \left(a_1^{f_1} a_2^{f_2} \cdots\right)$ when the size a_i occurs exactly f_i number of times in λ. Let P_2^n denote the set of all partitions of n with exactly two sizes.

3.1 A proof using Jacobi's formula

Proof. Let $\lambda = \left(a_1^{f_1} a_2^{f_2}\right) \in P_2^n$. Then we have $a_1 f_1 + a_2 f_2 = n$ with $a_1 > a_2$.

Case (i): $f_1 = f_2$. In this case we have $n = f_1 a_1 + f_2 a_2 = f(a_1 + a_2)$, where $f_1 = f_2 = f$ (say). Clearly, $f|n$ and the enumeration of the ordered pairs (a_1, a_2) satisfying the equality is $q\left(\frac{n}{f}, 2\right)$. Thus, the number of partitions belonging to this case is $\sum_{f|n} q\left(\frac{n}{f}, 2\right)$.

Case (ii): $f_1 = a_1$ and $f_2 = a_2$. We see that the number of partitions belonging to this case equals the number of ways that n can be written as the sum of two distinct nonzero squares; we denote this enumeration by $R_2(n)$.

Case (iii): Assume that $f_1 \neq f_2$ with either $a_1 \neq f_1$ or $a_2 \neq f_2$, and either $f_1 \neq a_2$ or $f_2 \neq a_1$. In this case, consider the mapping

$$T((a_1^{f_1} a_2^{f_2})) = \begin{cases} (f_1^{a_1} f_2^{a_2}), & \text{if } f_2 < f_1; \\ (f_2^{a_2} f_1^{a_1}), & \text{if } f_1 < f_2. \end{cases}$$

Clearly, T is an involutory mapping with no fixed point. Moreover, we see that if λ belongs to this case, then $T(\lambda)$ also belongs to this case and vice versa. Thus, the set of partitions belonging to this case can be got as the disjoint union of set of pairs of elements of the form $\{\lambda, T(\lambda)\}$, where λ runs over the partitions belonging to this case. Consequently, the number of partitions belonging to this case is even; thus, this enumeration contributes zero to the modulo 2.

Case (iv): $f_1 \neq f_2$ with $a_1 = f_2$ and $a_2 = f_1$. We note that this is a special case when $n \equiv 0 \pmod 2$. In this case, we have $n = 2a_1 a_2$. Consequently, the number of partitions belonging to this case is $\frac{\tau(\frac{n}{2}) - \delta(\frac{n}{2})}{2}$, where $\delta(\cdot)$ is the characteristic function of squares defined as follows:

$$\delta(n) = \begin{cases} 1, & \text{if } n \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}$$

Since the above cases are exhaustive, we have the following parity identity for $t(n, 2)$:

$$t(n,2) \equiv \begin{cases} \left(\sum_{d|n} q\left(\frac{n}{d}, 2\right) + R_2(n) + \frac{\tau(\frac{n}{2}) - \delta(\frac{n}{2})}{2}\right) \pmod 2, & \text{if } n \equiv 0 \pmod 2; \\ \left(\sum_{d|n} q\left(\frac{n}{d}, 2\right) + R_2(n)\right) \pmod 2, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{8}$$

The remaining part of the proof relies on a formula due to Jacobi. Jacobi [6] found a formula for the number of representations of a given positive integer n as the sum of two squares. This representation includes negative numbers and zero in the squaring process and also the order of the summands is taken into account. Jacobi's formula [6] is

$$r_2(n) = 4\left(d_1(n) - d_3(n)\right),$$

where $r_2(n)$ denotes the number of ways that a given positive integer n can be written as the sum of two squares and $d_1(n)$(resp.$d_3(n)$) denotes the number of divisors of n that are 1(resp. 3) modulo 4. Using this formula, we can write

$$\begin{aligned} R_2(n) &= \begin{cases} \frac{r_2(n)-4}{8}, & \text{if } \delta(n) = 1 \text{ or } \delta(\frac{n}{2}) = 1; \\ \frac{r_2(n)}{8}, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{d_1(n)-d_3(n)-1}{2}, & \text{if } \delta(n) = 1 \text{ or } \delta(\frac{n}{2}) = 1; \\ \frac{d_1(n)-d_3(n)}{2}, & \text{otherwise.} \end{cases} \end{aligned}$$

We now simplify the parity formula of $t(n,2)$ mentioned in (8) by using the above expression for $R_2(n)$.

To proceed further, we need a few calculations. If $n \equiv 1 \pmod 2$, then for every divisor d of n, we have

$$\lfloor \frac{d-1}{2} \rfloor \equiv \begin{cases} 0 \pmod 2, & \text{if } d \equiv 1 \pmod 4; \\ 1 \pmod 2, & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and hence it follows that

$$\sum_{d|n} q(d,2) = \sum_{d|n} \left\lfloor \frac{d-1}{2} \right\rfloor \equiv d_3(n) \pmod 2.$$

If $n \equiv 0 \pmod 2$, then we have

$$\sum_{d|n} q(d,2) = \sum_{d|n,\ d \equiv 1 \pmod 2} q(d,2) + \sum_{d|n,\ d \equiv 0 \pmod 2} q(d,2).$$

By the previous calculation, we see that

$$\sum_{d|n,\ d \equiv 1 \pmod 2} q(d,2) \equiv d_3(n) \pmod 2.$$

Let β be the highest power of 2 that divides n. Since $\lfloor \frac{2n-1}{2} \rfloor = n-1$, we have $\lfloor \frac{2^k n-1}{2} \rfloor = 2^{k-1}n - 1 \equiv 1 \pmod 2$ for every $k \geq 2$. And consequently we get

$$\sum_{d|n,\ d \equiv 0 \pmod 2} q(d,2) \equiv (\beta - 1)\tau\left(\frac{n}{2^\beta}\right) \pmod 2.$$

Accordingly, when $n \equiv 0 \pmod 2$ and β is the highest power of 2 that divides n, we have

$$\sum_{d|n} q(d,2) \equiv d_3(n) + (\beta-1)\tau\left(\frac{n}{2^\beta}\right) \pmod 2.$$

When these calculations are incorporated in the parity identity (8) we get

$$t(n,2) \equiv \begin{cases} d_3(n) + \frac{d_1(n)-d_3(n)-1}{2} \pmod 2, & \text{if } n \equiv 1 \pmod 2 \\ & \text{and } \delta(n)=1; \\ d_3(n) + \frac{d_1(n)-d_3(n)}{2} \pmod 2, & \text{if } n \equiv 1 \pmod 2 \\ & \text{and } \delta(n)=0; \\ d_3(n) + \frac{d_1(n)-d_3(n)-1}{2} + (\beta-1)\tau\left(\frac{n}{2^\beta}\right) + \frac{\tau(\frac{n}{2})-\delta(\frac{n}{2})}{2} \pmod 2, & \text{if } n \equiv 0 \pmod 2 \\ & \text{and } \delta(n)=1 \text{ or} \\ & \delta(\frac{n}{2})=1; \\ d_3(n) + \frac{d_1(n)-d_3(n)}{2} + (\beta-1)\tau\left(\frac{n}{2^\beta}\right) + \frac{\tau(\frac{n}{2})-\delta(\frac{n}{2})}{2} \pmod 2, & \text{if } n \equiv 0 \pmod 2 \\ & \text{and } \delta(n)=0. \end{cases}$$

Let $n \equiv 0 \pmod 2$. If $\delta(\frac{n}{2^\beta}) = 1$ then either $\delta(n) = 1$ or $\delta(\frac{n}{2}) = 1$. Conversely, if $\delta(\frac{n}{2^\beta}) = 0$, then $\delta(n) = \delta(\frac{n}{2}) = 0$. This simple observation transforms the above congruence as follows:

$$t(n,2) \equiv \begin{cases} \frac{\tau(n)-1}{2} \pmod 2, & \text{if } n \equiv 1 \pmod 2 \text{ and} \\ & \delta(n)=1; \\ \frac{\tau(n)}{2} \pmod 2, & \text{if } n \equiv 1 \pmod 2 \text{ and} \\ & \delta(n)=0; \\ (\beta-1)\tau(\frac{n}{2^\beta}) + \frac{\tau(\frac{n}{2^\beta})-1}{2} + \frac{\tau(\frac{n}{2})-\delta(\frac{n}{2})}{2} \pmod 2 & \text{if } n \equiv 0 \pmod 2 \text{ and} \\ & \delta(\frac{n}{2^\beta})=1; \\ (\beta-1)\tau(\frac{n}{2^\beta}) + \frac{\tau(\frac{n}{2^\beta})}{2} + \frac{\tau(\frac{n}{2})}{2} \pmod 2 & \text{if } n \equiv 0 \pmod 2 \text{ and} \\ & \delta(\frac{n}{2^\beta})=0. \end{cases}$$

As we see, the proof for the case $n \equiv 1 \pmod 2$ is finished. To finish the proof of the counterpart, we verify that the last two cases in the above congruence is equivalent to the congruence mentioned in the statement of the result. To that end, the following easily verifiable fact is required: using the multiplicative property of τ function, one can get that $\tau(\frac{n}{2^\beta}) + \tau(\frac{n}{2}) = \tau(\frac{n}{2^\beta}) + \tau(\frac{n}{2^\beta})\beta = \tau(\frac{n}{2^\beta})(\beta+1)$ when $n \equiv 0 \pmod 2$.

1. Let $n \equiv 0 \pmod 2$. If $\delta(n) = 0$ and $\delta(\frac{n}{2^\beta}) = 1$, then we have $\tau(\frac{n}{2^\beta}) \equiv 1 \pmod 2$ and

$\beta - 1 \equiv 0 \pmod 2$. This gives

$$\begin{aligned} t(n,2) &\equiv \frac{\tau(\frac{n}{2^\beta}) - 1}{2} + \frac{\tau(\frac{n}{2}) - 1}{2} \\ &\equiv \frac{\tau(\frac{n}{2^\beta})(\beta+1)}{2} - 1 \\ &\equiv \frac{\beta+1}{2} - 1 \\ &\equiv \frac{\beta-1}{2} \\ &\equiv \frac{\tau(\frac{n}{2^\beta})(\beta-1)}{2} \pmod 2. \end{aligned}$$

2. Let $n \equiv 0 \pmod 2$. If $\delta(n) = \delta(\frac{n}{2^\beta}) = 0$, then we have

$$t(n,2) \equiv \frac{\tau(\frac{n}{2^\beta})}{2} + \frac{\tau(\frac{n}{2})}{2} \equiv \frac{(\beta+1)\tau(\frac{n}{2^\beta})}{2} \equiv \frac{(\beta-1)\tau(\frac{n}{2^\beta})}{2} \pmod 2.$$

   Equivalence of the last two congruences follows from the fact that the parity of $\beta+1$ and $\beta-1$ are identical.

3. Let $n \equiv 0 \pmod 2$. If $\delta(n) = 1$, then

$$t(n,2) \equiv (\beta-1)\tau(\frac{n}{2^\beta}) + \frac{\tau(\frac{n}{2^\beta}) - 1}{2} + \frac{\tau(\frac{n}{2})}{2} \equiv 1 + \frac{\tau(\frac{n}{2^\beta})(\beta+1) - 1}{2} \pmod 2. \quad (9)$$

   Since $\tau(\frac{n}{2^\beta}) \equiv 1 \pmod 2$, we have $\frac{\tau(\frac{n}{2^\beta})(\beta+1)-1}{2} + 1 \equiv \frac{\beta+2}{2} \pmod 2$; on the other hand, we have $\frac{(\beta-1)\tau(\frac{n}{2^\beta})-1}{2} \equiv \frac{\beta-2}{2} \equiv \frac{\beta-2}{2} + 2 \equiv \frac{\beta+2}{2} \pmod 2$. Thus, the congruence (9) is equivalent to the congruence: $t(n,2) \equiv \frac{(\beta-1)\tau(\frac{n}{2^\beta})-1}{2} \pmod 2$ as expected.

The proof is now completed. □

3.2 A proof using conjugate mapping

Definition 11. Let $\lambda = (a_1^{f_1} a_2^{f_2}) \in P_2^n$. The conjugate of λ, denoted $C(\lambda)$, is the partition $((f_1+f_2)^{a_2} f_1^{a_1-a_2})$.

Proof. Define the conjugate mapping $C : P_2^n \to \mathcal{P}_2^n$. Since C is an involutory self mapping, arguments similar to that of in the previous proof gives the congruence $t(n,2) \equiv SC_2(n) \pmod 2$, where $SC_2(n)$ denotes the number of self conjugate partitions in P_2^n.

Now we derive a formula for $SC_2(n)$ in terms of the τ function. Define

$$D_n = \left\{ d \in \mathbb{N} : d | n, d < \sqrt{n} \text{ and } \frac{n}{d} - d \equiv 0 \pmod 2 \right\}.$$

We contend that $SC_2(n) = |D_n|$. Let $\lambda = (a_1^{f_1} a_2^{f_2}) \in P_2^n$ be a self conjugate partition. Then from the definition of self conjugate partition, we have $a_1 = f_1 + f_2$ and $a_2 = f_1$. Consequently, we can write $n = f_1^2 + 2f_1f_2$. This leads to the equality $\frac{n}{f_1} - f_1 = 2f_2$. Thus, f_1 is a divisor of n which is less than $\sqrt{n}$ and $\frac{n}{f_1} - f_1 \equiv 0 \pmod 2$. On the other hand, let $d \in D_n$. Since $d < \sqrt{n}$, we can write $\frac{n}{d} - d = 2k$ for some $k \geq 1$. From this we can construct the partition $((d+k)^d d^k)$ which is a self conjugate partition of n with two sizes. Accordingly, corresponding to each self conjugate partition in P_2^n, we can find an element in D_n; and to each element in D_n we can find a self conjugate partition in P_2^n. Thus, the claim follows. Now we find an expression for $|D_n|$.

Case (i): Assume that $n \equiv 1 \pmod 2$. Then apparently $\frac{n}{d} - d \equiv 0 \pmod 2$ for every $d|n$. Thus, in this case, D_n is the set of divisors of n that are less than $\sqrt{n}$. Let D'_n be the set of divisors of n that are greater than $\sqrt{n}$. Define the mapping $\phi : D_n \to D'_n$ by $\phi(d) = \frac{n}{d}$ for every $d \in D_n$. Clearly, ϕ is a non-fixed bijection and $D_n \cap D'_n = \emptyset$. Thus,

$$|D_n| = \begin{cases} \frac{\tau(n)}{2}, & \text{if } n \text{ is a non-square;} \\ \frac{\tau(n)-1}{2}, & \text{if } n \text{ is a square.} \end{cases}$$

Case (ii): Assume that $n \equiv 0 \pmod 2$. Let β be the highest power of 2 that divides n. For every odd divisor d of n, we have $\frac{n}{d} - d \equiv 1 \pmod 2$ and for every even divisor d of n such that $2^\beta | d$, we again have $\frac{n}{d} - d \equiv 1 \pmod 2$. So, the divisors of these types does not contribute to the set D_n. Consider every even divisor d of n such that $d < \sqrt{n}$ and 2^β does not divides d; then, for such d, we have $\frac{n}{d} - d \equiv 0 \pmod 2$. Thus, D_n is the set of all even divisors of n that are less than $\sqrt{n}$ and not a multiple of 2^β. Let D''_n be the set of all even divisors of n that are greater than $\sqrt{n}$ and not a multiple of 2^β. Now we define $\psi : D_n \to D''_n$ by $\psi(d) = \frac{n}{d}$ for every $d \in D_n$. Clearly, ψ is a bijective mapping and $D_n \cap D''_n = \emptyset$. Thus,

$$|D_n| = \begin{cases} \frac{(\beta-1)\tau(\frac{n}{2^\beta})-1}{2}, & \text{if } n \text{ is a square ;} \\ \frac{(\beta-1)\tau(\frac{n}{2^\beta})}{2}, & \text{if } n \text{ is not a square.} \end{cases}$$

The second proof is now completed. □

References

[1] G. E. Andrews, Stacked lattice boxes, *Ann. Comb.* **3** (1999), 115–130.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[3] A. David Christopher and M. Davamani Christober, Relatively prime uniform partitions, *Gen. Math. Notes.* **13** (2012), 1–12.

[4] A. David Christopher and M. Davamani Christober, Estimates of five restricted partition functions that are quasi polynomials, *Bull. Math. Sci.* **5** (2015), 1–11.

[5] A. David Christopher and M. Davamani Christober, On asymptotic formula of the partition function $p_A(n)$, *INTEGERS* **15** (2015), Article #A2.

[6] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Königsberg, 1829. Also in *Gesammelte Werke,* Band I, pp. 49–239.

[7] P. A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, *Proc. London Math. Soc.*, **19** (1919), 75–113. Also in *Collected Papers*, Vol. 2, pp. 303–341.

[8] Nesrine Benyahia Tani and Sadek Bouroubi, Enumeration of the partitions of an integer into parts of a specified number of different sizes and especially two sizes, *J. Integer Seq.* **14** (2011), Article 11.3.6.

[9] N. J. A. Sloane, *Online Encyclopedia of Integer Sequences*, `http://oeis.org`.

2010 *Mathematics Subject Classification*: Primary 05A17; Secondary 11P81.
Keywords: partition, size of a partition, parity, asymptotic estimate.

(Concerned with sequences A000005, A000203, A002133, A004526, A010052, A117955, and A117956.)

Received March 27 2015; revised versions received August 29 2015; October 5 2015; December 7 2015. Published in *Journal of Integer Sequences*, December 9 2015.

Return to Journal of Integer Sequences home page.