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Equivalence Classes of Motzkin Paths Modulo a Pattern of Length at Most Two

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Abstract

For any pattern α of length at most two, we enumerate equivalence classes of Motzkin paths where two paths of the same length are equivalent whenever they coincide on all occurrences of the pattern α.

1 Introduction and notation

A *Motzkin path* of length n, $n \geq 0$, is a lattice path starting at point $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$, down steps $D = (1,-1)$ and flat steps $F = (1,0)$. See Figure 1 for an illustration of a Motzkin path of length 18. Let $\mathcal{M}$ be the set of all Motzkin paths. A *pattern* of length one (resp., two) in $M \in \mathcal{M}$ consists of one step (resp., two consecutive steps). We will say that an *occurrence* of a pattern is at position i, $i \geq 1$, in M whenever the first step of this occurrence appears at the i-th step of the path. The *height* of this occurrence is the minimal ordinate reached by its points. For instance, the path $M = UUFDDFFUDUFUDDFFUD$ (see Figure 1) contains three occurrences of the pattern FU at positions 7, 11 and 16, respectively at heights 0, 1 and 0. A Motzkin path will be called *Dyck path* when it does not contain any flat step F. Let $\mathcal{D}$ be the set of all Dyck paths.

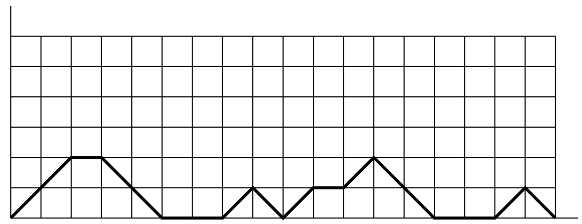

Figure 1: A Motzkin path $M = UUFDDFFUDUFUDDFFUD$.

Many statistics on Motzkin and Dyck paths have been studied. Almost always, it is shown how we can enumerate these paths according to several parameters, such as length, number of occurrences of a pattern, number of returns to the x-axis (see, for instance, [4, 7, 8, 9, 11, 12, 13, 15, 16, 19] for Dyck paths and [2, 3, 5, 6, 10, 14, 17] for Motzkin paths). Recently in [1], the authors investigate equivalence relations on the set $\mathcal{D}$ of Dyck paths where two Dyck paths of the same length are equivalent whenever they coincide on all occurrences of a given pattern.

In this paper, we study these equivalence relations on the set $\mathcal{M}$ of Motzkin paths for any pattern α of length at most two, i.e.,

two Motzkin paths of the same length are α-equivalent whenever all occurrences of the pattern α appear at the same positions in the two paths.

For instance, $UDUFDF\mathbf{FU}UF\mathbf{FU}DDD\mathbf{FU}D$ is FU-equivalent to the path in Figure 1 since all occurrences of the pattern FU (in boldface) appear at the same positions in the two paths.

For any pattern α of length at most two, we provide generating functions for the number of α-equivalence classes in $\mathcal{M}$, with respect to the length (see Table 1). The general method used consists of exhibiting one-to-one correspondences between some subsets of Motzkin paths and the different sets of equivalence classes by using combinatorial reasonings, and then, evaluating algebraically the generating functions for these subsets.

2 Equivalence classes modulo a pattern of length one

Throughout this section, we study the α-equivalence in $\mathcal{M}$ for $\alpha \in \{U, D, F\}$.

2.1 Modulo $\alpha = F$

Let $\mathcal{A}$ be the set of Motzkin paths without any points of ordinate greater than one. For instance, we have $UFDUDF \in \mathcal{A}$ and $UUDDUD \notin \mathcal{A}$.

Lemma 1. *There is a bijection between $\mathcal{A}$ and the set of F-equivalence classes of $\mathcal{M}$.*

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 10$
$\{U\}, \{D\}$	$\frac{2}{1-2x+\sqrt{1-4x^2}}$	A001405	$1, 2, 3, 6, 10, 20, 35, 70, 126, 252$
$\{F\}$	2^n	A000079	$1, 2, 4, 8, 16, 32, 64, 128, 256, 512$
$\{UU\}, \{DD\}$	$\frac{(x-1)^2-\sqrt{(1-3x^2)(x^2+1)}}{2x(x^3-(1-x)^2)}$	A191385	$1, 1, 1, 2, 3, 5, 7, 12, 18, 31$
$\{DU\}$	Shift of Fibonacci	A132916	$1, 1, 1, 2, 3, 5, 8, 13, 21, 34$
$\{UD\}$	Fibonacci	A000045	$1, 2, 3, 5, 8, 13, 21, 34, 55, 89$
$\{UF\}, \{DF\}$, $\{FU\}, \{FD\}$	$\frac{2}{1-2x-\sqrt{1-4x^3}}$	A165407	$1, 1, 2, 3, 4, 7, 11, 16, 27, 43$
$\{FF\}$	$\frac{x(1-x-x^2+3x^3-5x^4+3x^5-2x^6-x^7-x^9)}{(1-x-2x^4-x^6-x^7)(1-2x+x^2-x^3)}$	New	$1, 2, 2, 5, 9, 17, 29, 53, 94, 168$

Table 1: Number of α-equivalence classes for Motzkin paths.

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{A}$. Let us prove that there exists $M' \in \mathcal{A}$ (with the same length as M) such that M and M' belong to the same class. Since M does not belong to $\mathcal{A}$, we can write $M = RUQDS$, where R is of maximal length so that it does not contain any point of ordinate two, and Q is a (possibly empty) Motzkin path.

Now, we define a sequence of Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \geq 1$, where for any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

Let $M_{i-1} = RUQDS$ be the above decomposition. Then, $M_i = RDQUS$ is obtained from M_{i-1} by swapping the two steps U and D contiguous to Q.

The process finishes because it decreases the number of points of ordinate greater than one. All Motzkin paths $M_0 = M, M_1, \ldots, M_k = P'$ belong to the same equivalence class. So, at the end of the process M_k belongs to $\mathcal{A}$. For instance, if

$$M = UUUDUFDFUFFDFFDDUD,$$

then we obtain $M_1 = M' = UDUDUFDFUFFDFFUDUD$. See Figure 2 for an illustration of this example.

Now we will prove that any F-equivalence class contains at most one element in $\mathcal{A}$. For a contradiction, let M and M' be two different Motzkin paths in $\mathcal{A}$ belonging to the same class. We write $M = QR$ and $M' = QS$ where R and S start with two different steps. Since M and M' lie in the same class, these steps cannot be F. Without loss of generality, let us assume that the first step of R is an up step U and then, the first step of S is a down step D. This means that the last point of Q has its ordinate equal to zero (otherwise, R would contain a point of ordinate at least two). As S starts with a down step, S has its second point (from the left) with an ordinate equal to -1, which gives a contradiction. $\square$

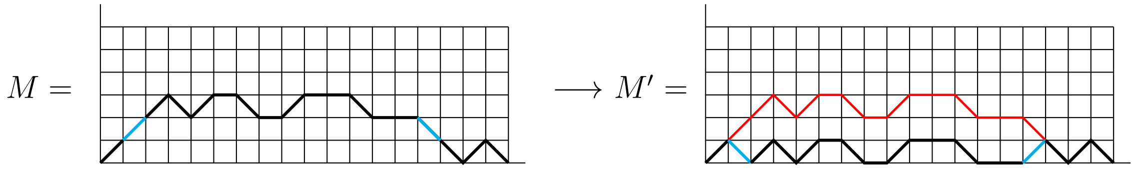

Figure 2: Illustration of the example described in the proof of Lemma 1.

Theorem 2. *The generating function for the set of F-equivalence classes in $\mathcal{M}$ with respect to the length is given by*

$$\frac{1-x}{1-2x}.$$

Proof. Using Lemma 1, it suffices to obtain the generating function $A(x)$ for the set $\mathcal{A}$. A non-empty Motzkin path $M \in \mathcal{A}$ can be written either $M = FQ$ where $Q \in \mathcal{A}$ or $M = UF^rDQ$ where $r \geq 0$ and $Q \in \mathcal{A}$. Then, we deduce the functional equation $A(x) = 1 + xA(x) + \frac{x^2}{1-x}A(x)$ which implies that $A(x) = \frac{1-x}{1-2x}$. □

2.2 Modulo $\alpha = U$ or $\alpha = D$

The result for the case $\alpha = D$ is deduced from $\alpha = U$ using a simple symmetry (the mirror transformation on Motzkin paths). So, we set $\alpha = U$ in this part.

Let $\mathcal{B}$ be the set of Motzkin paths without any flat steps at positive height. For instance, we have $UDFUD \in \mathcal{B}$ and $UFDUD \notin \mathcal{B}$.

Lemma 3. *There is a bijection between $\mathcal{B}$ and the set of U-equivalence classes of $\mathcal{M}$.*

Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{B}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{B}$ (with the same length as M) such that M and M' belong to the same class. Since M does not belong to $\mathcal{B}$, there exists a Motzkin path Q (possibly empty) such that $M = RFQDS$ where R is maximal so that it does not contain any flat step at positive height, its last point has an ordinate of at least one, and Q is a Motzkin path.

Now, we define a sequence of Motzkin paths $M_0 = M, M_1, M_2, \dots, M_{k-1}, M_k = M'$, $k \geq 1$, where for any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

Let $M_{i-1} = RFQDS$ be the above decomposition. Then, we define the Motzkin path $M_i = RDQFS$ obtained from M_{i-1} by swapping the two steps F and D contiguous to Q.

The process finishes because it decreases the height of the leftmost flat step having a positive height. All Motzkin paths $M_0 = M, M_1, \dots, M_k = M'$ belong to the same equivalence class. So, at the end of the process, M_k belongs to $\mathcal{B}$. For instance, if we perform the above process on $M = UUUDFFFFUUDDFFDDFF$, then we obtain $M_1 = UUUDDFFFUUDDFFFDFF$ and $M_2 = M' = UUUDDDFFUUDDFFFFFF$ (see Figure 3 for an illustration of this example).

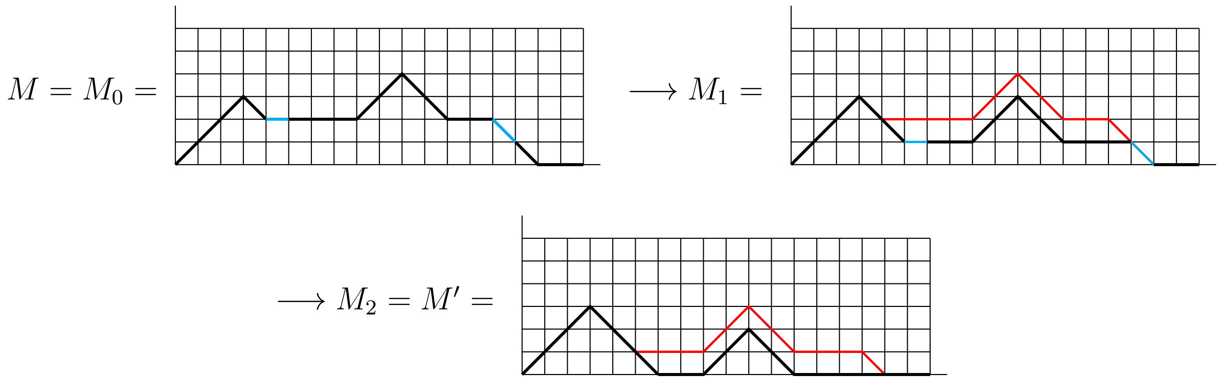

Figure 3: Illustration of the example described in the proof of Lemma 3.

Now we will prove that any U-equivalence class contains at most one element in B. For a contradiction, let M and M' be two different Motzkin paths in $\mathcal{B}$ belonging to the same class. We write $M = QR$ and $M' = QS$ where R and S start with two different steps. Since M and M' lie in the same class, these steps cannot be U. Without loss of generality, let us assume that the first step of R is a down step D and then, the first step of S is a flat step F. This means that the last point of Q has its ordinate equal to zero (otherwise M' could not belong to $\mathcal{B}$). As the first step of R is D, the second point of R has an ordinate equal to -1 which gives a contradiction. $\square$

Theorem 4. *The generating function for the set of U-equivalence (resp., D-equivalence) classes in $\mathcal{M}$ with respect to the length is given by*

$$\frac{2}{1 - 2x + \sqrt{1 - 4x^2}}.$$

Proof. Using Lemma 3, it suffices to obtain the generating function $B(x)$ for the set $\mathcal{B}$. A non-empty Motzkin path $M \in \mathcal{B}$ can be written either $M = FQ$ where $Q \in \mathcal{B}$ or $M = UQDQ'$ where Q is a Dyck path in $\mathcal{D}$ and $Q' \in \mathcal{B}$. Then, we deduce the functional equation $B(x) = 1 + xB(x) + x^2C(x^2)B(x)$ where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the set $\mathcal{D}$ of Dyck paths. A simple calculation gives the result. $\square$

3 Equivalence classes modulo a pattern of length two

Throughout this section, we study the α-equivalence in $\mathcal{M}$ for any pattern α of length two. By considering the mirror transformation of Motzkin paths, it suffices to examine the cases where $\alpha \in \{UU, UD, UF, FU, DU, FF\}$.

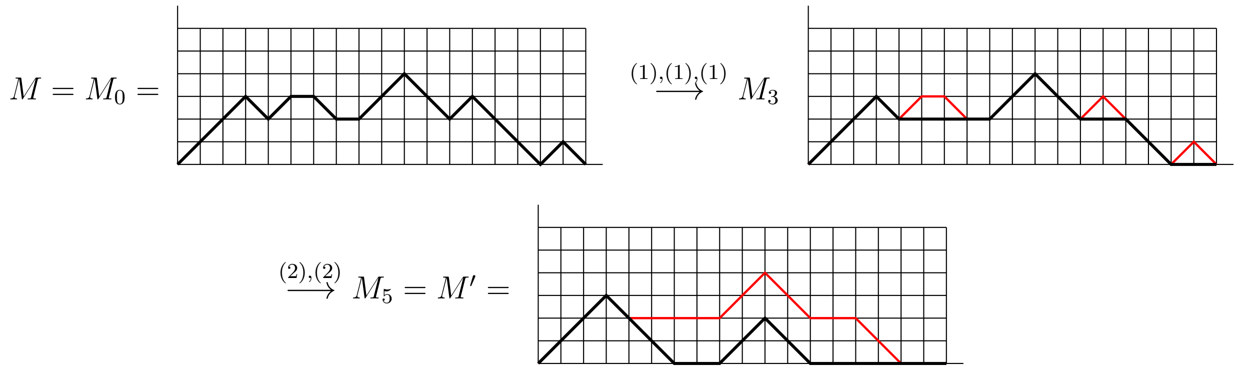

Figure 4: Illustration of the example described in the proof of Lemma 5.

3.1 Modulo $\alpha = UU$

Let $\mathcal{E}$ be the set of Motzkin paths without any flat steps at positive height and having no ascents of length one, i.e., no maximal consecutive up steps of length one. For instance, we have $UUDDF \in \mathcal{E}$ and $FUDFF \notin \mathcal{E}$.

Lemma 5. *There is a bijection between $\mathcal{E}$ and the set of UU-equivalence classes of $\mathcal{M}$.*

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{E}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{E}$ of the same length as M such that M and M' belong to the same class.

We define a sequence of Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \geq 1$, where for any i, $1 \leq i \leq k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

(1) If there is an ascent of length one in M_{i-1}, then we write $M_{i-1} = RUQDS$ where R is of maximal length so that it does not contains any ascent of length one, R does not finish with U, and Q is a Motzkin path (possibly empty) that does not start with an up step. So, we set $M_i = RFQFS$;

(2) Otherwise, if there is a flat step at positive height, then we perform the process described in the proof of Lemma 3.

This process finishes because part (1) of the process decreases by one the number of ascents of length one, and part (2) decreases the height of the leftmost flat step having a positive height (see the proof of Lemma 3). All Motzkin paths $M_0 = M, M_1, \ldots, M_k = M'$ belong to the same class. At the end of the process, M_k belongs to $\mathcal{E}$. For instance, if we perform the above process on $M = M_0 = UUUDUFDFUUDDUDDDUD$ then we obtain $M_3 = UUUDFFFFUUDDFFDDFF$ after three runs of (1), and we obtain $M_5 = UUUDDDFFUUDDFFFFFF$ after two runs of (2) (see Figure 4 for an illustration of this example).

For a contradiction, let M and M' be two different Motzkin paths in $\mathcal{E}$ belonging to the same class. We write $M = QR$ and $M' = QS$ where R and S start with two different steps, and R and S cannot start with UU. We distinguish two cases. If the last step of Q is a down step D or a flat step F, then R and S cannot start with UF and UD (and also with U) because an ascent of length one would be created; if the last step of Q is an up step U, then R and S cannot start with U because a pattern UU would be created. Without loss of generality, we can assume (for the two last cases) that R (resp., S) starts with D (resp., with F); so, the last point of Q has a positive ordinate that implies that the first step of S is a flat step of positive height, which is a contradiction. ☐

Theorem 6. *The generating function for the set of UU-equivalence classes of $\mathcal{M}$ with respect to the length is given by*

$$\frac{(x-1)^2 - \sqrt{(1-3x^2)(x^2+1)}}{2x(x^3-(1-x)^2)}.$$

Proof. Using Lemma 5, it suffices to give the generating function for the set $\mathcal{E}$, which is already known (see A191385 in [18]). ☐

3.2 Modulo $\alpha = DU$

Let $\mathcal{F}$ be the set of Motzkin paths without any points of ordinate greater than one, and without any flat steps at height zero. For instance, we have $UFDUFFDUD \in \mathcal{F}$ and $UDFUUDD \notin \mathcal{F}$.

Lemma 7. *There is a bijection between $\mathcal{F} \cup \{F\}$ and the set of DU-equivalence classes of $\mathcal{M}$.*

Proof. Let $M \neq F$ be a Motzkin path in $\mathcal{M}\backslash\mathcal{F}$ (the case where $M = F$ is clear). Let us prove that there exists a Motzkin path $M' \in \mathcal{F}$ of the same length as M such that M and M' belong to the same class.

Any Motzkin path $M \in \mathcal{M}$ can be uniquely written either

$$M = \beta_0 \qquad \text{or} \qquad M = \beta_0(DU\beta_1)\cdots(DU\beta_r),$$

where $r \geq 1$, and β_i, $0 \leq i \leq r$, (possibly empty) does not contain any pattern DU.

If $M = \beta_0$ then we set $M' = UF^{s_0-2}D$ where $s_0 \geq 2$ is the length of M; otherwise, we set $M' = UF^{s_0-1}(DUF^{s_1})\cdots(DUF^{s_{r-1}})(DUF^{s_r-1})D$ where s_i, $0 \leq i \leq r$, is the length of β_i. By construction, M and M' belong to the same equivalence class and M' lies in $\mathcal{F}$. For instance, if $M = FUFDUUUDUDDUUFFDDD$ then we obtain $M' = UFFDUFFDUFFDUFFFFD$ (see Figure 5 for an illustration of this example).

Any Motzkin path $M \in \mathcal{F}$ is characterized by the positions of its pattern DU. So, two different Motzkin paths in $\mathcal{F}$ cannot belong to the same class, which completes the proof. ☐

Theorem 8. *The generating function for the set of DU-equivalence classes of $\mathcal{M}$ with respect to the length is given by $F(x) = \frac{1-x^2-x^3}{1-x-x^2}$, that corresponds to a shift of the Fibonacci sequence.*

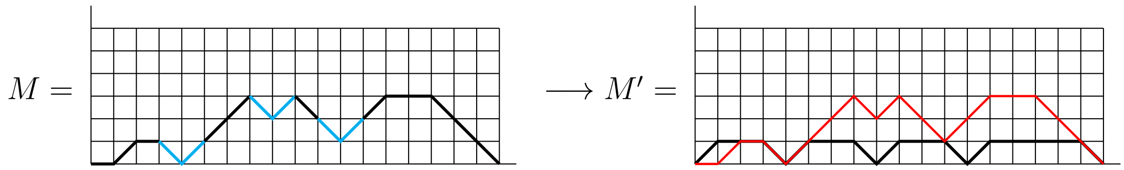

Figure 5: Illustration of the example in the proof of Lemma 9.

Proof. Using Lemma 7, it suffices to provide the generating function $F(x)$ for the set $\mathcal{F}\cup\{F\}$. Let M be a non-empty Motzkin path in $\mathcal{F}$. It can be written $M = UF^rDQ$ where $r \geq 0$ and $Q \in \mathcal{F}$. So, we have the functional equation $F(x) = 1 + x + \frac{x^2}{1-x}(F(x) - x)$ which implies the result. $\square$

3.3 Modulo $\alpha = UD$

Let $\mathcal{G}$ be the set of Motzkin paths without any points of ordinate greater than one, and without any flat steps at positive height. For instance, we have $FUDFUD \in \mathcal{G}$ and $UFDUUDD \notin \mathcal{G}$.

Lemma 9. *There is a bijection between $\mathcal{G}$ and the set of UD-equivalence classes of $\mathcal{M}$.*

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{G}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{G}$ of the same length as M such that M and M' belong to the same class.

Any Motzkin path $M \in \mathcal{M}$ can be uniquely written either

$$M = \beta_0 \qquad \text{or} \qquad M = \beta_0(UD\beta_1)\cdots(UD\beta_r),$$

where $r \geq 1$, and β_i, $0 \leq i \leq r$, (possibly empty) does not contain any pattern UD.

If $M = \beta_0$ then we set $M' = F^{s_0}$ where $s_0 \geq 0$ is the length of M; otherwise, we set $M' = F^{s_0}(UDF^{s_1})\cdots(UDF^{s_r})$ where s_i, $0 \leq i \leq r$, is the length of β_i. Obviously, M and M' belong to the same equivalence class and M' lies in $\mathcal{G}$. For instance, if $M = FUFDUUUDUDDUUUDDDD$ then we obtain $M' = FFFFFFUDUDFFFUDFFF$ (see Figure 6 for an illustration of this example).

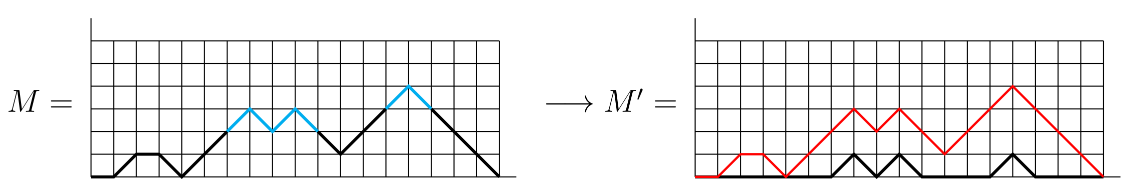

Figure 6: Illustration of the example in the proof of Lemma 9.

Any Motzkin path $M \in \mathcal{G}$ is characterized by the positions of its pattern UD. So, two different Motzkin paths in $\mathcal{G}$ cannot belong to the same class, which completes the proof. $\square$

Theorem 10. *The generating function for the set of UD-equivalence classes of $\mathcal{M}$ with respect to the length is given by $F(x) = \frac{1}{1-x-x^2}$, that corresponds to the Fibonacci sequence.*

Proof. Using Lemma 9, it suffices to provide the generating function $G(x)$ for the set $\mathcal{G}$. Let M be a non-empty Motzkin path in $\mathcal{G}$. It can be written either $M = FQ$ or $M = UDQ$ where $Q \in \mathcal{G}$ which implies that $G(x) = \frac{1}{1-x-x^2}$. □

3.4 Modulo $\alpha = UF$

Let $\mathcal{H}$ be the set of Motzkin paths without any patterns UD and UU, and such that all patterns FF and DF appear at height zero. For instance, we have $FFUFUFDDF \in \mathcal{H}$ and $UDUFUFDDF \notin \mathcal{H}$.

Lemma 11. *There is a bijection between $\mathcal{H}$ and the set of UF-equivalence classes of $\mathcal{M}$.*

Proof. Let M be a Motzkin path in $\mathcal{M}\backslash\mathcal{H}$. Let us prove that there exists a Motzkin path $M' \in \mathcal{H}$ of the same length as M such that M and M' belong to the same class.

Any Motzkin path $M \in \mathcal{M}$ can be uniquely written either

$$M = \beta_0 \qquad \text{or} \qquad M = \beta_0 \Pi_{i=1}^{r} (UF)^{k_i} \beta_i,$$

where $r \geq 1$, $k_i \geq 1$ for $1 \leq i \leq r$ and β_i, $0 \leq i \leq r$, (non-empty) does not contain any pattern UF.

If $M = \beta_0$ then we set $M' = F^{s_0}$ where $s_0 \geq 0$ is the length of M; since an element in $\mathcal{H}$ avoids UU and UD, M' is the only one element in $\mathcal{H}$ lying in the class of M (that avoids UF).

Otherwise, we set $M' = F^{s_0}\Pi_{i=1}^{r}(UF)^{k_i}\beta'_i$ with $\beta'_i = D^{t_i}F^{s_i-t_i}$ for $1 \leq i \leq r$, where s_i, $0 \leq i \leq r$, is the length of β_i and t_i, $i \geq 1$, is recursively defined by the equation

$$t_i = \min(s_i, \sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j).$$

It is worth noticing that $\sum_{j=1}^{i} k_j$ (resp., $\sum_{j=1}^{i-1} t_j$) is the number of up steps (resp., down steps) before β'_i in M'. So, setting $t_i = \min(s_i, \sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j)$ means that β'_i consists of the maximum possible of down steps, eventually completed with some flat steps at height zero. Then, M and M' belong to the same equivalence class and M' lies in $\mathcal{G}$. For instance, if we have $M = UUUFDFDUFUFFFDUDDD$ then we obtain $M' = FFUFDFFUFUFDDFFFFF$ (see Figure 7 for an illustration of this example).

Moreover, M' is the unique element in $\mathcal{H}$ in the same class as $M = \beta_0\Pi_{i=1}^{r}(UF)^{k_i}\beta_i$. Indeed, β_i, $i \geq 1$, does not contains any pattern UU, UF and UD implies that β_i does not contain any up steps; M does not contains FF and DF at positive height implies that $\beta_i = D^{k_i}F^{s_i-k_i}$ where k_i is either the length s_i of β_i or the difference between the numbers of up steps and down steps before β_i, that is precisely $\sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j$. Thus, we necessarily have $M = M'$ which completes the proof.

□

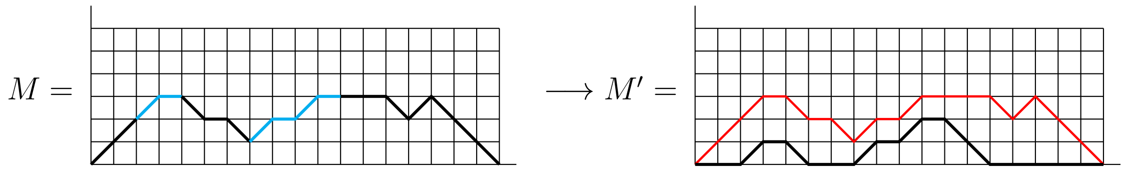

Figure 7: Illustration of the example in the proof of Lemma 11.

Theorem 12. *The generating function for the set of UF-equivalence classes of $\mathcal{M}$ with respect to the length is given by* $H(x) = \frac{2}{1-2x-\sqrt{1-4x^3}}$.

Proof. Using Lemma 11, it suffices to provide the generating function $H(x)$ for the set $\mathcal{H}$. Let M be a non-empty Motzkin path in $\mathcal{H}$. It can be written either $M = FQ$ or $M = UQDR$ where Q and R are two Motzkin paths.

If we have $M = FQ$, then M belongs to $\mathcal{H}$ if and only if $Q \in \mathcal{H}$. Otherwise, $M = UQDR$ belongs to $\mathcal{H}$ if and only if $R \in \mathcal{H}$ and $Q \in \mathcal{H}$ such that $Q = F\Pi_{i=1}^{r}(UF)^{k_i}D^{\ell_i}$ for some k_i and ℓ_i, with $1 \leq i \leq r$. Since Q is a Motzkin path of length $n = 1 + 3\sum_{i=1}^{r} k_i$, this implies that $Q' = \Pi_{i=1}^{r}U^{k_i}D^{\ell_i}$ is a Dyck path of semilength $\sum_{i=1}^{r} k_i = \frac{n-1}{3}$. Conversely, any Dyck path can be associated to a Motzkin path of the form of Q. So we deduce the functional equation $H(x) = 1 + xH(x) + x^3C(x^3)H(x)$ where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the set $\mathcal{D}$ of Dyck paths, which completes the proof. $\square$

3.5 Modulo $\alpha = FU$

Let $\mathcal{I}$ be the set of Motzkin paths starting with a flat step, without any patterns DU, UU, FD and such that all patterns FF appear at height zero. For instance, we have $FFUFUDFUDD \in \mathcal{I}$ and $FFUFUFDDFUD \notin \mathcal{I}$.

Lemma 13. *There is a bijection between $\mathcal{I}$ and the set of FU-equivalence classes of $\mathcal{M}$.*

Proof. This is a simple adaptation of the proof of Lemma 11 by replacing UF with FU in the decomposition $M = \beta_0\Pi_{i=1}^{r}(UF)^{k_i}\beta_i$. So, we do not present this proof here. $\square$

Theorem 14. *The generating function for the set of FU-equivalence classes of $\mathcal{M}$ with respect to the length is given by* $I(x) = \frac{2}{1-2x-\sqrt{1-4x^3}}$.

Proof. Using Lemma 13, it suffices to provide the generating function $I(x)$ for the set $\mathcal{I}$. Let M be a non-empty Motzkin path in $\mathcal{I}$. It can be uniquely decomposed $M' = F^{s_0}\Pi_{i=1}^{r}(FU)^{k_i}\beta'_i$ with $\beta'_i = D^{t_i}F^{s_i-t_i}$ for $1 \leq i \leq r$, where s_i, $0 \leq i \leq r$, is the length of β_i and t_i is recursively defined by $t_i = \min(s_i, \sum_{j=1}^{i} k_j - \sum_{j=1}^{i-1} t_j)$. The transformation that consists of replacing all occurrence FU into an occurrence UF, induces a one-to-one correspondence between $\mathcal{I}$ and $\mathcal{H}$. Therefore, the generating function $I(x)$ for $\mathcal{I}$ verifies $I(x) = H(x)$. Using Theorem 12, we complete the proof. $\square$

3.6 Modulo $\alpha = FF$

Let $M \in \mathcal{M}$ be a Motzkin path. A *long flat* in M is a maximal sequence of at least two consecutive flat steps in M. An *isolate flat step* is a flat step which is not contiguous to another flat step.

Let $\mathcal{J}$ be the set of Motzkin paths without any points of ordinate greater than one, without any isolate flat steps, such that two long flats are separated by at most two steps, and at most one step precedes (resp., follows) the first (resp., last) long flat. For instance, there are five Motzkin paths in $\mathcal{J}$ of length 6: $FFFFFF$, $FFUFFD$, $UFFDFF$, $UFFFFD$ and $FFUDFF$. For $r, s \in \{0, 1\}$, we define the set $\mathcal{J}_r^s$ of elements of $\mathcal{J}$ such that the height of the leftmost long flat is r and the height of the rightmost long flat is s. Obviously, we have $\mathcal{J} = \cup_{r,s\in\{0,1\}} \mathcal{J}_r^s$.

Let M be a Motzkin path without any points of ordinate greater than one. We denote by $|M|$ its length, i.e., the number of the steps in M.

Fact 1: There is a unique decomposition of the form

$$M = \beta_0 \Pi_{i=1}^{k} (Q_i \beta_i)$$

with Q_i of maximal length in $\mathcal{J}$ for all $1 \le i \le k$ and where $\beta_i \in \mathcal{M}$, $0 \le i \le k$, does not contain any long flat, and only β_0 and β_k can possibly be empty.

Fact 2: If $M = \beta_0 \Pi_{i=1}^{k} (Q_i \beta_i)$ is the decomposition obtained in Fact 1, then there exists M' in the same class as M such that $M' = \beta'_0 \Pi_{i=1}^{k} (Q_i \beta'_i)$ with $\beta'_i = F$ if $|\beta_i| = 1$, $\beta'_i = (UD)^{\frac{|\beta_i|}{2}}$ if $|\beta_i| \ge 2$ is even and $\beta'_i = UFD(UD)^{\frac{|\beta_i|-3}{2}}$ if $|\beta_i| \ge 3$ is odd. In the following, a Motzkin path β'_i of the previous form will be called a *hinge*. It is worth noticing that any hinge is entirely characterized by its length.

Let $\mathcal{K}$ be the set of Motzkin paths $M = \beta_0 \Pi_{i=1}^{k} (Q_i \beta_i)$ verifying the conditions described in Fact 2, and such that all Q_i starts with FF whenever β_{i-1} and β_i are non-empty. For instance, the third Motzkin path $UDUDUDFFUFFDUDFFUD$ in Figure 8 belongs to $\mathcal{K}$. Its decomposition provides $\beta_0 = UDUDUD$, $Q_1 = FFUFFD$, $\beta_1 = UD$, $Q_2 = FF$ and $\beta_2 = UD$.

Lemma 15. *There is a bijection between $\mathcal{K}$ and the set of FF-equivalence classes of $\mathcal{M}$.*

Proof. For any Motzkin path in $\mathcal{M} \backslash \mathcal{K}$, the process described in the proof of Lemma 1 ensures that there is a Motzkin path $M \in \mathcal{A}$ in the same F-equivalence class, and thus in the same FF-equivalence class.

Let $M = \beta_0 \Pi_{i=1}^{k} (Q_i \beta_i)$ be the decomposition described in Fact 2.

We define a sequence of Motzkin path $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, $k \ge 1$.

For any i, $1 \le i \le k$, the Motzkin path M_i is obtained from M_{i-1} by performing the following process.

(a) If β_{i-1} and β_i are non-empty and $Q_i = UWD$, then we replace Q_i with $\bar{W}$ where $\bar{W}$ is obtained from W by exchanging U and D, and we replace β_{i-1} (resp., β_i) with a hinge of length $|\beta_{i-1}| + 1$ (resp., of length $|\beta_i| + 1$);

(b) If β_{i-1} and β_i are non-empty and $Q_i = UWF$, then we replace Q_i with $\bar{W}FD$ where $\bar{W}$ is obtained from W by exchanging U and D, and we replace β_{i-1} (resp., β_i) with a hinge of length $|\beta_{i-1}|+1$ (resp., of length $|\beta_i|-1$).

By construction, all Motzkin paths $M_0 = M, M_1, M_2, \ldots, M_{k-1}, M_k = M'$, belong to the same class and M' belongs to $\mathcal{K}$. Moreover, M' is the unique element in $\mathcal{K}$ in the same class as M; indeed, let $M = \beta_0\Pi_{i=1}^{k}(Q_i\beta_i)$ and $M' = \beta'_0\Pi_{i=1}^{k'}(Q'_i\beta'_i)$ two different elements in $\mathcal{K}$ lying in the same class.

– From the definition of the sets $\mathcal{K}$ and $\mathcal{J}$, it is clear that $k = k'$.

– If the long steps in M appear in M' with the same height as in M, then we necessarily have $Q_i = Q'_i$ for $1 \le i \le k$. Thus, we deduce $|\beta_i| = |\beta'_i|$ for $i \ge 0$, and with the last remark of Fact 1, $\beta_i = \beta'_i$ for all $i \ge 0$, which is a contradiction with $M \ne M'$.

– If there is a long flat in M with height $h \in [0,1]$ and such that it appears in M' at height $1-h$. We choose the leftmost long flat with this property. Let $i \ge 1$ be the integer such that this long flat of M appears in Q_i. Since any element in $\mathcal{J}$ has at most two steps between two long flats, and at most one step before (resp., after) the first (resp., the last) long flat step, any long flat step of height $h' \in [0,1]$ in Q_i appears at height $1-h'$ in Q'_i. Since $Q_i \in \mathcal{J}$, we have either $Q_i = FFWFF$ and $Q'_i = UFF\bar{W}FFD$, or $Q_i = FFWFFD$ and $Q'_i = UFF\bar{W}FF$, or $Q_i = UFFWFF$ and $Q'_i = FF\bar{W}FFD$, or $Q_i = UFFWFFD$ and $Q'_i = FF\bar{W}FF$.

Observing these four cases, Q_i cannot be a prefix nor a suffix of M (otherwise M' could not have the leftmost FF at the same position as M). This means that β_{i-1} and β_i are non-empty. Thus, there is an element among M and M' where there is i such that Q_i whenever the element is M, Q'_i whenever the element is M', is contiguous with two non-empty hinges and such that the first long flat is of height zero. We obtain a contradiction with the fact that M and M' belong to $\mathcal{K}$ which completes the proof.

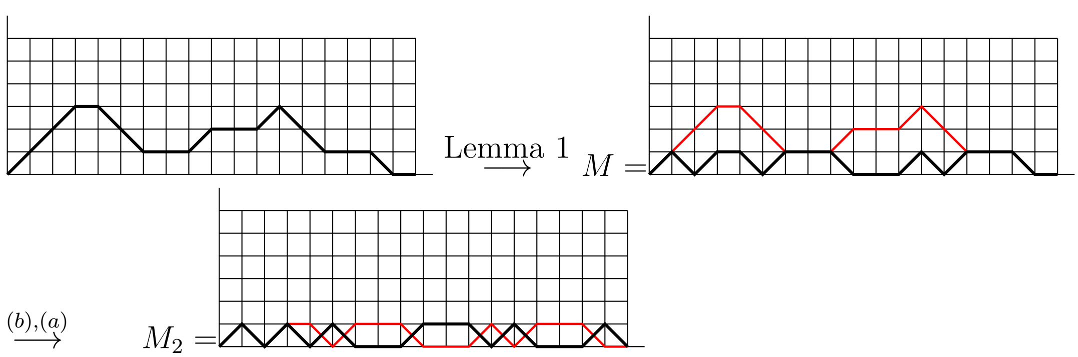

Figure 8: Illustration of the example in the proof of Lemma 15.

For instance, if $M = UFDUDUFFDFFUDUFFDF$ then the decomposition in Fact 2 implies $\beta_0 = UFDUD$, $Q_1 = UFFDFF$, $\beta_1 = UD$, $Q_2 = UFFD$ and $\beta_2 = F$. For

$i = 1$, β_0 and β_1 are non-empty and $Q_1 = UWF$ with $W = FFDF$; the process (b) creates $M_1 = UDUDUDFFUFFDFUFFDF$ where β_0 (resp., β_1 and Q_1) is replaced with $UDUDUD$ (resp., F and $FFUFFD$). For $i = 2$, $\beta_1 = F$ and $\beta_2 = F$ are non-empty and $Q_2 = UFFD$; the process (a) creates $M_2 = UDUDUDFFUFFDUDFFUD$ where β_1 (resp., β_2 and Q_2) is replaced with UD (resp., UD and FF). See Figure 8 for an illustration of this example. □

Lemma 16. *The generating function for the set $\mathcal{J}$, with respect to the length is given by* $J(x) = \frac{x^4-x^2+x-1}{x^7+x^6+2x^4+x-1}$.

Proof. In order to obtain the generating function $J(x)$ for the set $\mathcal{J}$, we calculate the generating functions $J_r^s(x)$ for the set $\mathcal{J}_r^s$ with $r, s \in \{0, 1\}$.

– If M is a Motzkin path in $\mathcal{J}_0^0$. We have either $M = F^k$ or $M = WF^k$ or $M = W'UDF^k$ with $k \geq 2$, $W \in \mathcal{J}_0^1$ and $W' \in \mathcal{J}_0^0$;

– if M is a Motzkin path in $\mathcal{J}_1^0$. We have either $M = WF^k$ or $M = W'UDF^k$ with $k \geq 2$, $W \in \mathcal{J}_1^1$ and $W' \in \mathcal{J}_1^0$;

– If M is a Motzkin path in $\mathcal{J}_0^1$. We have either $M = WUF^kD$ or $M = W'UF^kD$ with $k \geq 2$, $W \in \mathcal{J}_0^0$ and $W' \in \mathcal{J}_0^1$;

– If M is a Motzkin path in $\mathcal{J}_1^1$. We have either $M = UF^kD$ or $M = WUF^kD$ or $M = W'UF^kD$ with $k \geq 2$, $W \in \mathcal{J}_1^0$ and $W' \in \mathcal{J}_1^1$.

We deduce the following system of functional equations:

$$\begin{cases} J_0^0(x) = & \frac{x^2}{1-x}(J_0^1(x) + x^2 J_0^0(x) + 1) \\ J_1^0(x) = & \frac{x^2}{1-x}(J_1^1(x) + x^2 J_1^0(x)) \\ J_1^1(x) = & \frac{x^4}{1-x}(J_0^0(x) + J_0^1(x)) \\ J_1^0(x) = & \frac{x^4}{1-x}(J_1^0(x) + J_1^1(x) + 1). \end{cases}$$

A simple calculation (with Maple for instance) provides the generating functions for the four sets $\mathcal{J}_0^0, \mathcal{J}_0^1, \mathcal{J}_1^1$ and $\mathcal{J}_1^0$, and thus for the set $\mathcal{J} = \cup_{r,s\in\{0,1\}} \mathcal{J}_r^s$ which completes the proof. □

Theorem 17. *The generating function for the set of FF-equivalence classes of $\mathcal{M}$ with respect to the length is given by*

$$\frac{x(1 - x - x^2 + 3x^3 - 5x^4 + 3x^5 - 2x^6 - x^7 - x^9)}{(1 - x - 2x^4 - x^6 - x^7)(1 - 2x + x^2 - x^3)}.$$

Proof. With Lemma 15, it suffices to obtain the generating function $K(x)$ for the set $\mathcal{K}$. First, we calculate the generating functions for the sets $\mathcal{K}_r$ with $r, s \in \{0, 1, 2\}$, where $\mathcal{K}_0$ (resp., $\mathcal{K}_1$, resp., $\mathcal{K}_2$) is the subset of $\mathcal{K}$ of Motzkin path starting by FF (resp., UFF, resp., a hinge). Let $J_r^s(x)$, $r,s\in\{0,1\}$, be the generating function obtained in the proof of Lemma 16. We set $J_1(x)=J_1^0(x)+J_1^1(x)$ and $J_0(x)=J_0^0(x)+J_0^1(x)$.

– Let M be a Motzkin path in $\mathcal{K}_0$. We discuss on the number of hinges in M.

If M does not contains any hinge, then this means that $M\in\mathcal{J}_0$; If M has only one hinge, then either $M=QH_kS$, $k\geq 2$, or $M=QH_kT$, $k\geq 3$, or $M=RH_kS$, $k\geq 1$, or $M=RH_kT$, $k\geq 2$ with $Q\in\mathcal{J}_0^0$, $R\in\mathcal{J}_0^1$, $S\in\mathcal{J}_1^0\cup\mathcal{J}_1^1\cup\{\epsilon\}$, $T\in\mathcal{J}_0^0\cup\mathcal{J}_0^1$ and H_k is a hinge of length at least k; If M has at least two hinge, then either $M=QH_kW$, $k\geq 3$, or $M=RH_kW$, $k\geq 2$, with $Q\in\mathcal{J}_0^0$, $R\in\mathcal{J}_0^1$ and $W\in\mathcal{K}_0\backslash\mathcal{J}_0$.

Thus we have the functional equation:

$$\begin{array}{rl} K_0(x)= & J_0(x)+J_0^0(x)\frac{x^2}{1-x}(J_1(x)+1)+J_0^0(x)\frac{x^3}{1-x}J_0(x)+J_0^1(x)\frac{x}{1-x}(J_1(x)+1)+ \\ & J_0^1(x)\frac{x^2}{1-x}J_0(x)+J_0^0(x)\frac{x^3}{1-x}(K_0(x)-J_0(x))+J_0^1(x)\frac{x^2}{1-x}(K_0(x)-J_0(x)) \end{array}.$$

– Let M be a Motzkin path in $\mathcal{K}_1$. We discuss on the number of hinges in M. If M does not contain any hinge, then this means that $M\in\mathcal{J}_1$; If M has only one hinge, then either $M=Q'H_kS$, $k\geq 2$, or $M=Q'H_kT$, $k\geq 3$, or $M=R'H_kS$, $k\geq 1$, or $M=R'H_kT$, $k\geq 2$ with $Q'\in\mathcal{J}_1^0$, $R'\in\mathcal{J}_1^1$, $S\in\mathcal{J}_1^0\cup\mathcal{J}_1^1\cup\{\epsilon\}$, $T\in\mathcal{J}_0^0\cup\mathcal{J}_0^1$ and H_k is a hinge of length at least k; If M has at least two hinges, then either $M=Q'H_kW$, $k\geq 3$, or $M=R'H_kW$, $k\geq 2$, with $Q'\in\mathcal{J}_1^0$, $R'\in\mathcal{J}_1^1$ and $W\in\mathcal{K}_1\backslash\mathcal{J}_1$.

Thus we have the functional equation:

$$\begin{array}{rl} K_1(x)= & J_1(x)+J_1^0(x)\frac{x^2}{1-x}(J_1(x)+1)+J_1^0(x)\frac{x^3}{1-x}J_0(x)+J_1^1(x)\frac{x}{1-x}(J_1(x)+1)+ \\ & J_1^1(x)\frac{x^2}{1-x}J_0(x)+J_1^0(x)\frac{x^3}{1-x}(K_0(x)-J_0(x))+J_1^1(x)\frac{x^2}{1-x}(K_0(x)-J_0(x)) \end{array}.$$

– Let M be a Motzkin path in $\mathcal{K}_2$. We have either $M=H_kS$, $k\geq 1$, or $M=H_kT$, $k\geq 2$ with $S\in\mathcal{J}_1^0\cup\mathcal{J}_1^1$, $T\in\mathcal{J}_0^0\cup\mathcal{J}_0^1\cup\{\epsilon\}$ and H_k is a hinge of length at least k.

Thus we have the functional equation:

$$K_2(x)=J_1(x)\frac{x}{1-x}+(K_0(x)+1)\frac{x^2}{1-x}.$$

Using the Lemma 16, a simple calculation with Maple gives the generating function $K(x)=K_0(x)+K_1(x)+K_2(x)$ for the set $\mathcal{K}$. $\square$

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