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Polygonal, Sierpiński, and Riesel numbers

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Abstract

In this paper, we show that there are infinitely many Sierpiński numbers in the sequence of triangular numbers, hexagonal numbers, and pentagonal numbers. We also show that there are infinitely many Riesel numbers in the same sequences. Furthermore, we show that there are infinitely many n-gonal numbers that are simultaneously Sierpiński and Riesel.

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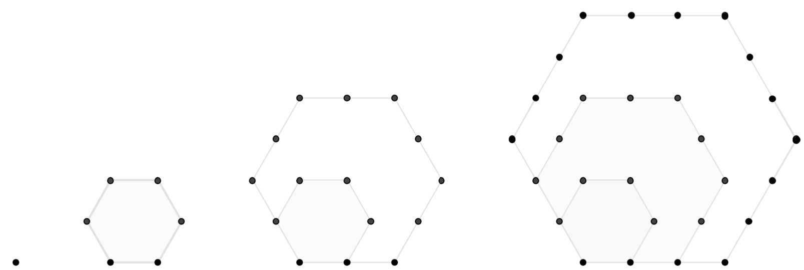
1 Introduction

Polygonal numbers are those that can be expressed geometrically by an arrangement of equally spaced points. For example, a positive integer n is a triangular number if n dots can be arranged in the form of an equilateral triangle. Similarly, n is a square number if n dots can be arranged in the form of a square. The diagram below represents the first four hexagonal numbers, which are 1, 6, 15, and 28.



In 1960, Sierpiński [11] showed that there are infinitely many odd positive integers k with the property that $k \cdot 2^n + 1$ is composite for all positive integers n. Such an integer k is called a Sierpiński number in honor of Sierpiński's work. Two years later, Selfridge (unpublished) showed that 78557 is a Sierpiński number. To this day, this is the smallest known Sierpiński number. As of this writing, there are six candidates smaller than 78557 to consider: 10223, 21181, 22699, 24737, 55459, 67607. See http://www.seventeenorbust.com for the most up-to-date information.

Riesel numbers are defined in a similar way: an odd positive integer k is Riesel if $k \cdot 2^n - 1$ is composite for all positive integers n. These were first investigated by Riesel in 1956 [10]. The smallest known Riesel number is 509203. As of this writing there are 50 remaining candidates smaller that 509203 to consider. See http://www.prothsearch.net/rieselprob.html for the most recent information.

The tool used to construct these numbers is a covering system. A collection of congruences

$$\begin{array}{ll} r_1 & \pmod{m_1} \\ r_2 & \pmod{m_2} \\ \vdots & \\ r_t & \pmod{m_t} \end{array}$$

is called a *covering system of congruences*, also called a *covering system*, if each integer n satisfies $n \equiv r_i \pmod{m_i}$ for some $1 \leq i \leq t$. This technique was first introduced by Erdős who later used the idea to show that there are infinitely many odd numbers that are not of the form $2^k + p$, where p is a prime [4].

Previous work has been done to show an intersection between Sierpiński or Riesel numbers with familiar integer sequences such as the Fibonacci numbers [8, 9] and the Lucas numbers [1]. Perfect power Sierpiński numbers and Riesel numbers have been studied in depth; in particular, there are infinitely many Sierpiński numbers of the form k^r for any fixed positive integer r [2, 5]. For Riesel numbers, there are infinitely many k such that k^r is Riesel for values of r with $\gcd(r, 12) \leq 3$ [2] and for $\gcd(r, 105) = 1$ [7]. In this paper we expand on these findings by considering the intersection of sequences of polygonal numbers with Sierpiński and Riesel numbers. As the kth polygonal number for an n-sided polygon is given by $\frac{1}{2}(k^2(n-2) - k(n-4))$, which is quadratic in k, we build on techniques for constructing perfect power Sierpiński numbers and binomial Sierpiński and Riesel numbers (cf. [5, 6]). Through the use of coverings, we construct polygonal numbers that are also Riesel numbers and Sierpiński numbers.

Cohen and Selfridge showed that there are infinitely many numbers that are simultaneously Sierpiński and Riesel [3]. The smallest Sierpiński-Riesel number that came from their construction has 26 digits. Several others also produced Sierpiński-Riesel numbers; for example, Brier (unpublished) produced an example with 41 digits in 1998, and Gallot (unpublished) produced an example with 27 digits in 2000. In 2008, an example with 24 digits was produced [5]. Recently, a Sierpiński-Riesel number that is 22 digits long was discovered by Clavier (unpublished). Entry A180247 in the Online Encyclopedia of Integer Sequences has more information about these results. (See `http://oeis.org/A180247`.) We use similar techniques in this paper to find polygonal numbers that are Sierpiński-Riesel.

All computations were performed using the computer algebra system Maple.

2 Triangular and hexagonal numbers

2.1 Triangular-Sierpiński numbers

Let T_k denote the k^{th} triangular number. That is, $T_k = \frac{k(k+1)}{2}$. Now, consider the implications in Table 1 below.

$$\begin{array}{lcllr}
n \equiv 1 \pmod{2} & \& & k \equiv 1 \pmod{3} & \Longrightarrow & 3 \mid (T_k \cdot 2^n + 1) \\
n \equiv 0 \pmod{3} & \& & k \equiv 3 \pmod{7} & \Longrightarrow & 7 \mid (T_k \cdot 2^n + 1) \\
n \equiv 2 \pmod{4} & \& & k \equiv 1 \text{ or } 3 \pmod{5} & \Longrightarrow & 5 \mid (T_k \cdot 2^n + 1) \\
n \equiv 4 \pmod{8} & \& & k \equiv 1 \text{ or } 15 \pmod{17} & \Longrightarrow & 17 \mid (T_k \cdot 2^n + 1) \\
n \equiv 8 \pmod{12} & \& & k \equiv 4 \text{ or } 8 \pmod{13} & \Longrightarrow & 13 \mid (T_k \cdot 2^n + 1) \\
n \equiv 16 \pmod{24} & \& & k \equiv 53 \text{ or } 187 \pmod{241} & \Longrightarrow & 241 \mid (T_k \cdot 2^n + 1)
\end{array}$$

Table 1

The congruences for n in Table 1 form a covering. In each row, the congruence for k results in $T_k \cdot 2^n + 1$ being divisible by one of the primes in $\mathcal{P} = \{3, 5, 7, 17, 13, 241\}$ for every positive integer in the congruence class for n. Assume that our values of k in each row will be chosen large enough from the congruence for k such that $T_k \cdot 2^n + 1$ is larger than its prime divisor in that row. It follows that each of these $T_k \cdot 2^n + 1$ must be composite for every positive integer in this congruence class. To ensure that T_k is odd (in order to satisfy the definition of a Sierpiński number), we also include the congruence $k \equiv 1$ or $2 \pmod 4$. If $k = 4\ell + 1$, we have

$$T_k = \frac{1}{2}k(k+1) = \frac{1}{2}(4\ell+1)(4\ell+2) = (4\ell+1)(2\ell+1),$$

and if $k = 4\ell + 2$, we have

$$T_k = \frac{1}{2}k(k+1) = \frac{1}{2}(4\ell+2)(4\ell+3) = (2\ell+1)(4\ell+3),$$

both of which are clearly odd.

Now, we find the intersection of all congruences for k to find a T_k that is a Sierpiński number. Using the Chinese remainder theorem, we have the following result.

Theorem 1. *There are infinitely many Sierpiński numbers in the sequence of triangular numbers.*

The smallest solution to the congruences for k that we find using the Chinese remainder theorem is 698953, and if k is a natural number with $k \equiv 698953 \pmod{4 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241}$, then T_k is a Sierpiński number. Thus, the smallest Triangular-Sierpiński number from this construction is 244267997581.

2.2 Triangular-Riesel numbers

Consider the implications in Table 2 below.

$n \equiv 0 \pmod 2$	&	$k \equiv 1 \pmod 3$	$\Longrightarrow$	$3 \mid (T_k \cdot 2^n - 1)$
$n \equiv 0 \pmod 3$	&	$k \equiv 1$ or $5 \pmod 7$	$\Longrightarrow$	$7 \mid (T_k \cdot 2^n - 1)$
$n \equiv 1 \pmod 4$	&	$k \equiv 2 \pmod 5$	$\Longrightarrow$	$5 \mid (T_k \cdot 2^n - 1)$
$n \equiv 7 \pmod 8$	&	$k \equiv 8 \pmod{17}$	$\Longrightarrow$	$17 \mid (T_k \cdot 2^n - 1)$
$n \equiv 11 \pmod{12}$	&	$k \equiv 5$ or $7 \pmod{13}$	$\Longrightarrow$	$13 \mid (T_k \cdot 2^n - 1)$
$n \equiv 16 \pmod{24}$	&	$k \equiv 5$ or $235 \pmod{241}$	$\Longrightarrow$	$241 \mid (T_k \cdot 2^n - 1)$

Table 2

We also include $k \equiv 1$ or $2 \pmod 4$ to ensure T_k is odd. Once again, the congruences for n in Table 2 form a covering. In each row, the congruences for k result in $T_k \cdot 2^n - 1$ being divisible by one of the primes in the set

$$\mathcal{P} = \{3, 7, 5, 17, 13, 241\}$$

for every positive integer n in the same row. As before, assume that our values of k will be chosen large enough from those in the congruence for k so that T_k is larger than any of the primes in the set $\mathcal{P}$. It follows that each of these $T_k \cdot 2^n - 1$ must be composite for every positive integer n. We then find the intersection of all congruences for k to find a T_k that is a Riesel number. By use of the Chinese remainder theorem, we see that the smallest residue class that is in the intersection of all congruences for k is $k \equiv 888802 \pmod{4 \cdot 3 \cdot 7 \cdot 5 \cdot 17 \cdot 13 \cdot 241}$. Thus, for every such value of k, the triangular number T_k is Riesel. Hence, we have the following theorem.

Theorem 2. *There are infinitely many Riesel numbers in the sequence of triangular numbers.*

2.3 Hexagonal-Sierpiński and Riesel numbers

Let H_k denote the k^{th} hexagonal number. That is, $H_k = 2k^2 - k$. Notice that if $k = 2\ell + 1$, we have $T_k = \frac{1}{2}k(k+1) = 2\ell^2 + 3\ell + 1 = H_{\ell+1}$. That is, when k is odd, the triangular number T_k is also a hexagonal number. Thus, if we include the congruences from Table 1 and $k \equiv 1 \pmod 2$, then we will have a subset of the triangular numbers that are also Sierpiński numbers in addition to hexagonal numbers. The smallest such k is 698953, and all positive integers k congruent to 698953 modulo $4 \cdot 3 \cdot 7 \cdot 5 \cdot 17 \cdot 13 \cdot 241$ also give T_k which are also Sierpiński and hexagonal. Similarly, the positive integers k that are congruent to 2916817 modulo $4 \cdot 3 \cdot 7 \cdot 5 \cdot 17 \cdot 13 \cdot 241$ yield T_k which are Riesel and also hexagonal. Thus, we have the following corollaries.

Corollary 3. *There are infinitely many Sierpiński numbers in the sequence of hexagonal numbers.*

Corollary 4. *There are infinitely many Riesel numbers in the sequence of hexagonal numbers.*

2.4 Triangular-Sierpiński-Riesel numbers

Table 3 below gives congruences for k to construct triangular numbers T_k that are simultaneously Sierpiński and Riesel.

$n \equiv 1 \pmod{2}$	&	$k \equiv 1 \pmod{3}$	$\Longrightarrow$	$3 \mid (T_k \cdot 2^n + 1)$
$n \equiv 1 \pmod{3}$	&	$k \equiv 2$ or $4 \pmod{7}$	$\Longrightarrow$	$7 \mid (T_k \cdot 2^n + 1)$
$n \equiv 5 \pmod{9}$	&	$k \equiv 23$ or $49 \pmod{73}$	$\Longrightarrow$	$73 \mid (T_k \cdot 2^n + 1)$
$n \equiv 6 \pmod{12}$	&	$k \equiv 1$ or $11 \pmod{13}$	$\Longrightarrow$	$13 \mid (T_k \cdot 2^n + 1)$
$n \equiv 8 \pmod{18}$	&	$k \equiv 6$ or $12 \pmod{19}$	$\Longrightarrow$	$19 \mid (T_k \cdot 2^n + 1)$
$n \equiv 2 \pmod{36}$	&	$k \equiv 15$ or $21 \pmod{37}$	$\Longrightarrow$	$37 \mid (T_k \cdot 2^n + 1)$
$n \equiv 20 \pmod{36}$	&	$k \equiv 24$ or $84 \pmod{109}$	$\Longrightarrow$	$109 \mid (T_k \cdot 2^n + 1)$
$n \equiv 4 \pmod{5}$	&	$k \equiv 13$ or $17 \pmod{31}$	$\Longrightarrow$	$31 \mid (T_k \cdot 2^n + 1)$
$n \equiv 6 \pmod{10}$	&	$k \equiv 3$ or $7 \pmod{11}$	$\Longrightarrow$	$11 \mid (T_k \cdot 2^n + 1)$
$n \equiv 8 \pmod{20}$	&	$k \equiv 9$ or $31 \pmod{41}$	$\Longrightarrow$	$41 \mid (T_k \cdot 2^n + 1)$
$n \equiv 0 \pmod{15}$	&	$k \equiv 69$ or $81 \pmod{151}$	$\Longrightarrow$	$151 \mid (T_k \cdot 2^n + 1)$
$n \equiv 12 \pmod{60}$	&	$k \equiv 20$ or $40 \pmod{61}$	$\Longrightarrow$	$61 \mid (T_k \cdot 2^n + 1)$
$n \equiv 0 \pmod{2}$	&	$k \equiv 1 \pmod{3}$	$\Longrightarrow$	$3 \mid (T_k \cdot 2^n - 1)$
$n \equiv 1 \pmod{4}$	&	$k \equiv 2 \pmod{5}$	$\Longrightarrow$	$5 \mid (T_k \cdot 2^n - 1)$
$n \equiv 7 \pmod{8}$	&	$k \equiv 8 \pmod{17}$	$\Longrightarrow$	$17 \mid (T_k \cdot 2^n - 1)$
$n \equiv 11 \pmod{16}$	&	$k \equiv 128 \pmod{257}$	$\Longrightarrow$	$257 \mid (T_k \cdot 2^n - 1)$
$n \equiv 11 \pmod{24}$	&	$k \equiv 90$ or $150 \pmod{241}$	$\Longrightarrow$	$241 \mid (T_k \cdot 2^n - 1)$
$n \equiv 3 \pmod{48}$	&	$k \equiv 41$ or $55 \pmod{97}$	$\Longrightarrow$	$97 \mid (T_k \cdot 2^n - 1)$
$n \equiv 19 \pmod{48}$	&	$k \equiv 315$ or $357 \pmod{673}$	$\Longrightarrow$	$673 \mid (T_k \cdot 2^n - 1)$

Table 3

In Table 3, the congruences for n above the horizontal line form a covering. This part of the table ensures that the congruences for k yield a Sierpiński number T_k. In addition, the congruences for n below the horizontal line also form a covering. Thus, the bottom part of the table ensures that the congruences for k yield a Riesel number. Notice that the congruences for k above the line and those below the line are compatible; the only modulus that is repeated in the two parts of the table is 3, and in both instances, we have $k \equiv 1 \pmod{3}$.

Now we include the congruence $k \equiv 1$ or $2 \pmod{4}$ to ensure that T_k is odd, and then use the Chinese remainder theorem to combine all of the congruences for k. The smallest solution to this set of congruences is

$$k \equiv 9229039712485870023302\quad (\text{mod } 27035115516102155476410389940).$$

We conclude there are infinitely many Sierpiński-Riesel numbers in the sequence of triangular numbers, and the smallest example resulting from this construction is

$$42587587007320635212044865468723864478997427 53.$$

We state this result as a theorem below.

Theorem 5. *There are infinitely many triangular numbers that are simultaneously Sierpiński numbers and Riesel numbers.*

2.5 Hexagonal-Sierpiński-Riesel numbers

If we again include the congruence $k \equiv 1 \pmod 2$ with the congruences in the previous subsection, we then have triangular numbers that are also hexagonal, in addition to being both Sierpiński and Riesel. Combining these congruences using the Chinese remainder theorem, we find the smallest solution to this set of congruences is

$$k \equiv 2474326773087797727457413 7 \pmod{270351155161021554764103899940},$$

then T_k is hexagonal, Sierpiński, and Riesel. We conclude with the following:

Theorem 6. *There are infinitely many hexagonal numbers that are simultaneously Sierpiński and Riesel numbers.*

3 Pentagonal numbers

Let P_k denote the k^{th} pentagonal number. We then have $P_k = \frac{1}{2}k(3k-1)$. In this section, we show that there are infinitely many pentagonal-Sierpiński numbers, infinitely many pentagonal-Riesel numbers, and infinitely many pentagonal numbers that are simultaneously Sierpiński and Riesel.

3.1 Pentagonal-Sierpiński numbers

Consider the implications in Table 4 below.

$n \equiv 1 \pmod 2$	&	$k \equiv 1 \pmod 3$	$\Longrightarrow$	$3 \mid (P_k \cdot 2^n + 1)$
$n \equiv 2 \pmod 3$	&	$k \equiv 2$ or $3 \pmod 7$	$\Longrightarrow$	$7 \mid (P_k \cdot 2^n + 1)$
$n \equiv 2 \pmod 4$	&	$k \equiv 1 \pmod 5$	$\Longrightarrow$	$5 \mid (P_k \cdot 2^n + 1)$
$n \equiv 4 \pmod 8$	&	$k \equiv 1$ or $5 \pmod{17}$	$\Longrightarrow$	$17 \mid (P_k \cdot 2^n + 1)$
$n \equiv 0 \pmod{12}$	&	$k \equiv 3$ or $6 \pmod{13}$	$\Longrightarrow$	$13 \mid (P_k \cdot 2^n + 1)$
$n \equiv 16 \pmod{24}$	&	$k \equiv 189$ or $213 \pmod{241}$	$\Longrightarrow$	$241 \mid (P_k \cdot 2^n + 1)$

Table 4

Observe that if $k \equiv 1$ or $2 \pmod 4$, then P_k is odd. To see this, notice that if $k = 4\ell + 1$, we have

$$P_k = \frac{1}{2}k(3k-1) = \frac{1}{2}(4\ell+1)(12\ell+3-1) = (4\ell+1)(6\ell+1),$$

and if $k = 4\ell + 2$ we have

$$P_k = \frac{1}{2}k(3k-1) = \frac{1}{2}(4\ell+2)(12\ell+6-1) = (2\ell+1)(12\ell+5),$$

which are both clearly odd. Thus, we also include $k \equiv 1$ or $2 \pmod 4$ in order to construct Sierpiński numbers in this sequence.

Once again, the congruences for n in Table 4 form a covering. In each row, the congruences for k and n result in $P_k \cdot 2^n + 1$ being divisible by one of the primes in the set $\mathcal{P} = \{3, 5, 7, 13, 17, 241\}$. As before, assume that our values of k will be chosen large enough from those in the congruence classes for k so that P_k is larger than any of the primes in the set $\mathcal{P}$. It follows that each of these $P_k \cdot 2^n - 1$ must be composite for every positive integer n. We deduce the intersection of all congruences for k to find a P_k that is a Sierpiński number. Using the Chinese remainder theorem for the congruences for k, we find that there are infinitely many such k, and the smallest solution that arises out of these congruences is

$$k \equiv 56101 \pmod{22369620}.$$

We conclude the following:

Corollary 7. *There are infinitely many pentagonal numbers that are Sierpiński numbers.*

3.2 Pentagonal-Riesel numbers

In this section, we demonstrate the following result:

Theorem 8. *There are infinitely many pentagonal numbers that are Riesel numbers.*

We again prove this statement using a covering of the integers, shown in Table 5 below.

$n \equiv 0 \pmod 2$	&	$k \equiv 1 \pmod 3$	$\Longrightarrow$	$3 \mid (P_k \cdot 2^n - 1)$
$n \equiv 2 \pmod 3$	&	$k \equiv 6 \pmod 7$	$\Longrightarrow$	$7 \mid (P_k \cdot 2^n - 1)$
$n \equiv 3 \pmod 4$	&	$k \equiv 3$ or $4 \pmod 5$	$\Longrightarrow$	$5 \mid (P_k \cdot 2^n - 1)$
$n \equiv 1 \pmod 8$	&	$k \equiv 10$ or $13 \pmod{17}$	$\Longrightarrow$	$17 \mid (P_k \cdot 2^n - 1)$
$n \equiv 1 \pmod{12}$	&	$k \equiv 11 \pmod{13}$	$\Longrightarrow$	$13 \mid (P_k \cdot 2^n - 1)$
$n \equiv 21 \pmod{24}$	&	$k \equiv 61$ or $100 \pmod{241}$	$\Longrightarrow$	$241 \mid (P_k \cdot 2^n - 1)$

Table 5

The congruences for n form a covering of the integers, so we again use the Chinese remainder theorem to combine the congruences for k (including $k \equiv 1$ or $2 \pmod 4$); the smallest solution for k that satisfies all of the congruences in the table is

$$k \equiv 590029 \pmod{22369620}.$$

For any of these solutions for k, the expression $P_k \cdot 2^n - 1$ is divisible by one of the primes in the set $\mathcal{P} = \{3, 5, 7, 13, 17, 241\}$.

3.3 Pentagonal-Sierpiński-Riesel

We show now that there are infinitely many pentagonal numbers that are simultaneously Sierpiński and Riesel. Consider the congruences in Table 6.

In Table 6, the congruences for n above the horizontal line form a covering of the integers. Thus, the congruences for k above this line yield pentagonal numbers P_k that are Sierpiński. Similarly, the congruences for n below the horizontal line also form a covering of the integers. Thus, the corresponding congruences for k in the bottom part of the table yield pentagonal numbers P_k that are also Riesel. Again, the congruences for k above and below the line are compatible; the only repeated modulus is 3, and in both parts of the table, we have $k \equiv 1 \pmod 3$. When we also include $k \equiv 1$ or $2 \pmod 4$ to make sure that the resulting P_k is also an odd integer, we find that there are 2^{17} solutions for k modulo

$$M = 4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 97 \cdot 109 \cdot 151 \cdot 241 \cdot 257 \cdot 673.$$

The smallest such solution is

$$k \equiv 1809725181412779246512\,18 \pmod M,$$

yielding the smallest pentagonal-Sierpiński-Riesel from this construction:

$$4912657848359275131577466718514577533146099967\,7.$$

Thus, we have shown the following result.

Theorem 9. *There are infinitely many pentagonal numbers that are simultaneously Sierpiński and Riesel numbers.*

4 Polygonal numbers

Theorem 10. *For infinitely many values of s, there are infinitely many s-gonal numbers that are Sierpiński numbers.*

$$
\begin{array}{llllr}
n \equiv 0 \pmod 2 & \& & k \equiv 2 \pmod 3 & \Longrightarrow & 3 \mid (P_k \cdot 2^n + 1) \\
n \equiv 1 \pmod 4 & \& & k \equiv 3 \text{ or } 4 \pmod 5 & \Longrightarrow & 5 \mid (P_k \cdot 2^n + 1) \\
n \equiv 1 \pmod{10} & \& & k \equiv 2 \pmod{11} & \Longrightarrow & 11 \mid (P_k \cdot 2^n + 1) \\
n \equiv 7 \pmod 8 & \& & k \equiv 9 \text{ or } 14 \pmod{17} & \Longrightarrow & 17 \mid (P_k \cdot 2^n + 1) \\
n \equiv 3 \pmod{18} & \& & k \equiv 15 \text{ or } 17 \pmod{19} & \Longrightarrow & 19 \mid (P_k \cdot 2^n + 1) \\
n \equiv 11 \pmod{24} & \& & k \equiv 162 \text{ or } 240 \pmod{241} & \Longrightarrow & 241 \mid (P_k \cdot 2^n + 1) \\
n \equiv 3 \pmod{16} & \& & k \equiv 140 \text{ or } 203 \pmod{257} & \Longrightarrow & 257 \mid (P_k \cdot 2^n + 1) \\
n \equiv 43 \pmod{48} & \& & k \equiv 32 \text{ or } 33 \pmod{97} & \Longrightarrow & 97 \mid (P_k \cdot 2^n + 1) \\
n \equiv 27 \pmod{48} & \& & k \equiv 112 \text{ or } 337 \pmod{673} & \Longrightarrow & 673 \mid (P_k \cdot 2^n + 1) \\
\hline
n \equiv 1 \pmod 2 & \& & k \equiv 2 \pmod 3 & \Longrightarrow & 3 \mid (P_k \cdot 2^n - 1) \\
n \equiv 6 \pmod{10} & \& & k \equiv 2 \pmod{11} & \Longrightarrow & 11 \mid (P_k \cdot 2^n - 1) \\
n \equiv 4 \pmod{12} & \& & k \equiv 4 \text{ or } 5 \pmod{13} & \Longrightarrow & 13 \mid (P_k \cdot 2^n - 1) \\
n \equiv 12 \pmod{18} & \& & k \equiv 15 \text{ or } 17 \pmod{19} & \Longrightarrow & 19 \mid (P_k \cdot 2^n - 1) \\
n \equiv 24 \pmod{36} & \& & k \equiv 29 \text{ or } 33 \pmod{37} & \Longrightarrow & 37 \mid (P_k \cdot 2^n - 1) \\
n \equiv 10 \pmod{20} & \& & k \equiv 19 \text{ or } 36 \pmod{41} & \Longrightarrow & 41 \mid (P_k \cdot 2^n - 1) \\
n \equiv 58 \pmod{60} & \& & k \equiv 50 \text{ or } 52 \pmod{61} & \Longrightarrow & 61 \mid (P_k \cdot 2^n - 1) \\
n \equiv 6 \pmod{36} & \& & k \equiv 83 \text{ or } 99 \pmod{109} & \Longrightarrow & 109 \mid (P_k \cdot 2^n - 1) \\
n \equiv 2 \pmod 3 & \& & k \equiv 6 \pmod 7 & \Longrightarrow & 7 \mid (P_k \cdot 2^n - 1) \\
n \equiv 0 \pmod 9 & \& & k \equiv 1 \text{ or } 48 \pmod{73} & \Longrightarrow & 73 \mid (P_k \cdot 2^n - 1) \\
n \equiv 4 \pmod 5 & \& & k \equiv 22 \text{ or } 30 \pmod{31} & \Longrightarrow & 31 \mid (P_k \cdot 2^n - 1) \\
n \equiv 7 \pmod{15} & \& & k \equiv 28 \text{ or } 73 \pmod{151} & \Longrightarrow & 151 \mid (P_k \cdot 2^n - 1)
\end{array}
$$

Table 6

Proof. Observe that the k^{th} s-gonal number is given by

$$S_k = S_k(s) = \frac{1}{2}k\big((s-2)k - (s-4)\big).$$

Using the congruences in Table 1, if $s \equiv 3 \pmod p$ for each p in the set

$$\mathcal{P} := \{3, 5, 7, 13, 17, 241\},$$

then we have

$$T_k \equiv S_k \pmod{\prod_{p \in \mathcal{P}} p}$$

and

$n \equiv 1 \pmod{2}$	&	$k \equiv 1 \pmod{3}$	$\Longrightarrow$	$3 \mid (S_k \cdot 2^n + 1)$
$n \equiv 0 \pmod{3}$	&	$k \equiv 3 \pmod{7}$	$\Longrightarrow$	$7 \mid (S_k \cdot 2^n + 1)$
$n \equiv 2 \pmod{4}$	&	$k \equiv 1$ or $3 \pmod{5}$	$\Longrightarrow$	$5 \mid (S_k \cdot 2^n + 1)$
$n \equiv 4 \pmod{8}$	&	$k \equiv 1$ or $15 \pmod{17}$	$\Longrightarrow$	$17 \mid (S_k \cdot 2^n + 1)$
$n \equiv 8 \pmod{12}$	&	$k \equiv 4$ or $8 \pmod{13}$	$\Longrightarrow$	$13 \mid (S_k \cdot 2^n + 1)$
$n \equiv 16 \pmod{24}$	&	$k \equiv 53$ or $187 \pmod{241}$	$\Longrightarrow$	$241 \mid (S_k \cdot 2^n + 1)$

Table 7

This implies that the expression $S_k \cdot 2^n + 1$ is composite for all positive integers n if k lies in the intersection of the congruence classes listed in the table above since the congruences for n form a covering of the integers. If we also include the congruence $k \equiv 1 \pmod{4}$, then the resulting polygonal number S_k is odd since $k = 4\ell + 1$ implies

$$S_k = (4\ell + 1)(2\ell s - 4\ell + 1).$$

The conclusion follows. □

Using the same technique, we also deduce the following results.

Theorem 11. *For infinitely many values of s, there are infinitely many s-gonal numbers that are Riesel numbers.*

Theorem 12. *For infinitely many values of s, there are infinitely many s-gonal numbers that are Sierpiński-Riesel numbers.*

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