Journal of Integer Sequences, Vol. 17 (2014),
Article 14.10.3

# Pattern Popularity in Multiply Restricted Permutations

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**Abstract**

We derive explicit formulae or generating functions for the popularity of all the length-3 patterns in multiply restricted permutations, and provide combinatorial interpretations for some non-trivial equipopular patterns as well.

## 1 Introduction

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in the symmetric group $S_n$. We say that $\sigma$ *contains a pattern* $q = q_1q_2\cdots q_k \in S_k$ if there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the entries $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ have the same relative order as the entries of $q$, i.e., $q_j < q_l$ if and only if $\sigma_{i_j} < \sigma_{i_l}$ whenever $1 \le j, l \le k$. We say that $\sigma$ avoids $q$ if $\sigma$ does not contain $q$ as a pattern. A permutation may contain multiple copies of a pattern. For example, permutation 43512 contains two copies of pattern 321, namely 431 and 432, but avoids pattern 123.

For a pattern $q$, let $S_n(q)$ denote the set of all permutations in $S_n$ that avoid the pattern $q$, and for $R \subseteq S_k$, let $S_n(R) = \bigcap_{q\in R} S_n(q)$ be the set of permutations in $S_n$ that avoid every pattern contained in $R$. For two permutations $\sigma$ and $q$, we set $f_q(\sigma)$ to be the number of copies of $q$ in $\sigma$ as a pattern. The *popularity* of pattern $q$ in $S_n(R)$ is defined as

$$f_q(S_n(R)) = \sum_{\sigma\in S_n(R)} f_q(\sigma).$$

| | | | |
|---|---|---|---|
| | $f_{213}(n) = (n-3)2^{n-2} + 1$ | | $\sum f_{231}(n)x^n = \sum f_{312}(n)x^n = \frac{x^3(1+2x)}{(1-x-x^2)^3}$ |
| $S_n(123, 132)$ | $f_{231}(n) = f_{312}(n) = (n^2 - 5n + 8)2^{n-3} - 1$ | $S_n(123, 132, 213)$ | $\sum f_{321}(n)x^n = \frac{x^3(1+6x+12x^2+8x^3)}{(1-x-x^2)^4}$ |
| | $f_{321}(n) = (n^3/3 - 2n^2 + 14n/3 - 5)2^{n-2} + 1$ | | $f_{213}(n) = f_{312}(n) = \binom{n}{3}$ |
| | $f_{123}(n) = (n-4)2^{n-1} + n + 2$ | $S_n(123, 132, 231)$ | $f_{321}(n) = (n-2)\binom{n}{3}$ |
| $S_n(132, 213)$ | $f_{231}(n) = f_{312}(n) = (\frac{n^2}{4} - \frac{7n}{4} + 4)2^n - n - 4$ | | $f_{123}(n) = f_{312}(n) = \binom{n+1}{4}$ |
| | $f_{321}(n) = (\frac{1}{12}n^3 - \frac{3}{4}n^2 + \frac{38}{12}n - 6)2^n + n + 6$ | $S_n(132, 213, 231)$ | $f_{321}(n) = \frac{1}{12}n(n-2)(n-1)^2$ |
| $S_n(132, 231)$ | $f_{123}(n) = f_{213}(n) = f_{312}(n) = f_{321}(n) = \frac{2^n}{8}\binom{n}{3}$ | | $f_{213}(n) = f_{231}(n) = \binom{n}{3}$ |
| $S_n(132, 312)$ | $f_{123}(n) = f_{213}(n) = f_{231}(n) = f_{321}(n) = \frac{2^n}{8}\binom{n}{3}$ | $S_n(123, 132, 312)$ | $f_{321}(n) = (n-2)\binom{n}{3}$ |
| | $f_{213}(n) = f_{231}(n) = f_{312}(n) = \binom{n+2}{5}$ | | $f_{132}(n) = f_{213}(n) = \binom{n+1}{4}$ |
| $S_n(132, 321)$ | $f_{123}(n) = \frac{7n^5}{120} - \frac{n^4}{3} + \frac{17n^3}{24} - \frac{2n^2}{3} + \frac{7}{30}$ | $S_n(123, 231, 312)$ | $f_{321}(n) = \frac{1}{12}n(n-2)(n-1)^2$ |

Table 1: Pattern popularity in doubly and triply restricted permutations.

We say that $p$ and $q$ are equipopular if $f_p(S_n(R)) = f_q(S_n(R))$ for all $n$.

The *complement* of $\sigma$ is given by $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$, its *reverse* is defined as $\sigma^r = \sigma_n \cdots \sigma_2\sigma_1$ and the *inverse* $\sigma^{-1}$ is the regular group-theoretic inverse permutation. For any set of permutations $R$, let $R^c$ be the set obtained by complementing each element of $R$, and the sets $R^r$ and $R^{-1}$ are defined analogously. It is well known that

**Lemma 1.** *Let $R \subseteq S_k$ be any set of permutations in $S_k$, and $\sigma \in S_n$, we have*

$$\sigma \in S_n(R) \Leftrightarrow \sigma^c \in S_n(R^c) \Leftrightarrow \sigma^r \in S_n(R^r) \Leftrightarrow \sigma^{-1} \in S_n(R^{-1}).$$

Cooper [6] first raised the problem of determining the total number $f_q(S_n(r))$, and Bóna [2] derived the generating function of the sequence $(f_q(S_n(132)))_{n\geq 1}$ for monotone pattern, i.e., $q = 12\cdots k$ or $q = k(k-1)\cdots 21$. Further, Bóna [3] studied the generating functions for other length-3 patterns in $S_n(132)$, and showed both algebraically and bijectively that

$$f_{231}(S_n(132)) = f_{312}(S_n(132)) = f_{213}(S_n(132)).$$

According to the correspondence between 132-avoiding permutations and binary plane trees, Rudolph [13] showed that patterns of equal length are equipopular if their associated binary plane trees have identical spine structure. For the converse direction, Chua and Sankar [4] gave a complete classification of 132-avoiding permutations into equipopularity classes. Moreover, Homberger [9] presented exact formulae for the occurrences of each length-3 pattern in $S_n(123)$. From Lemma 1 and the existing results on $S_n(123)$ and $S_n(132)$, we can obtain the popularity of each length-3 pattern for the singly restricted permutations $S_n(r)$ with $r = 213, 231, 312, 321$. Therefore, it is well-studied for the popularity of length-3 patterns in singly restricted permutations, whereas it remains open for multiply restricted permutations.

In this paper, we focus on counting the number of occurrences of length-3 patterns in multiply restricted permutations $S_n(R)$ for $R \subset S_3$, especially for double and triple restrictions.

We obtain exact formulae or generating functions for popularity of each length-3 pattern, and the detailed results are summarized in Table 1. Moreover, we present combinatorial proofs for non-trivial equalities between the number of occurrences of different patterns. It is routine to consider the restricted permutations of higher multiplicity since there are only finite permutations, as shown in [14, Proposition 17]. Therefore, this work gives a complete study on the popularity of length-3 patterns in the multiply restricted permutations. For the distributions of other statistics in multiply restricted permutations, see [7, 8, 10, 11, 12].

# 2 Doubly restricted permutations

This section deals with the enumeration of the popularity for length-3 patterns in the doubly restricted permutations, i.e., permutations avoiding two different patterns in $S_3$. For doubly restricted permutations, we have the following proposition from [14].

**Proposition 2.** ([14, Lemma 5]) *For every symmetric group $S_n$,*

1. $|S_n(123,132)| = |S_n(123,213)| = |S_n(231,321)| = |S_n(312,321)| = 2^{n-1}$;
2. $|S_n(132,213)| = |S_n(231,312)| = 2^{n-1}$;
3. $|S_n(132,231)| = |S_n(213,312)| = 2^{n-1}$;
4. $|S_n(132,312)| = |S_n(213,231)| = 2^{n-1}$;
5. $|S_n(132,321)| = |S_n(123,231)| = |S_n(123,312)| = |S_n(213,321)| = \binom{n}{2} + 1$;
6. $|S_n(123,321)| = 0$ *for* $n \geq 5$.

Thus it is sufficient to consider the pattern popularity for the first set from class 1 to class 5, and the pattern popularity for the other sets can be derived by taking complement, reverse or inverse.

A *composition* of $n$ is an expression of $n$ as an ordered sum of positive integers, and we say that $c$ has $k$ parts or $c$ is a $k$-composition if there are exactly $k$ summands appeared in composition $c$. Let $\mathcal{C}_n$ and $\mathcal{C}_{n,k}$ denote the set of all compositions of $n$ and the set of $k$-compositions of $n$, respectively. It is known that $|\mathcal{C}_0| = 1$, and for $n \geq 1$, $1 \leq k \leq n$, $|\mathcal{C}_n| = 2^{n-1}$ and $|\mathcal{C}_{n,k}| = \binom{n-1}{k-1}$. For more details on compositions, see [16]. It is helpful to introduce a lemma as follows:

**Lemma 3.** *For $n \geq 1$, we have*

$$\begin{aligned}
a(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} c_k = 2^n - 1, \\
b(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} c_k(c_k-1) = 2^{n+1} - 2n - 2, \\
c(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} k = (n+1)2^{n-2}, \\
d(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} k(k-1) = (n^2+n-2)2^{n-3},
\end{aligned}$$

*where the sums are taken over all compositions of $n$.*

*Proof.* For $c_k = m$ , we can regard $c_1 + c_2 + \cdots + c_{k-1}$ as a composition of $n - m$. Since the number of compositions of $n - m$ is $2^{n-m-1}$ for $1 \leq m \leq n-1$ and the number of compositions of $n$ with $k$ parts is $\binom{n-1}{k-1}$, we have

$$a(n) = n + \sum_{m=1}^{n-1} m2^{n-m-1}, \quad b(n) = n(n-1) + \sum_{m=1}^{n-1} m(m-1)2^{n-m-1},$$

and

$$c(n) = \sum_{k=1}^{n} k\binom{n-1}{k-1}, \quad d(n) = \sum_{k=1}^{n} k(k-1)\binom{n-1}{k-1}.$$

Let $g(x) = \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$ and $h(x) = x\sum_{i=1}^{n}\binom{n-1}{i-1}x^{i-1} = x(1+x)^{n-1}$. We have

$$\begin{aligned}
g'(x) &= \sum_{i=1}^{n-1} ix^{i-1} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}, \\
g''(x) &= \sum_{i=1}^{n-1} i(i-1)x^{i-2} = \frac{(3n-n^2-2)x^n + (2n^2-4n)x^{n-1} + (n-n^2)x^{n-2} + 2}{(1-x)^3}, \\
h'(x) &= \sum_{i=1}^{n} i\binom{n-1}{i-1}x^{i-1} = (nx+1)(1+x)^{n-2}, \\
h''(x) &= \sum_{i=1}^{n} i(i-1)\binom{n-1}{i-1}x^{i-2} = \left[n^2x + n(2-x) - 2\right](1+x)^{n-3}.
\end{aligned}$$

It follows that

$$a(n) = 2^{n-2}g'(1/2) + n, \quad b(n) = 2^{n-3}g''(1/2) + n(n-1),$$

and

$$c(n) = h'(1), \quad d(n) = h''(1).$$

Lemma 3 holds by simple computations. $\square$

## 2.1 Pattern popularity in $(123, 132)$-avoiding permutations

In this subsection, we calculate the popularity of all length-3 patterns in $S_n(123, 132)$. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, $\sigma_i$ is said to be a *left-to-right maximum* (resp., *right-to-left maximum*) if $\sigma_i > \sigma_j$ for all $j < i$ (resp., $j > i$). We first recall a correspondence between $S_n(123, 132)$ and $\mathcal{C}_n$ as implicitly shown in [10].

**Lemma 4.** ([10, Theorem 3]) *There is a bijection $\varphi_1$ between $S_n(123, 132)$ and $\mathcal{C}_n$.*

*Proof.* Given $\sigma \in S_n(123, 132)$, let $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ be the $k$ right-to-left maxima with $i_1 < i_2 < \cdots < i_k$. Then $c = i_1 + (i_2 - i_1) + \cdots + (i_{k-1} - i_{k-2}) + (i_k - i_{k-1})$ is a composition of $n$ since $i_k = n$. On the converse, let $m_i = n - (c_1 + \cdots + c_{i-1})$ for any given composition $n = c_1 + c_2 + \cdots + c_k \in \mathcal{C}_n$. Set $\tau_i = m_i - 1, m_i - 2, \ldots, m_i - c_i + 1, m_i$ for $1 \le i \le k$. It is easy to check that $\sigma = \tau_1\,\tau_2\,\cdots\,\tau_k \in S_n(123, 132)$. $\square$

For example, $\sigma = 8\,9\,7\,5\,4\,3\,6\,1\,2$ corresponds to the composition $9 = 2 + 1 + 4 + 2$.

Given a pattern $q$, for simplicity, let $f_q(n) := \sum_{\sigma\in S_n(123,132)} f_q(\sigma)$ be the number of occurrences of pattern $q$ in $S_n(123, 132)$, and we will use this notation in subsequent sections when the set in question is unambiguous. A *factor* of $\sigma$ is a subsequence consisting of contiguous letters in $\sigma$. From Lemma 4, we have

**Proposition 5.** *For $n \ge 3$,*

$$f_{213}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i - 1}{2}, \tag{1}$$

$$f_{231}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j(c_i - 1). \tag{2}$$

*Proof.* For each permutation $\sigma \in S_n(123, 132)$ with $\varphi_1(\sigma) = c_1 + c_2 + \cdots + c_k$, we can rewrite $\sigma$ as $\sigma = \tau_1\,\tau_2\,\cdots\,\tau_k$ from Lemma 4. We say that $\tau_i > \tau_j$ if all the elements in $\tau_i$ are larger than that in $\tau_j$. We see that the pattern 213 can only occur in every factor $\tau_i$ since the elements except the last one are decreasing in $\tau_i$ and $\tau_i > \tau_j$ for $j > i$. Thus, there are $\binom{c_i-1}{2}$ choices to select two elements in $\tau_i$ to play the role of "21", and the last element of $\tau_i$ plays the role of "3". If $c_i \le 2$, then there is no copy of the pattern 213 in $\tau_i$, this coincides with the value $\binom{c_i-1}{2} = 0$ for $c_i = 1$ or 2. Summing up all the number of 213-patterns in factors $\tau_1, \tau_2, \ldots, \tau_k$ yields formula (1).

For pattern 231, we have $c_i - 1$ choices in factor $\tau_i$ to select one element to play the role of "2" and one choice (always the last element of $\tau_i$) for "3". After this, we have $c_{i+1} + \cdots + c_k$ choices to select one element in $\tau_{i+1}, \ldots, \tau_k$ for the role of "1" since all the elements after $\tau_i$ are smaller than those in $\tau_i$. Summing up all the number of 231-patterns according to the position of "3" gives formula (2). $\square$

**Theorem 6.** *For $n \geq 3$, in the set $S_n(123, 132)$, we have*

$$\begin{aligned} f_{213}(n) &= (n-3)2^{n-2} + 1, && (3) \\ f_{231}(n) &= f_{312}(n) = (n^2 - 5n + 8)2^{n-3} - 1, && (4) \\ f_{321}(n) &= (n^3/3 - 2n^2 + 14n/3 - 5)2^{n-2} + 1. && (5) \end{aligned}$$

*Proof.* From $S_3(123, 132) = \{213, 231, 312, 321\}$, we have

$$f_{213}(3) = f_{231}(3) = 1.$$

To prove formula (3), Proposition 5 gives that, for $n \geq 3$,

$$f_{213}(n+1) = \sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2} + \sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2}.$$

If $c_k = 1$, then $k \geq 2$, and we have

$$\sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2} = \sum_{c_1+c_2+\cdots+c_{k-1}=n} \sum_{i=1}^{k-1} \binom{c_i - 1}{2} = f_{213}(n).$$

If $c_k \geq 2$, then we set $c_k = 1 + r_k$ with $r_k \geq 1$. From Lemma 3, we find that

$$\begin{aligned} \sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2} &= \sum_{c_1+\cdots+c_{k-1}+r_k=n} \left[ \sum_{i=1}^{k-1} \binom{c_i - 1}{2} + \binom{r_k - 1}{2} + (r_k - 1) \right] \\ &= f_{213}(n) + \sum_{c_1+\cdots+c_{k-1}+r_k=n} (r_k - 1) \\ &= f_{213}(n) + a(n) - 2^{n-1}. \end{aligned}$$

Combining the above two cases, we have

$$f_{213}(n+1) = 2f_{213}(n) + 2^{n-1} - 1,$$

which proves formula (3) by solving the recurrence with initial value $f_{213}(3) = 1$.

For formula (4), we first have $f_{231}(n) = f_{312}(n)$ from $231^{-1} = 312$ and $\sigma \in S_n(123, 132) \Leftrightarrow \sigma^{-1} \in S_n(123, 132)$. Using the same method as in the proof of formula (3), we can show

$$\sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j(c_i - 1) = f_{231}(n) - c(n) + n2^{n-1},$$

and

$$\sum_{\substack{c_k\geq 2\\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} c_j(c_i-1) = f_{231}(n) - a(n) - c(n) + (n+1)2^{n-1}.$$

It follows that, from Lemma 3,

$$f_{231}(n+1) = 2f_{231}(n) + (n-2)2^{n-1} + 1.$$

Formula (4) is proved by solving this recurrence using $f_{231}(3) = 1$.

Since the total number of all length-3 patterns in a permutation $\sigma \in S_n$ is $\binom{n}{3}$, we have

$$f_{213}(n) + 2f_{231}(n) + f_{321}(n) = \binom{n}{3}2^{n-1},$$

and formula (5) holds. □

The first few values of $f_q(S_n(123, 132))$ for $q$ of length 3 are shown below. Moreover, we observe that they appear in the On-Line Encyclopedia of Integer Sequences [15] as follows: $(f_{213}(n))_{n\geq 3}$ form sequence A000337, $(f_{231}(n))_{n\geq 3}$ form sequence A055580.

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 1 | 1 | 1 | 1 | 6 | 0 | 0 | 49 | 111 | 111 | 369 |
| 4 | 0 | 0 | 5 | 7 | 7 | 13 | 7 | 0 | 0 | 129 | 351 | 351 | 1409 |
| 5 | 0 | 0 | 17 | 31 | 31 | 81 | 8 | 0 | 0 | 321 | 1023 | 1023 | 4801 |

## 2.2 Pattern popularity in $(132, 213)$-avoiding permutations

We first recall a correspondence between $S_n(132, 213)$ and $\mathcal{C}_n$ as follows:

**Lemma 7.** ([10, Theorem 8]) *There is a bijection $\varphi_2$ between $S_n(132, 213)$ and $\mathcal{C}_n$.*

*Proof.* Given $\sigma \in S_n(132, 213)$, let $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ be the $k$ right-to-left maxima with $i_1 < i_2 < \cdots < i_k$. It follows that $c = i_1 + (i_2 - i_1) + \cdots + (i_{k-1} - i_{k-2}) + (i_k - i_{k-1})$ is a composition of $n$ since $i_k = n$. On the converse, given a composition $n = c_1 + c_2 + \cdots + c_k \in \mathcal{C}_n$, let $m_i = n - (c_1 + \cdots + c_{i-1})$ and $\tau_i = m_i - c_i + 1, m_i - c_i + 2, \ldots, m_i - 1, m_i$ for $1 \leq i \leq k$. Set $\sigma = \tau_1\,\tau_2\, \cdots\, \tau_k$, and it is easy to check that $\sigma \in S_n(132, 213)$. □

For example, for the composition $9 = 3 + 3 + 1 + 2$, we get $\sigma = 7\,8\,9\,4\,5\,6\,3\,1\,2$. From this lemma, we have

**Proposition 8.** *For $n \geq 3$,*

$$f_{123}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{3}, \tag{6}$$

$$f_{231}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}. \tag{7}$$

*Proof.* For a permutation $\sigma \in S_n(132, 213)$ with $\varphi_2(\sigma) = c_1 + c_2 + \cdots + c_k$, we rewrite $\sigma$ as $\sigma = \tau_1 \tau_2 \cdots \tau_k$. The pattern 123 can only occur in every factor $\tau_i$ as $\tau_i > \tau_j$ for $j > i$ and the elements in $\tau_i$ are increasing. Thus, we have $\binom{c_i}{3}$ choices to select three elements in $\tau_i$ to play the role of "123", and formula (6) follows by summing up all 123-patterns in factors $\tau_1, \tau_2, \ldots, \tau_k$.

For the pattern 231, we have $\binom{c_i}{2}$ choices in factor $\tau_i$ to select two elements to play the role of "23". After this, we have $c_{i+1} + \cdots + c_k$ choices to select one element in $\tau_{i+1}, \ldots, \tau_k$ for the role of "1" since $\tau_j < \tau_i$ for all $j > i$. Summing up all the number of 231-patterns according to the position of "23" gives formula (7). $\square$

**Theorem 9.** *For $n \geq 3$, in the set $S_n(132, 213)$, we have*

$$f_{123}(n) = (n-4)2^{n-1} + n + 2, \tag{8}$$
$$f_{231}(n) = f_{312}(n) = (n^2 - 7n + 16)2^{n-2} - n - 4, \tag{9}$$
$$f_{321}(n) = (n^3/3 - 3n^2 + 38n/3 - 24)2^{n-2} + n + 6. \tag{10}$$

*Proof.* From Proposition 8, it follows that

$$f_{123}(n+1) = \sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i}{3} + \sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i}{3}.$$

An argument similar to the proof of Theorem 6 shows that

$$f_{123}(n+1) = 2f_{123}(n) + 2^n - n - 1.$$

Solving this recurrence with initial value $f_{123}(3) = 1$ leads to formula (8).

From Lemma 1, we see that $\sigma \in S_n(132, 213) \Leftrightarrow \sigma^{-1} \in S_n(132, 213)$, which implies $f_{231}(n) = f_{312}(n)$ as $231^{-1} = 312$.

To calculate $f_{231}(n)$, by Proposition 8, we arrive at

$$f_{231}(n+1) = \sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} + \sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}.$$

If $c_k = 1$, then $k \geq 2$, and we have

$$\sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} = f_{231}(n) + \alpha(n),$$

where

$$\begin{aligned}
\alpha(n) &= \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{2} = \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k} \left[\binom{c_i-1}{2} + c_i - 1\right] \\
&= f_{213}(S_n(123,132)) + \sum_{c_1+\cdots+c_k=n} (n-k) \\
&= f_{213}(S_n(123,132)) - c(n) + n2^{n-1}.
\end{aligned}$$

Here we have used the deduced expression (1).
If $c_k \geq 2$, then we can derive that

$$\sum_{\substack{c_k\geq 2 \\ c_1+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} = f_{231}(n) + \beta(n),$$

where

$$\begin{aligned}
\beta(n) &= \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k-1} \binom{c_i}{2} \\
&= \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{2} - \sum_{c_1+\cdots+c_k=n} \frac{c_k(c_k-1)}{2} = \alpha(n) - b(n)/2.
\end{aligned}$$

From Lemma 3, we get

$$f_{231}(n+1) = 2f_{231}(n) + (2n-6)2^{n-1} + n + 3.$$

Formula (9) holds by solving this recurrence with initial condition $f_{213}(3) = 1$.

Finally, formula (10) follows from $f_{123}(n) + 2f_{231}(n) + f_{321}(n) = \binom{n}{3}2^{n-1}$. □

The first few values of $f_q(S_n(132,213))$ for $q$ of length 3 are shown below. They appear in [15] as follows: $(f_{123}(n))_{n\geq 3}$ form sequence A045618, $(f_{231}(n))_{n\geq 3}$ form sequence A055581 and $(f_{321}(n))_{n\geq 3}$ form sequence A055586.

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 0 | 1 | 1 | 1 | 6 | 72 | 0 | 0 | 150 | 150 | 268 |
| 4 | 6 | 0 | 0 | 8 | 8 | 10 | 7 | 201 | 0 | 0 | 501 | 501 | 1037 |
| 5 | 23 | 0 | 0 | 39 | 39 | 59 | 8 | 522 | 0 | 0 | 1524 | 1524 | 3598 |

## 2.3 Pattern popularity in $(132, 231)$-avoiding permutations

For each $\sigma \in S_n(132, 231)$, we observe that $n$ must lie in the beginning or the end of $\sigma$, and $n-1$ must lie in the beginning or the end of $\sigma\backslash\{n\}$,..., and so on. Here $\sigma\backslash\{n\}$ denotes the sequence obtained from $\sigma$ by deleting the element $n$. In view of such special structure, we can derive the pattern popularity in $(132, 231)$-avoiding permutations directly.

**Theorem 10.** *For $n \geq 3$, in the set $S_n(132, 231)$, we have*

$$f_{123}(n) = f_{213}(n) = f_{312}(n) = f_{321}(n) = \binom{n}{3} 2^{n-3}. \tag{11}$$

*Proof.* Suppose that $q$ is a length-3 pattern in $\{123, 213, 312, 321\}$, and $abc$ is a copy of the pattern $q$. Set

$$[n]\backslash\{a, b, c\} := \{r_1 > r_2 > \cdots > r_{n-4} > r_{n-3}\}.$$

We will construct a permutation in the set $S_n(132, 231)$ which contains $abc$ as a copy of the pattern $q$. Start with the subsequence $\sigma^0 := abc$, and for $i$ from 1 to $n-3$, $\sigma^i$ is obtained by inserting $r_i$ into $\sigma^{i-1}$ such that

- If there are at least two elements in $\sigma^{i-1}$ that are smaller than $r_i$, then choose the two elements $A$ and $B$ such that $A$ is the leftmost one and $B$ is the rightmost one. We put $r_i$ immediately to the left of $A$ or immediately to the right of $B$;

- If there is only one element $A$ in $\sigma^{i-1}$ such that $A < r_i$, then we put $r_i$ immediately to the left or to the right of $A$;

- If all the elements in $\sigma^{i-1}$ are larger than $r_i$, then choose $A$ the smallest one, and put $r_i$ immediately to the left or to the right of $A$.

Finally, we set $\sigma := \sigma^{n-3}$ and $\sigma \in S_n(132, 231)$ from the above construction. It can be seen that, the number of permutations having a copy $abc$ is $2^{n-3}$ since each $r_i$ has 2 choices in the inserting procedure. Moreover, there are $\binom{n}{3}$ choices to select three elements $a, b, c$ as an appearance of the pattern $q$ in $\{123, 213, 312, 321\}$. Hence we deduce $f_q(n) = \binom{n}{3} 2^{n-3}$. $\square$

Here we give an illustration for constructing a permutation in $S_8(132, 231)$ which contains $abc = 256$ as a copy of the pattern 123. Set $\sigma^0 := 2\,5\,6$, we may have $\sigma^1 = 8\,2\,5\,6$, $\sigma^2 = 8\,7\,2\,5\,6$, $\sigma^3 = 8\,7\,2\,4\,5\,6$, $\sigma^4 = 8\,7\,3\,2\,4\,5\,6$, $\sigma := \sigma^5 = 8\,7\,3\,2\,1\,4\,5\,6$.

We can also give a combinatorial proof for Theorem 10. Since $\sigma \in S_n(132, 231) \Leftrightarrow \sigma^r \in S_n(132, 231)$, it is easy to show $f_{123}(n) = f_{321}(n)$ and $f_{213}(n) = f_{312}(n)$ from $123^r = 321$ and $213^r = 312$. It remains to give a bijection for $f_{213}(n) = f_{123}(n)$, and our construction is motivated from Bóna [3].

We first introduce some notation about trees. A *binary plane tree* is a rooted unlabelled tree in which each vertex has at most two children, and each child is a left child or a right child of its parent. For each $\sigma \in S_n(132)$, we can construct a binary plane tree $T(\sigma)$ as follows:

the root of $T(\sigma)$ corresponds to the entry $n$ of $\sigma$, the left subtree of the root corresponds to the string of entries of $\sigma$ on the left of $n$, and the right subtree of the root corresponds to the string of entries of $\sigma$ on the right of $n$. Both subtrees are constructed recursively by the same rule. For more details, see [1, 3, 13].

A *left descendant* (resp., *right descendant*) of a vertex $x$ in a binary plane tree is a vertex in the left (resp., right) subtree of $x$. Similarly, an *ascendant* of a vertex $x$ in a binary plane tree is a vertex whose subtree contains $x$. Given a tree $T$ and a vertex $v \in T$, let $T_v$ be the subtree of $T$ rooted at $v$. Let $R$ be an occurrence of the pattern 123 in $\sigma \in S_n(132)$, and let $R_1, R_2, R_3$ be the three vertices of $T(\sigma)$ that correspond to $R$, going left to right. Then, $R_1$ is a left descendant of $R_2$, and $R_2$ is a left descendant of $R_3$.

According to the correspondence between 132-avoiding permutations and binary plane trees, we see that for $\sigma \in S_n(132, 231)$, $T(\sigma)$ is a binary plane tree on $n$ vertices such that each vertex has at most one child from the forbiddance of the pattern 231. For simplicity, let $\mathcal{T}_n$ be the set of such binary plane trees on $n$ vertices. Let $Q$ be an occurrence of the pattern 213 in $\sigma \in S_n(132, 231)$, and let $Q_2, Q_1, Q_3$ be the three vertices of $T(\sigma)$ that correspond to $Q$, going left to right. From the characterization of trees in $\mathcal{T}_n$, $Q_2$ is a left descendant of $Q_3$, and $Q_1$ is a right descendant of $Q_2$.

*Combinatorial proof for $f_{213}(n) = f_{123}(n)$.* Let $\mathcal{A}_n$ be the set of binary plane trees in $\mathcal{T}_n$ where three vertices forming a 213-pattern are colored black. Let $\mathcal{B}_n$ be the set of all binary plane trees in $\mathcal{T}_n$ where three vertices forming a 123-pattern are colored black. We define a map $\rho : \mathcal{A}_n \to \mathcal{B}_n$ as follows.

Given a tree $T \in \mathcal{A}_n$ with $Q_2, Q_1, Q_3$ being the three black vertices as a 213-pattern, we define $\rho(T)$ be the tree obtained from $T$ by changing the right subtree of $Q_2$ to be its left subtree. See Figure 1 for an illustration.

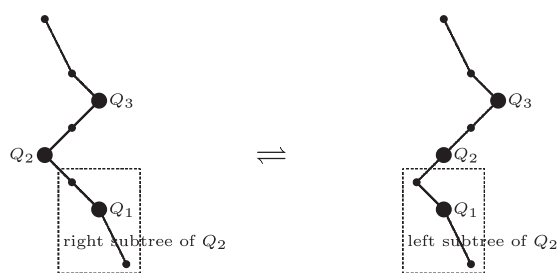


Figure 1: The bijection $\rho$.

In the tree $\rho(T)$, the relative positions of $Q_2$ and $Q_3$ keep the same, and $Q_1$ is a left descendant of $Q_2$. Therefore, points $Q_1Q_2Q_3$ form a 123-pattern in $\rho(T)$, and $\rho(T) \in \mathcal{B}_n$. On the converse, it is routine to verify that changing left subtree of $Q_2$ to be its right subtree is the desired reverse map. Therefore, $\rho$ is a bijection between $\mathcal{A}_n$ and $\mathcal{B}_n$.

The initial values for $f_q(S_n(132, 231))$ are

$$1, 8, 40, 160, 560, 1792, \ldots,$$

and this is essentially the sequence A001789 in [15].

## 2.4 Pattern popularity in $(132, 312)$-avoiding permutations

We first present a lemma as follows:

**Lemma 11.** *There is a bijection $\varphi_4$ between $S_n(132, 312)$ and $\mathcal{C}_n$.*

*Proof.* For $\sigma \in S_n(132, 312)$, let $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ be the $k$ left-to-right maxima with $i_1 < i_2 < \cdots < i_k$. Then $c = (i_2 - i_1) + (i_3 - i_2) + \cdots + (i_k - i_{k-1}) + (n+1-i_k)$ is a composition of $n$ since $i_1 = 1$. On the converse, let $n = c_k + c_{k-1} + \cdots + c_2 + c_1 \in \mathcal{C}_n$. For $1 \le i \le k$, if $c_i = 1$ then set $\tau_i = n-i+1$; otherwise, set $m_i = c_1 + \cdots + c_{i-1} - i + 2$ and $\tau_i = n-i+1, m_i + c_i - 2, \ldots, m_i + 1, m_i$. It is easy to get $\sigma = \tau_k\, \tau_{k-1} \,\cdots\, \tau_2\, \tau_1 \in S_n(132, 312)$. □

For example, if $9 = 3 + 1 + 2 + 3$, then $\sigma = 6\,5\,4\,7\,8\,3\,9\,2\,1$.

**Proposition 12.** *For $n \ge 3$,*

$$f_{123}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-2} c_i \binom{k-i}{2}. \tag{12}$$

*Proof.* Let $\sigma = \tau_k \,\cdots\, \tau_2\, \tau_1$ be a permutation in $S_n(132, 312)$ whose composition is given by $n = c_k + c_{k-1} + \cdots + c_2 + c_1$. It is evident that, for $i + 1 \le j \le k$, the first element in $\tau_i$ is larger than all the elements in $\tau_j$, whereas the other elements in $\tau_i$ are smaller than that in $\tau_j$. Furthermore, the left-to-right maxima form an increasing subsequence and the other elements form a decreasing subsequence. Thus we have $c_i$ choices to select one element in $\tau_i$ to play the role of "1", and then $\binom{i-1}{2}$ choices to select two left-to-right maxima after $\tau_i$ to paly the role of "23". Summing up all the number of 123-patterns in factors $\tau_k, \ldots, \tau_2, \tau_1$ yields that

$$f_{123}(n) = \sum_{c_k+\cdots+c_2+c_1=n} \sum_{i=3}^{k} c_i \binom{i-1}{2}.$$

By setting $i := k - i + 1$ and using the symmetry of the summands in compositions, it is equivalent to formula (12). □

**Theorem 13.** *For $n \ge 3$, in the set $S_n(132, 312)$, we have*

$$f_{123}(n) = f_{321}(n) = \binom{n}{3} 2^{n-3}, \tag{13}$$

$$f_{213}(n) = f_{231}(n) = \binom{n}{3} 2^{n-3}. \tag{14}$$

*Proof.* From Lemma 1, we know that $\sigma \in S_n(132, 312) \Leftrightarrow \sigma^c \in S_n(132, 312)$. Hence it is obvious that $f_{123}(n) = f_{321}(n)$ and $f_{213}(n) = f_{231}(n)$ as $123^c = 321$ and $213^c = 231$.

To calculate $f_{123}(n)$, by using Proposition 12 and the similar argument in the proof of Theorem 6, we have

$$f_{123}(n+1) = 2f_{123}(n) + (n^2 - n)2^{n-3}.$$

Formula (13) holds by solving the recurrence with initial value $f_{123}(3) = 1$, and formula (14) is a direct computation of $2f_{123}(n) + 2f_{213}(n) = \binom{n}{3}2^{n-1}$. $\square$

We will give a combinatorial interpretation for $f_{231}(n) = f_{123}(n)$. For each $\sigma \in S_n(132, 312)$, we construct a binary plane tree $T(\sigma)$ on $n$ vertices such that each vertex with a right descendant of some vertex does not have a left descendant from the forbiddance of the pattern 312. Let $\mathscr{T}_n$ denote the set of such trees on $n$ vertices. Let $Q$ be an occurrence of the pattern 231 in $\sigma \in S_n(132, 312)$, and let $Q_2, Q_3, Q_1$ be the three vertices of $T(\sigma)$ that correspond to $Q$, going left to right. Then, $Q_2$ is a left descendant of $Q_3$, and there exists a lowest ascendant $x$ of $Q_3$ or $x = Q_3$ so that $Q_1$ is a right descendant of $x$.

*Combinatorial proof for $f_{231}(n) = f_{123}(n)$.* Let $\mathscr{A}_n$ be the set of binary plane trees in $\mathscr{T}_n$ in which three vertices forming a 231-pattern are colored black. Let $\mathscr{B}_n$ be the set of all binary plane trees in $\mathscr{T}_n$ in which three vertices forming a 123-pattern are colored black. We define a map $\varrho : \mathscr{A}_n \to \mathscr{B}_n$ as follows.

Given a tree $T \in \mathscr{A}_n$ with $Q_2, Q_3, Q_1$ being the three black vertices forming a 231-pattern, let $y$ be the parent of $x$ if it exists. We can see that $x$ is the left child of $y$ from $T \in \mathscr{A}_n$. Let $T^u := T - T_x$ be the tree obtained from $T$ by deleting the subtree $T_x$, and $T^d := T_x - T_{Q_1}$ be the tree obtained from $T_x$ by deleting $T_{Q_1}$. Now we define $\varrho(T)$ to be the tree obtained from $T$ by first adjoining $T_{Q_1}$ to the vertex $y$ as its left subtree, then adjoining $T^d$ to $Q_1$ as its left subtree and keeping all three black vertices the same if $y$ exits; otherwise, we adjoin $T^d$ to $Q_1$ as its left subtree directly. An illustration is given in Figure 2.

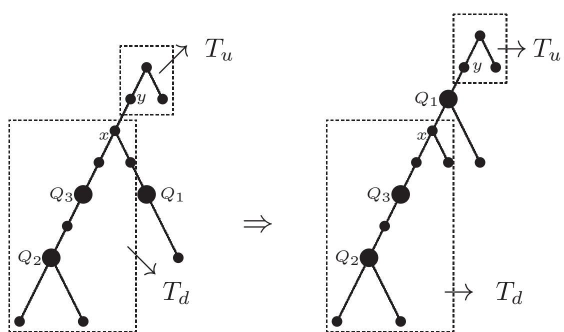


Figure 2: The bijection $\varrho$.

In the tree $\varrho(T)$, the relative positions of $Q_2$ and $Q_3$ are unchanged, and $Q_3$ is a left descendant of $Q_1$, thus the three black points $Q_2Q_3Q_1$ form a 123-pattern in $\varrho(T)$, and $\varrho(T) \in \mathscr{B}_n$. It is easy to describe the inverse map and we omit here.

## 2.5 Pattern popularity in $(132, 321)$-avoiding permutations

We first introduce a lemma as follows:

**Lemma 14.** [14, Proposition 13] *There is a bijection $\varphi_5$ between $S_n(132, 321)\backslash\{id\}$ and the set of 2-element subsets of $[n]$.*

*Proof.* For a permutation $\sigma \in S_n(132, 321)\backslash\{\mathrm{id}\}$, suppose $\sigma_k = m$ $(k < m)$ and define $\varphi_5(\sigma) = \{k, m\}$. On the converse, given two elements $1 \le k < m \le n$, set $\tau_1 = m-k+1, m-k+2, \ldots, m-1, m$, $\tau_2 = 1, 2, \ldots, m-k$ and $\tau_3 = m+1, m+2, \ldots, n-1, n$. We have $\sigma = \varphi_5^{-1}(k, m) = \tau_1\,\tau_2\,\tau_3$. ☐

For example, if $k = 4, m = 6$, then $\sigma = 3\,4\,5\,6\,1\,2\,7\,8$.

**Proposition 15.** *For $n \ge 3$,*

$$f_{213}(n) = \sum_{1 \le k < m \le n} k(m-k)(n-m), \tag{15}$$

$$f_{312}(n) = \sum_{1 \le k < m \le n} k\binom{m-k}{2}. \tag{16}$$

*Proof.* Given a permutation $\sigma = \tau_1\,\tau_2\,\tau_3$ in $S_n(132, 321)$ with $\varphi_5(\sigma) = \{k, m\}$, we see that the elements in each $\tau_i$ $(1 \le i \le 3)$ are increasing, and $\tau_2 < \tau_1 < \tau_3$. Hence we have $k$ choices to select one element in $\tau_1$ to play the role of "2", $m-k$ choices to select one element in $\tau_2$ to play the role of "1", and $n-m$ choices to select one element in $\tau_3$ to play the role of "3". Summing up all possible $k$ and $m$ gives formula (15).

For the pattern 312, we have $k$ choices to select one element in factor $\tau_1$ to play the role of "3", and then have $\binom{m-k}{2}$ choices to select two elements in factor $\tau_2$ to play the role of "12". Summing up all $k$ and $m$ proves formula (16). ☐

We now derive the exact formulae for the popularity of patterns in $S_n(132, 321)$ as follows.

**Theorem 16.** *For $n \ge 3$, in the set $S_n(132, 321)$, we have*

$$f_{213}(n) = f_{231}(n) = f_{312}(n) = \binom{n+2}{5}, \tag{17}$$

$$f_{123}(n) = n(7n^4 - 40n^3 + 85n^2 - 80n + 28)/120. \tag{18}$$

*Proof.* It is simple to prove $f_{312}(n) = f_{231}(n)$ from Lemma 1 and $312^{-1} = 231$. By Proposition 15, we have

$$
\begin{aligned}
f_{312}(n) &= \sum_{1\le k<m\le n} k\binom{m-k}{2} \\
&= \sum_{k=1}^{n-1} k \sum_{m=k+1}^{n} \binom{m-k}{2} = \sum_{k=1}^{n-1} k\binom{n-k+1}{3} \\
&= \sum_{k=1}^{n-1} \left[(n^3-n)k + (1-3n^2)k^2 + 2nk^3 - k^4\right],
\end{aligned}
$$

and

$$
\begin{aligned}
f_{213}(n) &= \sum_{1\le k<m\le n} k(m-k)(n-m) = \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} k(m-k)(n-m) \\
&= \sum_{k=1}^{n-1}\sum_{m'=1}^{n-k} km'(n-m'-k) = \sum_{k=1}^{n-1} k(n-k)\sum_{m'=1}^{n-k} m' - \sum_{k=1}^{n-1} k \sum_{m'=1}^{n-k} m'^2 \\
&= \sum_{k=1}^{n-1} \left[\left(\frac{n^3}{6}-\frac{n}{6}\right)k + \left(\frac{1}{6}-\frac{n^2}{2}\right)k^2 + \frac{n}{2}k^3 - \frac{1}{6}k^4\right].
\end{aligned}
$$

We get formula (17) by substituting the closed forms of $\sum_{k=1}^{n} k^p$ $(p = 1, 2, 3, 4)$ into the above expressions, and this theorem holds from $2f_{231}(n) + f_{213}(n) + f_{123}(n) = \binom{n}{3}\left[\binom{n}{2}+1\right]$. $\square$

Notice that $f_{213}(n) = f_{231}(n)$ can be proved by Bóna's bijection [3] on the set of binary plane trees on $n$ vertices such that the vertex which is a right descendant of some node has no right descendant.

The first few values of $f_q(S_n(132, 321))$ for $q$ of length 3 are shown below, and $(f_{213}(n))_{n\ge3}$ form sequence A000389 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 1 | 1 | 1 | 0 | 6 | 152 | 0 | 56 | 56 | 56 | 0 |
| 4 | 10 | 0 | 6 | 6 | 6 | 0 | 7 | 392 | 0 | 126 | 126 | 126 | 0 |
| 5 | 47 | 0 | 21 | 21 | 21 | 0 | 8 | 868 | 0 | 252 | 252 | 252 | 0 |

# 3 Triply restricted permutations

This section studies the pattern popularity in the permutations which avoid simultaneously any three patterns of length 3. We begin with the following proposition from [14].

**Proposition 17.** ([14, Lemma 6]) *The numbers of triply restricted permutations in $S_n$ satisfy the following equalities:*

1. $|S_n(123, 132, 213)| = |S_n(231, 312, 321)| = F_{n+1}$;
2. $|S_n(123, 132, 231)| = |S_n(123, 213, 312)| = |S_n(132, 231, 321)| = |S_n(213, 312, 321)| = n$;
3. $|S_n(132, 213, 231)| = |S_n(132, 213, 312)| = |S_n(132, 231, 312)| = |S_n(213, 231, 312)| = n$;
4. $|S_n(123, 132, 312)| = |S_n(123, 213, 231)| = |S_n(132, 312, 321)| = |S_n(213, 231, 321)| = n$;
5. $|S_n(123, 231, 312)| = |S_n(132, 213, 321)| = n$;
6. $|S_n(R)| = 0$ *for all* $R \supset \{123, 321\}$ *if* $n \geq 5$, *where* $F_n$ *is the Fibonacci number given by* $F_0 = 0, F_1 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$ *for* $n \geq 2$.

An argument similar to the one used for doubly restricted permutations shows that we only need to consider the pattern popularity for the first set of class 1 to class 5.

## 3.1 Pattern popularity in $(123, 132, 213)$-avoiding permutations

It is well-known that Fibonacci number $F_{n+1}$ counts the number of 0-1 sequences of length $n-1$ in which there are no consecutive ones, see [5]. We call such a sequence a *Fibonacci binary word* for convenience. Let $B_n$ denote the set of all Fibonacci binary words of length $n$. Simion and Schmidt [14] showed that

**Lemma 18.** ([14, Proposition 15*]) *There is a bijection* $\psi_1$ *between* $S_n(123, 132, 213)$ *and* $B_{n-1}$.

*Proof.* For $w = w_1 w_2 \cdots w_{n-1} \in B_{n-1}$, we construct the permutation $\sigma$ as follows. For $1 \leq i \leq n-1$, let $X_i = [n] - \{\sigma_1, \ldots, \sigma_{i-1}\}$, and set

$$\sigma_i = \begin{cases} \text{largest element in } X_i, & \text{if } w_i = 0, \\ \text{second largest element in } X_i, & \text{if } w_i = 1. \end{cases}$$

Finally, $\sigma_n$ is the unique element in $X_n$. □

For example, if $w = 01001010$, then $\psi_1(w) = 9\,7\,8\,6\,4\,5\,2\,3\,1$.

Given a word $w = w_1 w_2 \cdots w_n \in B_n$, the index $i$ $(1 \leq i < n)$ is an *ascent* of $w$ if $w_i < w_{i+1}$. Let $\mathrm{asc}(w) = \{i \,|w_i < w_{i+1}\}$ be the set of ascents of $w$, and let $\mathrm{maj}(w) = \sum_{i \in \mathrm{asc}(w)} i$.

**Proposition 19.** *For* $n \geq 3$,

$$f_{312}(n) = \sum_{w \in B_{n-1}} \mathrm{maj}(w). \tag{20}$$

*Proof.* Suppose $\sigma \in S_n(123, 132, 213)$ and $\psi_1(\sigma) = w_1 w_2 \cdots w_{n-1}$. If $k$ is an ascent of $w$, then $w_k w_{k+1} = 01$ and $\sigma_k > \sigma_{k+1}$. From bijection $\psi_1$, we see that for all $i \in [n-1]$, there is at most one $j > i$ such that $\sigma_j > \sigma_i$. This implies that $\sigma_i > \sigma_{k+1}$ for all $i < k$. Since $\sigma_k$ is the largest element in $X_k$, we have $\sigma_i > \sigma_j$ for all $i < k+1$ and $j > k+1$. On the other hand, since $\sigma_{k+1}$ is the second largest element in $X_{k+1}$, there exists a unique $l > k+1$ such that $\sigma_l > \sigma_{k+1}$. Thus, we find that $\sigma_i\sigma_{k+1}\sigma_l$ forms a 312-pattern for all $i \leq k$, that is the ascent $k$ will produce $k$'s copies of 312-pattern in which $\sigma_{k+1}$ plays the role of "1". Summing up all the ascents, we derive that the number of copies of 312-pattern in $\sigma$ is $\mathrm{maj}(\psi_1(\sigma))$. $\square$

Recall that the generating function of the Fibonacci number $F_n$ is given by

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}.$$

Hence we can deduce that

$$\sum_{n\geq 3} F_{n+1}x^n = x\sum_{n\geq 2} F_{n+2}x^n = \frac{1}{x}\left(\frac{x}{1-x-x^2} - x - x^2 - 2x^3\right) = \frac{x^3(3+2x)}{1-x-x^2}, \tag{21}$$

$$\sum_{n\geq 2} nF_{n+2}x^n = x\left(\frac{x^2(3+2x)}{1-x-x^2}\right)' = \frac{x^2(6+3x-4x^2-2x^3)}{(1-x-x^2)^2}, \tag{22}$$

$$\sum_{n\geq 3} \binom{n}{3} F_{n+1}x^n = \frac{x^3}{6}\left(\sum_{n\geq 3} F_{n+1}x^n\right)''' = \frac{x^3(3+8x+6x^2+4x^3)}{(1-x-x^2)^4}. \tag{23}$$

**Theorem 20.** *For $n \geq 3$, in the set $S_n(123, 132, 213)$, we have*

$$\sum_{n\geq 3} f_{231}(n)x^n = \sum_{n\geq 3} f_{312}(n)x^n = \frac{x^3(1+2x)}{(1-x-x^2)^3}, \tag{24}$$

$$\sum_{n\geq 3} f_{321}(n)x^n = \frac{x^3(1+6x+12x^2+8x^3)}{(1-x-x^2)^4}. \tag{25}$$

*Proof.* From Lemma 1, we have $f_{231}(n) = f_{312}(n)$ as $\sigma \in S_n(123, 132, 213) \Leftrightarrow \sigma^{-1} \in S_n(123, 132, 213)$ and $231^{-1} = 312$. By Proposition 19, we can write

$$\sum_{n\geq 3} f_{312}(n)x^n = \sum_{n\geq 3} x^n \sum_{w\in B_{n-1}} \mathrm{maj}(w) = x\sum_{n\geq 3}\sum_{w\in B_{n-1}} \mathrm{maj}(w)x^{n-1} = xu(x),$$

where $u(x) = \sum_{n\geq 2}\sum_{w\in B_n} \mathrm{maj}(w)x^n$. To calculate $u(x)$, we set

$$M_n(q) = \sum_{w\in B_n} q^{\mathrm{maj}(w)} \text{ and } M(x,q) = \sum_{n\geq 2} M_n(q)x^n.$$

It is easy to get

$$u(x) = \frac{\partial M(x,q)}{\partial q}\ |_{q=1}\ .$$

Given a word $w = w_1w_2\cdots w_n \in B_n$, if $w_n = 0$, then $\mathrm{maj}(w) = \mathrm{maj}(w_1w_2\cdots w_{n-1})$; otherwise, $w_{n-1}w_n = 01$ and $\mathrm{maj}(w) = \mathrm{maj}(w_1w_2\cdots w_{n-2}) + n - 1$. Hence, we have

$$M_n(q) = M_{n-1}(q) + q^{n-1}M_{n-2}(q) \text{ for } n \geq 4,$$

with $M_2(q) = 2+q$ and $M_3(q) = 2+q+2q^2$. Multiplying the recursion by $x^n$ and summing over $n \geq 4$ yields that

$$M(x,q) - (2+q)x^2 - (2+q+2q^2)x^3 = x\left[M(x,q) - (2+q)x^2\right] + qx^2M(xq,q).$$

Therefore

$$(1-x)M(x,q) = qx^2M(xq,q) + (2+q)x^2 + 2q^2x^3.$$

Differentiate both sides with respect to $q$, we get

$$(1-x)\frac{\partial M(x,q)}{\partial q} = x^2\left[M(xq,q) + q\frac{\partial M(xq,q)}{\partial q}\right] + x^2 + 4qx^3.$$

Setting $q = 1$ gives

$$(1-x)u(x) = x^2\left[M(x,1) + \frac{\partial M(xq,q)}{\partial q}\,|_{q=1}\right] + x^2 + 4x^3.$$

Notice that

$$M(x,1) = \sum_{n\geq 2}|B_n|x^n = \sum_{n\geq 2}F_{n+2}x^n,$$

and

$$\begin{aligned}
\frac{\partial M(xq,q)}{\partial q}\,|_{q=1} &= \left(\sum_{n\geq 2}\sum_{w\in B_n}(n+\mathrm{maj}(w))q^{n+\mathrm{maj}(w)-1}x^n\right)|_{q=1}\\
&= \sum_{n\geq 2}x^n\sum_{w\in B_n}(n+\mathrm{maj}(w))\\
&= \sum_{n\geq 2}nF_{n+2}x^n + u(x).
\end{aligned}$$

Invoking formulae (21) and (22), this implies that

$$(1-x)u(x) = x^2\left[\frac{x^2(3+2x)}{1-x-x^2} + \frac{x^2(6+3x-4x^2-2x^3)}{(1-x-x^2)^2} + u(x)\right] + x^2 + 4x^3.$$

Therefore, $u(x) = x^2(1+2x)/(1-x-x^2)^3$. Multiplying $u(x)$ by $x$, we arrive at formula (24).

As for formula (25), we notice that

$$\sum_{n\geq 3}f_{321}(n)x^n = \sum_{n\geq 3}\binom{n}{3}F_{n+1}x^n - 2\sum_{n\geq 3}f_{312}(n)x^n \tag{26}$$

from the observation $2f_{312}(n) + f_{321}(n) = \binom{n}{3}F_{n+1}$. Thus formula (25) is obtained by substituting equation (23) and the generating function of $f_{312}(n)$ into formula (26). $\square$

The first few values of $f_q(S_n(123,132,213))$ for $q$ of length 3 are shown below, and $(f_{231}(n))_{n\geq 3}$ form sequence A152881 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 1 | 1 | 1 | 6 | 0 | 0 | 0 | 40 | 40 | 180 |
| 4 | 0 | 0 | 0 | 5 | 5 | 10 | 7 | 0 | 0 | 0 | 95 | 95 | 545 |
| 5 | 0 | 0 | 0 | 15 | 15 | 50 | 8 | 0 | 0 | 0 | 213 | 213 | 1478 |

## 3.2 Pattern popularity in other triply restricted permutations

This subsection deals with the popularity of length-3 patterns in the other four classes of triply restricted permutations. We begin with a helpful lemma from [14] as follows:

**Lemma 21.** ([14, Proposition 16*]) *We have*

$$\sigma \in S_n(123,132,231) \Leftrightarrow \sigma = n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k \text{ for some } k. \tag{27}$$
$$\sigma \in S_n(132,213,231) \Leftrightarrow \sigma = n, n-1, \ldots, k+1, 1, 2, 3, \ldots, k-1, k \text{ for some } k. \tag{28}$$
$$\sigma \in S_n(123,132,312) \Leftrightarrow \sigma = n-1, n-2, \ldots, k+1, n, k, k-1, \ldots, 1 \text{ for some } k. \tag{29}$$
$$\sigma \in S_n(123,231,312) \Leftrightarrow \sigma = k-1, k-2, \ldots, 3, 2, 1, n, n-1 \ldots, k \text{ for some } k. \tag{30}$$

Appealing to the above structural characterizations, we can derive the pattern popularity in those classes as follows.

**Theorem 22.** *For $n \geq 3$, in the set $S_n(123,132,231)$, we have*

$$f_{213}(n) = f_{312}(n) = \binom{n}{3}, \tag{31}$$
$$f_{321}(n) = (n-2)\binom{n}{3}. \tag{32}$$

*Proof.* According to the structural formula (27), the identity $f_{213}(n) = f_{312}(n)$ can be proved by a direct bijection.

Let $q = abc$ $(b < a < c)$ be a copy of 213-pattern in $\sigma \in S_n(123,132,231)$. We have $\sigma(n) = c$ since $b < c$ and $\sigma \in S_n(123,132,231)$ has only one ascent at position $n-1$. Therefore, $q$ is a 213-pattern in the sole permutation

$$\sigma = n, n-1, \ldots, c+1, c-1, \ldots, \underline{a}, \ldots, \underline{b}, \ldots, 2, 1, \underline{c}.$$

For the sake of clarity, we underline the occurrence of the assumed pattern.

For $q' = cba$ (312-pattern), we find similarly that $q'$ is a 312-pattern in

$$\sigma' = n, n-1, \ldots, \underline{c}, \ldots, a+1, a-1, \ldots, \underline{b}, \ldots, 2, 1, \underline{a}.$$

For example, if $n=7$ and $q=326$, then $\sigma = 7\,5\,4\,\underline{3}\,\underline{2}\,1\,\underline{6}$, $q'=623$ and $\sigma' = 7\,\underline{6}\,5\,4\,\underline{2}\,1\,\underline{3}$.

Hence, for every copy of 213-pattern $(q,\sigma)$, there is a unique copy of 312-pattern $(q',\sigma')$, and the converse is also true. This implies that $f_{213}(n) = f_{312}(n)$.

To calculate $f_{312}(n)$, we suppose $\sigma = n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k$ for some $k$. We construct a 312-pattern as follows: Choose one element from the first $n-k$ elements to play the role of "3", then choose one element from the next $k-1$ elements to play the role of "1", and the last element plays the role of "2". Thus, summing up $k$ gives

$$f_{312}(n) = \sum_{k=1}^{n}(n-k)(k-1) = -n^2 + (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 = \frac{n(n-1)(n-2)}{6} = \binom{n}{3}.$$

The proof is completed by the relation $f_{213}(n) + f_{312}(n) + f_{321}(n) = n\binom{n}{3}$. $\square$

The first few values of $f_q(S_n(123,132,231))$ for $q$ of length 3 are shown below, and $(f_{213}(n))_{n\geq 3}$ form sequence A000292, $(f_{321}(n))_{n\geq 3}$ form sequence A002417 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 1 | 0 | 1 | 1 | 6 | 0 | 0 | 20 | 0 | 20 | 80 |
| 4 | 0 | 0 | 4 | 0 | 4 | 8 | 7 | 0 | 0 | 35 | 0 | 35 | 175 |
| 5 | 0 | 0 | 10 | 0 | 10 | 30 | 8 | 0 | 0 | 56 | 0 | 56 | 336 |

**Theorem 23.** *For $n \geq 3$, in the set $S_n(132,213,231)$, we have*

$$f_{123}(n) = f_{312}(n) = \binom{n+1}{4}, \tag{33}$$

$$f_{321}(n) = \frac{n(n-2)(n-1)^2}{12}. \tag{34}$$

*Proof.* Based on structural formula (28), we could also prove $f_{123}(n) = f_{312}(n)$ directly. Let $abc$ be a 123-pattern in

$$\sigma = n, n-1, \ldots, k+1, 1, \ldots, \underline{a}, a+1, \ldots, \underline{b}, b+1, \ldots, c-1, \underline{c}, c+1, \ldots, k-1, k.$$

Set

$$\sigma' = n, n-1, \ldots, \underline{n-k+c}, \ldots, c, 1, 2, \ldots, \underline{a}, a+1, \ldots, \underline{b}, b+1, \ldots, c-1.$$

It is easy to check that $(n-k+c)\,a\,b$ is a 312-pattern of $\sigma'$. For example, if $\sigma = 9\,8\,7\,1\,\underline{2}\,\underline{3}\,4\,\underline{5}\,6$, then $\sigma' = 9\,\underline{8}\,7\,6\,5\,1\,\underline{2}\,\underline{3}\,4$.

To calculate $f_{123}(n)$, we suppose $\sigma = n, n-1, \ldots, k+1, 1, 2, \ldots, k-1, k$ for some $k$. A 123-pattern can be obtained by picking three elements from the last $k$ elements to play the role of "123". Thus, summing up all possible $k$ gives

$$f_{123}(n) = \sum_{k=1}^{n}\binom{k}{3} = \binom{n+1}{4}.$$

We complete the proof from $f_{123}(n) + f_{312}(n) + f_{321}(n) = n\binom{n}{3}$. $\square$

The first few values of $f_q(S_n(132, 213, 231))$ for $q$ of length 3 are shown below, and $(f_{123}(n))_{n\geq 3}$ form sequence A000332, $(f_{321}(n))_{n\geq 3}$ form sequence A002415 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 0 | 0 | 1 | 1 | 6 | 35 | 0 | 0 | 0 | 35 | 50 |
| 4 | 5 | 0 | 0 | 0 | 5 | 6 | 7 | 70 | 0 | 0 | 0 | 70 | 105 |
| 5 | 15 | 0 | 0 | 0 | 15 | 20 | 8 | 126 | 0 | 0 | 0 | 126 | 196 |

**Theorem 24.** *For $n \geq 3$, in the set $S_n(123, 132, 312)$, we have*

$$f_{213}(n) = f_{231}(n) = \binom{n}{3}, \tag{35}$$

$$f_{321}(n) = (n-2)\binom{n}{3}. \tag{36}$$

*Proof.* In view of the structural formula (29), the equality $f_{213}(n) = f_{231}(n)$ can be proved by a direct correspondence. Let $abn$ be a copy of 213-pattern in

$$\sigma = n-1, \ldots, \underline{a}, a+1, \ldots, \underline{b}, b+1, \ldots, k+1, \underline{n}, k, k-1, \ldots, 2, 1.$$

Set

$$\sigma' = n-1, \ldots, \underline{n-a+b}, \ldots, n-a+k+1, \underline{n}, n-a+k, n-a+k-1, \ldots, \underline{n-a}, \ldots, 2, 1.$$

Then $n-a+b, n, n-a$ is a 231-pattern of $\sigma'$. For example, if $\sigma = 8\,\underline{7}\,6\,\underline{5}\,4\,\underline{9}\,3\,2\,1$, then $\sigma' = \sigma = 8\,\underline{7}\,6\,\underline{9}\,5\,4\,3\,\underline{2}\,1$.

To calculate $f_{213}(n)$, we suppose that $\sigma = n-1, n-2, \ldots, k+1, n, k, k-1, \ldots, 2, 1$ for some $k$. A 213-pattern can be obtained by choosing two elements from the first $n-k-1$ elements to play the role of "21", and let $n$ play the role of "3". Thus, summing up all possible $k$, we have

$$f_{213}(n) = \sum_{k=0}^{n-1} \binom{n-k-1}{2} = \binom{n}{3}.$$

The proof is completed by using the relation $f_{213}(n) + f_{231}(n) + f_{321}(n) = n\binom{n}{3}$. $\square$

**Theorem 25.** *For $n \geq 3$, in the set $S_n(123, 231, 312)$, we have*

$$f_{132}(n) = f_{213}(n) = \binom{n+1}{4}, \tag{37}$$

$$f_{321}(n) = \frac{n(n-2)(n-1)^2}{12}. \tag{38}$$

*Proof.* From Lemma 1, we see that

$$\sigma \in S_n(123,231,312) \Leftrightarrow \sigma^r \in S_n(321,132,213) \Leftrightarrow (\sigma^r)^c \in S_n(123,231,312).$$

As a consequence, we have $f_{213}(n) = f_{132}(n)$ from $(213^r)^c = 312^c = 132$.

For $f_{213}(n)$, we will employ the structure in formula (30). Suppose $\sigma = k-1, k-2, \ldots, 3, 2, 1, n, n-1 \ldots, k$ for some $k$. A 213-pattern can be obtained as follows: Choose two elements from the first $k-1$ elements to play the role of "21", and choose one element from the last $n-k+1$ elements to play the role of "3". Thus, summing up all possible $k$, we have

$$f_{213}(n) = \sum_{k=1}^{n} \binom{k-1}{2}(n-k+1) = \sum_{k=0}^{n-1} \binom{k}{2}(n-k) = \binom{n+1}{4}.$$

The formula for $f_{321}(n)$ is obtained by the relation $2f_{213}(n) + f_{321}(n) = n\binom{n}{3}$. □

# 4 Acknowledgments

The author thanks the anonymous referee and the editor for their helpful comments. This work was supported by the National Natural Science Foundation of China (No. 11401316), the Natural Science Foundation of Jiangsu Province (No. BK20131393) and Program of Natural Science Research of Jiangsu Higher Education Institutions of China (No. 13KJB110019).

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2010 *Mathematics Subject Classification*: Primary 05A05; Secondary 05A15; 05A19.
*Keywords:* permutation, pattern, composition, binary plane tree, Fibonacci number.

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(Concerned with sequences A000292, A000332, A000337, A000389, A001789, A002415, A002417, A045618, A055580, A055581, A055586, and A152881.)

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Received April 8 2014; revised versions received August 30 2014; September 19 2014; September 22 2014. Published in *Journal of Integer Sequences*, November 2 2014.

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