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On the Dirichlet Convolution of Completely Additive Functions

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Abstract

Let k and l be non-negative integers. For two completely additive functions f and g, we consider various identities for the Dirichlet convolution of the kth powers of f and the lth powers of g. Furthermore, we derive some asymptotic formulas for sums of convolutions on the natural logarithms.

1 Statements of results

Let f and g be two arithmetical functions that are completely additive. That is, these functions satisfy $f(mn) = f(m) + f(n)$ and $g(mn) = g(m) + g(n)$ for all positive integers m and n. We shall consider the arithmetical function

$$D_{k,l}(n; f, g) := \sum_{d|n} f^k(d) g^l\left(\frac{n}{d}\right), \tag{1}$$

which represents the Dirichlet convolution of the kth power of f and the lth power of g for non-negative integers k and l. The above function provides a certain generalization of the classical number-of-divisors function $d(n)$. In fact,

$$D_{0,0}(n; f, g) = d(n).$$

The first purpose of this study is to investigate some recurrence formulas for $D_{k,l}(n;f,g)$ with respect to k and l. Since

$$\sum_{d|n} f(d) = \frac{1}{2}d(n)f(n), \tag{2}$$

where f is a completely additive function, we have

$$D_{1,1}(n;f,g) = \frac{1}{2}d(n)f(n)g(n) - \sum_{d|n} f(d)g(d). \tag{3}$$

Similarly, as in (3), we use (1) for $D_{k,l+1}(n;f,g)$ to obtain

$$\begin{aligned} D_{k,l+1}(n;f,g) &= \sum_{d|n} f^k(d)g^l\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right) \\ &= g(n)\sum_{d|n} f^k(d)g^l\left(\frac{n}{d}\right) - \sum_{d|n} f^k(d)g(d)g^l\left(\frac{n}{d}\right). \end{aligned}$$

Hence, we deduce the following two recurrence formulas.

Theorem 1. *Let k and l be non-negative integers, and let f and g be completely additive functions. Then we have*

$$D_{k,l+1}(n;f,g) + \sum_{d|n} f^k(d)g^l\left(\frac{n}{d}\right)g(d) = g(n)D_{k,l}(n;f,g), \tag{4}$$

$$D_{k+1,l}(n;f,g) + \sum_{d|n} f^k(d)f\left(\frac{n}{d}\right)g^l\left(\frac{n}{d}\right) = f(n)D_{k,l}(n;f,g). \tag{5}$$

Now, we put $f = g$ in (4) (or (5)), and set $D_{k,l}(n;f) := D_{k,l}(n;f,f)$. Then, we deduce the following corollary.

Corollary 2. *Using the same notation given above, we have*

$$D_{k,l+1}(n;f) + D_{k+1,l}(n;f) = f(n)D_{k,l}(n;f). \tag{6}$$

Particularly, if $k = l$, we have

$$D_{k+1,k}(n;f) = D_{k,k+1}(n;f) = \frac{1}{2}f(n)D_{k,k}(n;f). \tag{7}$$

Because the symmetric property $D_{k,l}(n;f) = D_{l,k}(n;f)$, we only consider the function $D_{k,k+j}(n,f)$ for $j = 1, 2, \ldots$.

Example 3. The formulas (6) and (7) imply that

$$\begin{aligned}
D_{k,k+2}(n;f) &= \frac{1}{2}f^2(n)D_{k,k}(n;f) - D_{k+1,k+1}(n;f),\\
D_{k,k+3}(n;f) &= \frac{1}{2}f^3(n)D_{k,k}(n;f) - \frac{3}{2}f(n)D_{k+1,k+1}(n;f),\\
D_{k,k+4}(n;f) &= \frac{1}{2}f^4(n)D_{k,k}(n;f) - 2f^2(n)D_{k+1,k+1}(n;f) + D_{k+2,k+2}(n;f),\\
D_{k,k+5}(n;f) &= \frac{1}{2}f^5(n)D_{k,k}(n;f) - \frac{5}{2}f^3(n)D_{k+1,k+1}(n;f) + \frac{5}{2}f(n)D_{k+2,k+2}(n;f),\\
D_{k,k+6}(n;f) &= \frac{1}{2}f^6(n)D_{k,k}(n;f) - 3f^4(n)D_{k+1,k+1}(n;f) + \frac{9}{2}f^2(n)D_{k+2,k+2}(n;f)\\
&\quad - D_{k+3,k+3}(n;f),\\
D_{k,k+7}(n;f) &= \frac{1}{2}f^7(n)D_{k,k}(n;f) - \frac{7}{2}f^5(n)D_{k+1,k+1}(n;f) + 7f^3(n)D_{k+2,k+2}(n;f)\\
&\quad - \frac{7}{2}f(n)D_{k+3,k+3}(n;f),\\
D_{k,k+8}(n;f) &= \frac{1}{2}f^8(n)D_{k,k}(n;f) - 4f^6(n)D_{k+1,k+1}(n;f) + 10f^4(n)D_{k+2,k+2}(n;f)\\
&\quad - 8f^2(n)D_{k+3,k+3}(n;f) + D_{k+4,k+4}(n;f).
\end{aligned}$$

Next, we shall demonstrate that the explicit evaluation of the function $D_{k,k+m}(n;f)$ $(m = 2, 3, \ldots)$ can be expressed as a combination of the functions $D_{k,k}(n;f)$, $D_{k+1,k+1}(n;f)$, $D_{k+2,k+2}(n;f)$, $\ldots$, $D_{k+\lfloor\frac{m}{2}\rfloor,k+\lfloor\frac{m}{2}\rfloor}(n;f)$. Hence, we shall give a recurrence formula between $D_{k,k}(n;f), \ldots, D_{k+\lfloor\frac{m}{2}\rfloor,k+\lfloor\frac{m}{2}\rfloor}(n;f)$ and $D_{k,k+m}(n;f)$.

Theorem 4. *Let k and m be positive integers, and let $D_{k,k+m}(n;f)$ be the function defined by the above formula. Then we have*

$$D_{k,k+m}(n;f) = \sum_{j=0}^{\lfloor\frac{m}{2}\rfloor} c_{k,j}^{(m)} f^{m-2j}(n) D_{k+j,k+j}(n;f), \tag{8}$$

where

$$c_{k,j}^{(m)} = \begin{cases} \frac{1}{2}, & \text{if } j = 0;\\ -\frac{m}{2}, & \text{if } j = 1;\\ (-1)^j \frac{m}{2\cdot j!}\prod_{i=1}^{j-1}\left(m-(j+i)\right), & \text{if } 2 \le j \le \lfloor\frac{m}{2}\rfloor. \end{cases}$$

Proof. By (7) in Corollary 2, the equality (8) holds for $m = 1$ and all $k \in \mathbb{N}$. Now, we assume that (8) is true for $m = 1, 2, \ldots, l$ and $k \in \mathbb{N}$. Using this assumption and (6) in

Corollary 2, we observe that

$$D_{k,k+l+1}(n;f) = \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} c_{k,j}^{(l)} f^{l+1-2j}(n) D_{k+j,k+j}(n;f) \\ - \sum_{j=0}^{\lfloor \frac{l-1}{2} \rfloor} c_{k+1,j}^{(l-1)} f^{l-1-2j}(n) D_{k+1+j,k+1+j}(n;f).$$

For even $l = 2q$, we have

$$D_{k,k+2q+1}(n;f) = \frac{1}{2} f^{2q+1}(n) D_{k,k}(n;f) \\ + \sum_{j=1}^{q} \left(c_{k,j}^{(2q)} - c_{k+1,j-1}^{(2q-1)} \right) f^{2q+1-2j}(n) D_{k+j,k+j}(n;f)$$

and

$$c_{k,j}^{(2q)} - c_{k+1,j-1}^{(2q-1)} = (-1)^j \frac{2q+1}{2 \cdot j!} \prod_{i=1}^{j-2} (2q-(j+i))(2q-j) = c_{k,j}^{(2q+1)}.$$

For odd $l = 2q-1$, we observe that

$$D_{k,k+2q}(n;f) = \frac{1}{2} f^{2q}(n) D_{k,k}(n;f) \\ + \sum_{j=1}^{q-1} \left(c_{k,j}^{(2q-1)} - c_{k+1,j-1}^{(2q-2)} \right) f^{2q-2j}(n) D_{k+j}(n;f) \\ + (-1)^{\lfloor \frac{2q}{2} \rfloor} D_{k+\lfloor \frac{2q}{2} \rfloor, k+\lfloor \frac{2q}{2} \rfloor}(n;f).$$

By our assumption, since

$$c_{k,j}^{(2q-1)} - c_{k+1,j-1}^{(2q-2)} = (-1)^j \frac{2q}{2 \cdot j!} \prod_{i=1}^{j-2} (2q-1-(j+i))(2q-1-j) = c_{k,j}^{(2q)},$$

we obtain the assertion (8) for all k and $m \in \mathbb{N}$. $\square$

Now, we consider another expression for $D_{k,l}(n;f,g)$ using the arithmetical function

$$H_{k,m}(n;f,g) := \sum_{d|n} f^k(d) g^m(d). \tag{9}$$

If $f = g$, we set $H_{k+m}(n; f) = H_{k,m}(n; f, f)$. The right-hand side of (9) implies the Dirichlet convolution of 1 and $f^k g^m$. Since g is a completely additive function, we have

$$\begin{aligned} D_{k,l}(n; f, g) &= \sum_{d|n} f^k(d) \left(g(n) - g(d)\right)^l \\ &= \sum_{d|n} f^k(d) \sum_{m=0}^{l} (-1)^m \binom{l}{m} g^{l-m}(n) g^m(d). \end{aligned}$$

From (9) and the above, we obtain the following theorem.

Theorem 5. *Let k and l be non-negative integers, and let f and g be completely additive functions. Then we have*

$$D_{k,l}(n; f, g) = \sum_{m=0}^{l} (-1)^m \binom{l}{m} g^{l-m}(n) H_{k,m}(n; f, g),$$

where the function $H_{k,m}(n; f, g)$ is defined by (9).

We immediately obtain the following corollary.

Corollary 6. *Let k and l be non-negative integers, and let f and g be completely additive functions. Then we have*

$$D_{k,l}(n; f) = \sum_{m=0}^{l} (-1)^m \binom{l}{m} f^{l-m}(n) H_{k+m}(n; f). \tag{10}$$

Note that

$$\begin{aligned} H_{k,m}(n; f, g) &= \sum_{d|n} f^k \left(\frac{n}{d}\right) g^m \left(\frac{n}{d}\right) \\ &= \sum_{i=0}^{k} \sum_{j=0}^{m} (-1)^{i+j} \binom{k}{i} \binom{m}{j} f^{k-i}(n) g^{m-j}(n) \sum_{d|n} f^i(d) g^j(d). \end{aligned} \tag{11}$$

Applying (11) to Theorem 5, we have the following theorem.

Theorem 7. *Let k and l be non-negative integers, and let f and g be completely additive functions. Then we have*

$$\begin{aligned} &D_{k,l}(n; f, g) \\ &= \sum_{m=0}^{l} \sum_{i=0}^{k} \sum_{j=0}^{m} (-1)^{m+i+j} \binom{l}{m} \binom{k}{i} \binom{m}{j} f^{k-i}(n) g^{l-j}(n) \sum_{d|n} f^i(d) g^j(d). \end{aligned}$$

In the case where $f = g$, note that

$$\begin{aligned} H_{k+m}(n;f) &= \sum_{d|n} (f(n) - f(d))^{k+m} \\ &= \sum_{j=0}^{k+m} (-1)^j \binom{k+m}{j} f^{k+m-j}(n) \sum_{d|n} f^j(d). \end{aligned}$$

From (10) and the above, we obtain the following corollary.

Corollary 8. *Let k and l be non-negative integers, and let f be a completely additive function. Then we have*

$$D_{k,l}(n;f) = \sum_{m=0}^{l} \sum_{j=0}^{k+m} (-1)^{m+j} \binom{l}{m} \binom{k+m}{j} f^{k+l-j}(n) \sum_{d|n} f^j(d). \tag{12}$$

2 Recurrence formula connecting $D_{k,l}(n;f)$ with $\sum_{d|n} f^j(d)$

The second purpose of this study is to derive another expression for $D_{k,l}(n;f)$ that involves the divisor function $d(n)$. Before stating Theorem 10, we prepare the following lemma.

Lemma 9. *Let f be a completely additive function. There exist the constants $e_{q,q}, e_{q,q-1}, \ldots, e_{q,1}$ $(q = 1, 2, \ldots)$ that satisfy the equation*

$$\sum_{d|n} f^{2q-1}(d) = e_{q,q} d(n) f^{2q-1}(n) + \sum_{j=1}^{q-1} e_{q,q-j} f^{2q-2j-1}(n) \sum_{d|n} f^{2j}(d). \tag{13}$$

Moreover, the relations among sequences $(e_{q,q-j})_{j=1}^{q}$ are as follows.

$$\begin{aligned} e_{q,q} &= \frac{1}{2}\left(1 - \sum_{j=1}^{q-1} \binom{2q-1}{2j-1} e_{j,j}\right) = \frac{\left(2^{2q}-1\right) B_{2q}}{q}, \\ e_{q,q-j} &= \frac{1}{2}\left(\binom{2q-1}{2j} - \sum_{\substack{i=2 \\ i-j\geq 1}}^{q-1} \binom{2q-1}{2i-1} e_{i,i-j}\right), \end{aligned} \tag{14}$$

where B_n denotes the nth Bernoulli number, which is defined by the Taylor expansion

$$\frac{z}{e^z - 1} = \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n, \quad (|z| < 2\pi).$$

Proof. By (2), the case $q=1$ in (13) is trivial. Assume that there exist $e_{p,p}$, $e_{p,p-1}, \ldots, e_{p,1}$ $(p \leq q)$ such that

$$\sum_{d|n} f^{2p-1}(d) = e_{p,p} d(n) f^{2p-1}(n) + \sum_{j=1}^{p-1} e_{p,p-j} f^{2p-2j-1}(n) \sum_{d|n} f^{2j}(d). \tag{15}$$

Since

$$\sum_{d|n} f^{2q+1}(d) = \sum_{j=0}^{2q+1} (-1)^j \binom{2q+1}{j} f^{2q+1-j}(n) \sum_{d|n} f^j(d),$$

we have

$$\begin{aligned}\sum_{d|n} f^{2q+1}(d) = \frac{1}{2} d(n) f^{2q+1}(n) + \frac{1}{2} \sum_{j=1}^{q} \binom{2q+1}{2j} f^{2q+1-2j}(n) \sum_{d|n} f^{2j}(d) \\ - \frac{1}{2} \sum_{j=1}^{q} \binom{2q+1}{2j-1} f^{2q+2-2j}(n) \sum_{d|n} f^{2j-1}(d).\end{aligned} \tag{16}$$

Applying (15) to (16), we obtain

$$\begin{aligned}\sum_{d|n} f^{2q+1}(d) =& \frac{1}{2}\left(1 - \sum_{j=1}^{q} \binom{2q+1}{2j-1} e_{j,j}\right) d(n) f^{2q+1}(n) \\ &+ \frac{1}{2} \sum_{j=1}^{q} \left(\binom{2q+1}{2j} - \sum_{\substack{i=2 \\ i-j\geq 1}}^{q} \binom{2q+1}{2i-1} e_{i,i-j} \right) f^{2q-2j+1}(n) \sum_{d|n} f^{2j}(d).\end{aligned}$$

By induction, this completes the proof, except for the second term on the right-hand side of (14).

The first term on the right-hand side of (14) implies

$$e_{q,q} = 1 - \sum_{k=1}^{q} \binom{2q-1}{2k-1} e_{k,k} = 1 - \sum_{k=1}^{q} \binom{2q}{2k} \frac{k}{q} e_{k,k}.$$

Here we put $a(k) = k e_{k,k}$. Then we have

$$a(q) = q - \sum_{k=1}^{q} \binom{2q}{2k} a(k). \tag{17}$$

Since $a(1) = e_{1,1} = 1/2$ and $(2^2-1) B_2 = 1/2$, we only need to show that $\left(2^{2k}-1\right) B_{2k}$ $(k = 1, \ldots, q)$ satisfies the recurrence formula (17). Consider the nth Bernoulli polynomial

$B_n(x)$, which is defined by the following Taylor expansion:

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \quad (|z| < 2\pi).$$

The following relations are known among $B_n(1)$, $B_n(1/2)$ and B_n,

$$B_n(1) = B_n, \qquad B_n\left(\frac{1}{2}\right) = -\left(1-2^{1-n}\right) B_n.$$

By the formula [1, Thm. 12.12, p. 264]

$$B_n(y) = \sum_{k=0}^{n} \binom{n}{k} B_k y^{n-k}, \tag{18}$$

we observe that

$$B_{2q}(y) = y^{2q} - qy^{2q-1} + \sum_{k=2}^{2q} \binom{2q}{k} B_k y^{2q-k}.$$

In this equation, we consider $y=1$ and $y=1/2$; then

$$B_{2q} = 1 - q + \sum_{k=1}^{q} \binom{2q}{2k} B_{2k} \tag{19}$$

and

$$\begin{aligned} 2^{2q} B_{2q}\left(\frac{1}{2}\right) &= \left(2-2^{2q}\right) B_{2q} \\ &= 1 - 2q + \sum_{k=1}^{q} \binom{2q}{2k} B_{2k} 2^{2k}. \end{aligned} \tag{20}$$

Subtracting (20) from (19), we obtain

$$\left(2^{2q}-1\right) B_{2q} = q - \sum_{k=1}^{q} \binom{2q}{2k} \left(2^{2k}-1\right) B_{2k}.$$

This recurrence formula for $\left(2^{2k}-1\right) B_{2k}$'s is equivalent to (17). This completes the proof of (13). $\square$

Applying Lemma 9 to (12) in Corollary 8, we have

$$\begin{aligned} D_{k,l}(n;f) = &\sum_{m=0}^{l} (-1)^m \binom{l}{m} \sum_{j=0}^{\lfloor \frac{k+m}{2} \rfloor} \binom{k+m}{2j} f^{l+k-2j}(n) \sum_{d|n} f^{2j}(d) \\ &- \sum_{m=0}^{l} (-1)^m \binom{l}{m} \sum_{j=1}^{\lfloor \frac{k+m+1}{2} \rfloor} \binom{k+m}{2j-1} f^{l+k+1-2j}(n) \sum_{d|n} f^{2j-1}(d). \end{aligned} \tag{21}$$

The second term on the right-hand side of (21) gives us

$$
\begin{aligned}
&-\left(\sum_{m=0}^{l}(-1)^m\binom{l}{m}\sum_{j=1}^{\lfloor\frac{k+m}{2}\rfloor} e_{j,j}\binom{k+m}{2j-1}\right) f^{l+k}(n)d(n)\\
&-\sum_{m=0}^{l}\sum_{j=1}^{\lfloor\frac{k+m+1}{2}\rfloor}\sum_{i=1}^{j-1}(-1)^m\binom{l}{m}\binom{k+m}{2j-1}e_{j,j-i}f^{l+k-2i}(n)\sum_{d|n}f^{2i}(d)
\end{aligned}
\tag{22}
$$

using (13). From (14), (21) and (22), we have the following theorem.

Theorem 10. *Let k and l be non-negative integers, and let f be a completely additive function. There exist the constants $e_{j,j}$, $e_{j,j-i}$ ($j = 1, 2, \ldots, \lfloor\frac{k+m+1}{2}\rfloor$, $1 \le j-i < j$) and $A_{k,l}$ such that*

$$
\begin{aligned}
D_{k,l}(n;f) =& A_{k,l}f^{l+k}(n)d(n)\\
&+\sum_{m=0}^{l}\sum_{j=0}^{\lfloor\frac{k+m}{2}\rfloor}(-1)^m\binom{l}{m}\binom{k+m}{2j}f^{l+k-2j}(n)\sum_{d|n}f^{2j}(d) \\
&-\sum_{m=0}^{l}\sum_{j=1}^{\lfloor\frac{k+m}{2}\rfloor}\sum_{i=1}^{j-1}(-1)^m\binom{l}{m}\binom{k+m}{2j-1}e_{j,j-i}f^{l+k-2i}(n)\sum_{d|n}f^{2i}(d),
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
A_{k,l} &= \sum_{m=0}^{l}(-1)^{m-1}\binom{l}{m}\sum_{j=1}^{\lfloor\frac{k+m+1}{2}\rfloor}\frac{(2^{2j}-1)B_{2j}}{j}\binom{k+m}{2j-1}\\
&= 2\sum_{m=0}^{l}(-1)^m\binom{l}{m}\frac{2^{k+m+1}-1}{k+m+1}B_{k+m+1}.
\end{aligned}
\tag{24}
$$

Proof. We only need to show (24) to complete the proof of Theorem 10. We set

$$
A_{k,l} = \sum_{m=0}^{l}(-1)^{m-1}\binom{l}{m}J_{k,m}, \qquad J_{k,m} = \sum_{j=1}^{\lfloor\frac{k+m+1}{2}\rfloor}\frac{(2^{2j}-1)B_{2j}}{j}\binom{k+m}{2j-1}.
$$

From the identity $\frac{1}{j}\binom{k+m}{2j-1} = \frac{2}{k+m+1}\binom{k+m+1}{2j}$, we have

$$
J_{k,m} = \frac{2}{k+m+1}\sum_{j=1}^{\lfloor\frac{k+m+1}{2}\rfloor}(2^{2j}-1)B_{2j}\binom{k+m+1}{2j}.
$$

Since $B_{2j+1} = 0$ if $j \geq 1$ and $B_1 = -\frac{1}{2}$, we have

$$J_{k,m} = \frac{2}{k+m+1} \sum_{j=0}^{k+m+1} (2^{2j} - 1) B_j \binom{k+m+1}{j} + 1. \tag{25}$$

Taking $y = \frac{1}{2}$ and $y = 1$ in (18), we get

$$2^n B_n \left(\frac{1}{2}\right) = \sum_{j=0}^{n} 2^j \binom{n}{j} B_j \qquad \text{and} \qquad B_n = \sum_{j=0}^{n} \binom{n}{j} B_j,$$

respectively. Hence we have

$$2^n B_n \left(\frac{1}{2}\right) - B_n = \sum_{j=0}^{n} \left(2^j - 1\right) \binom{n}{j} B_j. \tag{26}$$

On the other hand, it is known that

$$B_n(mx) = m^{n-1} \sum_{j=0}^{m-1} B_n \left(x + \frac{j}{m}\right). \tag{27}$$

See [1, p. 275]. Taking $x = 0$ and $m = 2$ in (27), we obtain

$$2^n B_n \left(\frac{1}{2}\right) - B_n = -\left(2^n - 1\right) B_n.$$

Then we have, from (26) and the above,

$$-\left(2^n - 1\right) B_n = \sum_{j=0}^{n} \left(2^j - 1\right) B_j \binom{n}{j}.$$

Substituting this relation in (25), we have

$$J_{k,m} = -\frac{2}{k+m+1}(2^{k+m+1} - 1) B_{k+m+1} + 1.$$

Hence we have

$$A_{k,l} = 2 \sum_{m=0}^{l} (-1)^m \binom{l}{m} \frac{2^{k+m+1} - 1}{k+m+1} B_{k+m+1}.$$

□

From (23), we obtain the following corollary.

Corollary 11. *For every positive integer k, there are constants $c_k, c_{k-1}, \ldots, c_0$ such that*

$$D_{k,k}(n; f) = c_k d(n) f^{2k}(n) + \sum_{j=1}^{k} c_{k-j} f^{2k-2j}(n) \sum_{d|n} f^{2j}(d). \tag{28}$$

From (8) and (28), we obtain the following corollary.

Corollary 12. *For every positive integer k and m (≥ 2), there are constants $c_0, c_1, \ldots, c_k$, $\ldots, c_{k+\lfloor \frac{m}{2} \rfloor}$ as in Corollary 11 and $c_{k,1}^{(m)} = -\frac{m}{2}$, $c_{k,2}^{(m)} = \frac{m(m-3)}{4}$, $c_{k,3}^{(m)}$, $\ldots$, $c_{k,\lfloor \frac{m}{2} \rfloor}^{(m)}$ as in Theorem 4 such that*

$$\begin{aligned} D_{k,k+m}(n; f) = &\left(\frac{1}{2} c_k + \sum_{p=1}^{\lfloor \frac{m}{2} \rfloor} c_{k,p}^{(m)} c_{k+p} \right) d(n) f^{2k+m}(n) \\ &+ \frac{1}{2} \sum_{j=1}^{k} c_{k-j} f^{2k+m-2j}(n) \sum_{d|n} f^{2j}(d) \\ &+ \sum_{p=1}^{\lfloor \frac{m}{2} \rfloor} c_{k,p}^{(m)} \sum_{j=1}^{k+p} c_{k+p-j} f^{2k+m-2j}(n) \sum_{d|n} f^{2j}(d). \end{aligned}$$

Example 13. Corollaries 11 and 12 give us

$$\begin{aligned} D_{1,1}(n; f) &= \frac{1}{2} d(n) f^2(n) - \sum_{d|n} f^2(d), \\ D_{2,1}(n; f) &= \frac{1}{4} d(n) f^3(n) - \frac{1}{2} f(n) \sum_{d|n} f^2(d), \\ D_{3,1}(n; f) &= -\frac{1}{4} d(n) f^4(n) + \frac{3}{2} f^2(n) \sum_{d|n} f^2(d) - \sum_{d|n} f^4(d), \\ D_{2,2}(n; f) &= \frac{1}{2} d(n) f^4(n) - 2 f^2(n) \sum_{d|n} f^2(d) + \sum_{d|n} f^4(d), \\ D_{4,1}(n; f) &= -\frac{1}{2} d(n) f^5(n) + \frac{5}{2} f^3(n) \sum_{d|n} f^2(d) - \frac{3}{2} f(n) \sum_{d|n} f^4(d), \\ D_{3,2}(n; f) &= \frac{1}{4} d(n) f^5(d) - f^2(n) \sum_{d|n} f^2(d) + \frac{1}{2} f(n) \sum_{d|n} f^4(d). \end{aligned}$$

3 Applications

The second author [2, p. 330] showed an asymptotic formula for $\sum_{n\le x} D_{1,1}(n;\log)$

$$\sum_{n\le x} D_{1,1}(n;\log) = \frac{1}{6}x\log^3 x - \frac{1}{2}x\log^2 x + (1-2A_1)x\log x \\ + (2A_1 - 4A_2 - 1)x + O_\varepsilon\left(x^{\frac{1}{3}+\varepsilon}\right) \tag{29}$$

for $x > 2$ and all $\varepsilon > 0$. Here the constants A_1 and A_2 are coefficients of the Laurent expansion of the Riemann zeta-function $\zeta(s)$ in the neighbourhood $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + A_0 + A_1(s-1) + A_2(s-1)^2 + A_3(s-1)^3 + \cdots.$$

We use (7), (29) and Abel's identity [1, Thm. 4.2, p. 77] to obtain

$$\sum_{n\le x} D_{2,1}(n;\log) = \frac{1}{12}x\log^4 x - \frac{1}{3}x\log^3 x + (1+A_1)x\log^2 x \\ - 2(1+A_2)x\log x + 2(1+A_2)x + O_\varepsilon\left(x^{\frac{1}{3}+\varepsilon}\right).$$

Furthermore, a generalization of (29) for the partial sums of $D_{k,k}(n;\log)$ for positive integers k was considered by the second author [2, Thm. 1.2, p. 326], who demonstrated that there exists a polynomial P_{2k+1} of degree $2k+1$ such that

$$\sum_{n\le x} D_{k,k}(n;\log) = xP_{2k+1}(\log x) + O_{k,\varepsilon}\left(x^{\frac{1}{3}+\varepsilon}\right) \tag{30}$$

for every $\varepsilon > 0$. Applying Theorem 4 and the above formula (30), we have the following theorem.

Theorem 14. *There exists a polynomial U_{2k+m+1} of degree $2k+m+1$ such that*

$$\sum_{n\le x} D_{k,k+m}(n;\log) = xU_{2k+m+1}(\log x) + O_{k,m,\varepsilon}\left(x^{\frac{1}{3}+\varepsilon}\right)$$

for $m = 2, 3, \ldots$ and every $\varepsilon > 0$.

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