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# Periodicity of the Last Digits of Some Combinatorial Sequences[1]

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**Abstract**

In 1962 O. A. Gross proved that the last digits of the Fubini numbers (or surjective numbers) have a simple periodicity property. We extend this result to a wider class of combinatorial numbers coming from restricted set partitions.

## 1 The Stirling numbers of the second kind and the Fubini numbers

The $n$th Fubini number or surjection number [18, 22, 31, 34, 35], $F_n$, counts all the possible partitions of $n$ elements such that the order of the blocks matters. The $\left\{ {n \atop k} \right\}$ Stirling number of the second kind with parameters $n$ and $k$ enumerates the partitions of $n$ elements into $k$ blocks. Thus, $F_n$ has the following expression:

$$F_n = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}. \tag{1}$$

The first Fubini numbers are presented in the next table:

[1] This scientific work was financed by Proyecto Prometeo de la Secretaría Nacional de Ciencia, Tecnología e Innovación (Ecuador).

| $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 13 | 75 | 541 | 4 683 | 47 293 | 545 835 | 7 087 261 | 102 247 563 |

We may realize that the last digits form a periodic sequence of length four. This is true up to infinity. That is, for all $n \geq 1$

*Congruence* 1.

$$F_{n+4} \equiv F_n \pmod{10}.$$

A proof was given by Gross [20] which uses backward differences. Later we give a simple combinatorial proof.

We note that if the order of the blocks does not matter, we get the Bell numbers [11]:

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

This sequence does not possess this "periodicity property".

Now we introduce other classes of numbers which do share this periodicity.

## 2 The $r$-Stirling and $r$-Fubini numbers

The $r$-Stirling number of the second kind with parameters $n$ and $k$ is denoted by $\left\{ {n \atop k} \right\}_r$ and enumerates the partitions of $n$ elements into $k$ blocks *such that* all the first $r$ elements are in different blocks. So, for example, $\{1, 2, 3, 4, 5\}$ can be partitioned into $\{1, 4, 5\} \cup \{2, 3\}$ but $\{1, 2\} \cup \{3, 5\} \cup \{4\}$ is forbidden, if $r \geq 2$. An introductory paper on $r$-Stirling numbers is due to Broder [5]. A good source of combinatorial identities on $r$-Stirling numbers is a book of Charalambides [6]. (Note that in this book these numbers are called noncentral Stirling numbers.)

Similarly to (1) we can introduce the $r$-Fubini numbers as

$$F_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r. \tag{2}$$

(It is worthwile to shift the indices, since the upper and lower parameters have to be at least $r$, so $n$ and $k$ can run from zero in the formula.)

We note that the literature of these numbers is rather thin. From the analytical point of view they were discussed by Corcino and his co-authors [12]. The present author with Nyul [28] is preparing a manuscript which studies these numbers in a combinatorial way.

The first 2-Fubini and 3-Fubini numbers are

| $F_{1,2}$ | $F_{2,2}$ | $F_{3,2}$ | $F_{4,2}$ | $F_{5,2}$ | $F_{6,2}$ | $F_{7,2}$ | $F_{8,2}$ |
|---|---|---|---|---|---|---|---|
| 10 | 62 | 466 | 4 142 | 42 610 | 498 542 | 6 541 426 | 95 160 302 |

and

| $F_{1,3}$ | $F_{2,3}$ | $F_{3,3}$ | $F_{4,3}$ | $F_{5,3}$ | $F_{6,3}$ | $F_{7,3}$ | $F_{8,3}$ |
|---|---|---|---|---|---|---|---|
| 42 | 342 | 3 210 | 34 326 | 413 322 | 5 544 342 | 82 077 450 | 1 330 064 406 |

We see that the sequence of the last digits is also periodic with period four:

*Congruence* 2.

$$F_{n+4,r} \equiv F_{n,r} \pmod{10} \quad (n, r \geq 1).$$

If the order of the blocks does not interest us, we get the $r$-Bell numbers:

$$B_{n,r} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r .$$

These numbers were discussed combinatorially in a paper of the present author [26] and analytically by Corcino and his co-authors [13, 14], and also by Dil and Kurt [15]. The table of these numbers [26] shows that there is possibly no periodicity in the last digits of the $r$-Bell numbers.

# 3 Restricted Stirling numbers and three derived sequences

Let us go further and introduce another class of Stirling numbers. The $\left\{ {n \atop k} \right\}_{\leq m}$ restricted Stirling number of the second kind [1, 8, 9, 10] gives the number of partitions of $n$ elements into $k$ subsets under the restriction that *none* of the blocks contain more than $m$ elements. The notation reflects this restrictive property.

The sum of restricted Stirling numbers gives the restricted Bell numbers [30]:

$$B_{n,\leq m} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leq m} .$$

Their tables are as follows:

| $B_{1,\leq 2}$ | $B_{2,\leq 2}$ | $B_{3,\leq 2}$ | $B_{4,\leq 2}$ | $B_{5,\leq 2}$ | $B_{6,\leq 2}$ | $B_{7,\leq 2}$ | $B_{8,\leq 2}$ | $B_{9,\leq 2}$ | $B_{10,\leq 2}$ | $B_{11,\leq 2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 10 | 26 | 76 | 232 | 764 | 2 620 | 9 496 | 35 696 |

| $B_{1,\leq 3}$ | $B_{2,\leq 3}$ | $B_{3,\leq 3}$ | $B_{4,\leq 3}$ | $B_{5,\leq 3}$ | $B_{6,\leq 3}$ | $B_{7,\leq 3}$ | $B_{8,\leq 3}$ | $B_{9,\leq 3}$ | $B_{10,\leq 3}$ | $B_{11,\leq 3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 14 | 46 | 166 | 652 | 2 780 | 12 644 | 61 136 | 312 676 |

| $B_{1,\leq 4}$ | $B_{2,\leq 4}$ | $B_{3,\leq 4}$ | $B_{4,\leq 4}$ | $B_{5,\leq 4}$ | $B_{6,\leq 4}$ | $B_{7,\leq 4}$ | $B_{8,\leq 4}$ | $B_{9,\leq 4}$ | $B_{10,\leq 4}$ | $B_{11,\leq 4}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 15 | 51 | 196 | 827 | 3 795 | 18 755 | 99 146 | 556 711 |

If we eliminate the first elements $B_{1,\leq 2}$ and $B_{1,\leq 3}, B_{2,\leq 3}, B_{3,\leq 3}$ we can see that these sequences, for $m = 2, 3$, seem to be periodic of length five:

*Congruence* 3.

$$B_{n,\leq 2} \equiv B_{n+5,\leq 2} \pmod{10} \quad (n > 1).$$

$$B_{n,\leq 3} \equiv B_{n+5,\leq 3} \pmod{10} \quad (n > 3).$$

The third table shows that such kind of congruence cannot be proven if $m = 4$. The proof of the above congruence for $m = 2, 3$ is contained in the 5.3 subsection. We will also explain why this property fails to hold whenever $m > 3$.

What happens, if we take restricted Fubini numbers, as

$$F_{n,\leq m} = \sum_{k=0}^{n} k! \left\{ n \atop k \right\}_{\leq m} ? \tag{3}$$

The next congruences hold.

*Congruence* 4.

$$\begin{aligned} F_{n,\leq 1} &\equiv 0 \pmod{10} \quad (n > 4), \\ F_{n,\leq m} &\equiv 0 \pmod{10} \quad (n > 4, m = 2, 3, 4), \qquad (4) \\ F_{n,\leq m} &\equiv 0 \pmod{2} \quad (n > m, m > 4). \qquad (5) \end{aligned}$$

The first congruence is trivial, since $F_{n,\leq 1} = n!$. The others will be proved in subsection 5.4.

If we consider restricted Stirling numbers of the first kind, a similar conjecture can be phrased. These numbers with parameter $n, k$ and $m$ count all the permutations on $n$ elements with $k$ cycles *such that* all cycles contain at most $m$ items. Let us denote these numbers by $\left[ n \atop k \right]_{\leq m}$. Then let

$$A_{n,\leq m} = \sum_{k=0}^{n} \left[ n \atop k \right]_{\leq m} .$$

We may call these as restricted factorials, since if $m = n$ (there is no restriction) we get that $A_{n,\leq m} = n!$. Note that the sequence $(n!)$ is clearly periodic in the present sense, because $n! \equiv 0 \pmod{10}$ if $n > 4$. The tables

| $A_{1,\leq 3}$ | $A_{2,\leq 3}$ | $A_{3,\leq 3}$ | $A_{4,\leq 3}$ | $A_{5,\leq 3}$ | $A_{6,\leq 3}$ | $A_{7,\leq 3}$ | $A_{8,\leq 3}$ | $A_{9,\leq 3}$ | $A_{10,\leq 3}$ | $A_{11,\leq 3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 18 | 66 | 276 | 1 212 | 5 916 | 31 068 | 171 576 | 1 014 696 |

| $A_{1,\leq 4}$ | $A_{2,\leq 4}$ | $A_{3,\leq 4}$ | $A_{4,\leq 4}$ | $A_{5,\leq 4}$ | $A_{6,\leq 4}$ | $A_{7,\leq 4}$ | $A_{8,\leq 4}$ | $A_{9,\leq 4}$ | $A_{10,\leq 4}$ | $A_{11,\leq 4}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 24 | 96 | 456 | 2 472 | 14 736 | 92 304 | 632 736 | 4 661 856 |

suggest that $A_{n,\leq m}$ is perhaps periodic of order five. The next result can be phrased as follows:

*Congruence* 5.

$$A_{n,\leq m} \equiv A_{n+5,\leq m} \pmod{10} \quad (n > 2, m = 2, 3, 4).$$

Our argument presented in the subsection 5.5 will show that the restricted factorial numbers all terminate with digit 0 if $n > 4$ and $m > 4$, this is the reason why we excluded above the case $m > 4$.

Note that $A_{n,\leq 2} = B_{n,\leq 2}$ and this number equals to the number of involutions on $n$ elements. (Involution is a permutation $\pi$ such that $\pi^2 = 1$, the identity permutation.).

# 4 The associated Stirling numbers

At the end we turn to the definition of the associated Stirling numbers. The $m$-associated Stirling number of the second kind with parameters $n$ and $k$, denoted by $\left\{ {n \atop k} \right\}_{\geq m}$, gives the number of partitions of an $n$ element set into $k$ subsets *such that* every block contains at least $m$ elements [11, p. 221].

The associated Bell numbers are

$$B_{n,\geq m} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\geq m}.$$

The tables for $m = 2, 3, 4$ ($B_{n,\geq 1} = B_n$, the $n$th Bell number) follow.

| $B_{1,\geq 2}$ | $B_{2,\geq 2}$ | $B_{3,\geq 2}$ | $B_{4,\geq 2}$ | $B_{5,\geq 2}$ | $B_{6,\geq 2}$ | $B_{7,\geq 2}$ | $B_{8,\geq 2}$ | $B_{9,\geq 2}$ | $B_{10,\geq 2}$ | $B_{11,\geq 2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 4 | 11 | 41 | 162 | 715 | 3 425 | 17 722 | 98 253 |

| $B_{1,\geq 3}$ | $B_{2,\geq 3}$ | $B_{3,\geq 3}$ | $B_{4,\geq 3}$ | $B_{5,\geq 3}$ | $B_{6,\geq 3}$ | $B_{7,\geq 3}$ | $B_{8,\geq 3}$ | $B_{9,\geq 3}$ | $B_{10,\geq 3}$ | $B_{11,\geq 3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 11 | 36 | 92 | 491 | 2 557 | 11 353 |

| $B_{1,\geq 4}$ | $B_{2,\geq 4}$ | $B_{3,\geq 4}$ | $B_{4,\geq 4}$ | $B_{5,\geq 4}$ | $B_{6,\geq 4}$ | $B_{7,\geq 4}$ | $B_{8,\geq 4}$ | $B_{9,\geq 4}$ | $B_{10,\geq 4}$ | $B_{11,\geq 4}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 36 | 127 | 337 | 793 |

Here one cannot observe periodicity in the last digits. However, this is not the case, if we take the associated Fubini numbers, where the order of the blocks counts:

$$F_{n,\geq m} = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\geq m}.$$

Since $F_{n,\geq 1} = F_n$, the $n$th Fubini number, we present the table of these numbers for $m = 2, 3, 4$:

| $F_{1,\geq 2}$ | $F_{2,\geq 2}$ | $F_{3,\geq 2}$ | $F_{4,\geq 2}$ | $F_{5,\geq 2}$ | $F_{6,\geq 2}$ | $F_{7,\geq 2}$ | $F_{8,\geq 2}$ | $F_{9,\geq 2}$ | $F_{10,\geq 2}$ | $F_{11,\geq 2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 7 | 21 | 141 | 743 | 5 699 | 42 241 | 382 153 | 3 586 155 |

| $F_{1,\geq 3}$ | $F_{2,\geq 3}$ | $F_{3,\geq 3}$ | $F_{4,\geq 3}$ | $F_{5,\geq 3}$ | $F_{6,\geq 3}$ | $F_{7,\geq 3}$ | $F_{8,\geq 3}$ | $F_{9,\geq 3}$ | $F_{10,\geq 3}$ | $F_{11,\geq 3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 21 | 71 | 183 | 2 101 | 13 513 | 64 285 |

| $F_{1,\geq 4}$ | $F_{2,\geq 4}$ | $F_{3,\geq 4}$ | $F_{4,\geq 4}$ | $F_{5,\geq 4}$ | $F_{6,\geq 4}$ | $F_{7,\geq 4}$ | $F_{8,\geq 4}$ | $F_{9,\geq 4}$ | $F_{10,\geq 4}$ | $F_{11,\geq 4}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 71 | 253 | 673 | 1 585 |

The next special values are trivial:

$$F_{0,\geq m} = 1, \quad F_{n,\geq m} = 0 \ (0 < n < m), \quad F_{m,\geq m} = 1.$$

We will prove that the associated Fubini numbers are always odd, when $n \geq m$:

*Congruence* 6.

$$F_{n,\geq m} \equiv 1 \pmod 2 \quad (n \geq m),$$

and that the last digits form a periodic sequence of length 20:

*Congruence* 7.

$$F_{n,\geq m} \equiv F_{n+20,\geq m} \pmod{10} \quad (n \geq 5, m = 2, 3, 4, 5).$$

This last congruence is rather unusual, because the length of the period is 20, larger than for the other treated sequences, and, in addition, if $m = 1$ (Fubini number case), the period is just 4.

## 5 Proof of the congruences

Now we start to give the proofs of the above congruences.

### 5.1 The Fubini numbers – Congruence 1

We prove Congruence 1 of Gross, which states that

$$F_{n+4} \equiv F_n \pmod{10}.$$

Let $n > 4$. By the definition,

$$F_{n+4} - F_n = \sum_{k=0}^{n+4} k! \left\{ {n+4 \atop k} \right\} - \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} =$$

$$\sum_{k=5}^{n+4} k! \left\{ {n+4 \atop k} \right\} - \sum_{k=5}^{n} k! \left\{ {n \atop k} \right\} + \sum_{k=0}^{4} k! \left\{ {n+4 \atop k} \right\} - \sum_{k=0}^{4} k! \left\{ {n \atop k} \right\} \equiv$$

$$\left\{ {n+4 \atop 0} \right\} + \left\{ {n+4 \atop 1} \right\} + 2\left\{ {n+4 \atop 2} \right\} + 6\left\{ {n+4 \atop 3} \right\} + 24\left\{ {n+4 \atop 4} \right\}$$

$$-\left\{ {n \atop 0} \right\} - \left\{ {n \atop 1} \right\} - 2\left\{ {n \atop 2} \right\} - 6\left\{ {n \atop 3} \right\} - 24\left\{ {n \atop 4} \right\} \pmod{10}.$$

Because of the special values $\left\{ {n \atop 0} \right\} = 0$, $\left\{ {n \atop 1} \right\} = 1$, the first two members cancel. The remaining terms are divisible by two, so it is enough to prove that

$$5 \left| \left( \left\{ {n+4 \atop 2} \right\} + 3\left\{ {n+4 \atop 3} \right\} + 12\left\{ {n+4 \atop 4} \right\} - \left\{ {n \atop 2} \right\} - 3\left\{ {n \atop 3} \right\} - 12\left\{ {n \atop 4} \right\} \right) \right. .$$

The special values [4, p. 12.]

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1, \quad \left\{ {n \atop 3} \right\} = \frac{1}{2}\left(3^{n-1} - 2^n + 1\right), \quad \left\{ {n \atop 4} \right\} = \frac{1}{6}4^{n-1} - \frac{1}{2}3^{n-1} + 2^{n-2} - \frac{1}{6}$$

give that

$$\left\{ {n+4 \atop 2} \right\} - \left\{ {n \atop 2} \right\} = 2^{n+3} - 1 - (2^{n-1} - 1) = 15 \cdot 2^{n-1},$$

$$3\left\{ {n+4 \atop 3} \right\} - 3\left\{ {n \atop 3} \right\} = \frac{3}{2}\left(3^{n-1}(3^4 - 1) - 2^n(2^4 - 1)\right),$$

$$12\left\{ {n+4 \atop 4} \right\} - 12\left\{ {n \atop 4} \right\} = \frac{12}{2}\left(\frac{1}{3}4^{n-1}(4^4 - 1) - 3^{n-1}(3^4 - 1) + 2^{n-2}(2^4 - 1)\right).$$

All of these numbers – independently from $n$ – are divisible by 5, so we proved the periodicity of Fubini numbers.

### 5.2 The $r$-Fubini numbers – Congruence 2

The generalization of the congruence of Gross to the $r$-Stirling case states that

$$F_{n+4,r} \equiv F_{n,r} \pmod{10} \quad (n, r \geq 1).$$

Definition (2) immediately gives that if $r > 5$, $10 \mid F_{n,r}$ for all $n > 0$, so the periodicity is trivial. If $r = 1$, we get back the ordinary Fubini numbers (up to a shifting), so we can restrict ourselves to $2 \leq r \leq 4$. So

$$\begin{aligned}
F_{n+4,r} - F_{n,r} &\overset{r=2}{\equiv} 2\left\{ {n+4+r \atop r} \right\}_r + 6\left\{ {n+4+r \atop r+1} \right\}_r + 24\left\{ {n+4+r \atop r+2} \right\}_r \\
&\quad - 2\left\{ {n+r \atop r} \right\}_r - 6\left\{ {n+r \atop r+1} \right\}_r - 24\left\{ {n+r \atop r+2} \right\}_r, \\
&\overset{r=3}{\equiv} 6\left\{ {n+4+r \atop r} \right\}_r + 24\left\{ {n+4+r \atop r+1} \right\}_r \\
&\quad - 6\left\{ {n+r \atop r} \right\}_r - 24\left\{ {n+r \atop r+1} \right\}_r, \\
&\overset{r=4}{\equiv} 24\left\{ {n+4+r \atop r} \right\}_r - 24\left\{ {n+r \atop r} \right\}_r \pmod{10}.
\end{aligned}$$

We need the following special values:

$$\left\{ {n+r \atop r} \right\}_r = r^n, \tag{6}$$

$$\left\{ {n+r \atop r+1} \right\}_r = (r+1)^n - r^n, \tag{7}$$

$$\left\{ {n+r \atop r+2} \right\}_r = \frac{1}{2}(r+2)^n - (r+1)^n + \frac{1}{2}r^n. \tag{8}$$

The first identity can be proven easily: the left hand side counts the partitions of $n + r$ elements into $r$ subsets such that the first $r$ elements are in different subsets. Such partitions can be formed in the following way: we put the first $r$ elements into singletons and the remaining $n$ elements go to these $r$ blocks independently: we have $r^n$ possibilities.

The proof of the second special value is similar, but now we have an additional block. We put again our first $r$ elements into $r$ different blocks, and the remaining $n$ elements go to these and to the additional block. Up to now we have $(r+1)^n$ possibilities. But the last block cannot be empty, so we have to exclude the cases when all the $n$ elements go to the first $r$ partition. The number of such cases is $r^n$. These considerations give (7).

The left hand side of the third identity is the number of partitions of $n + r$ elements into $r + 2$ blocks with the usual restriction. Let us suppose that the two additional blocks contain $k$ elements from $n$. We can choose these elements in $\binom{n}{k}$ ways and then we construct a partition with the two blocks: $\left\{{k \atop 2}\right\}$ possibilities. (Note that $k \geq 2$.) The remaining $n-k$ elements go to the first $r$ block independently in $r^{n-k}$ ways. We sum on $k$ to get

$$\left\{{n+r \atop r+2}\right\}_r = \sum_{k=2}^{n} \binom{n}{k} \left\{{k \atop 2}\right\} r^{n-k}.$$

Since $\left\{{k \atop 2}\right\} = 2^{k-1} - 1$, the binomial theorem yields the desired identity.

We go back to prove our congruence of the $r$-Fubini numbers. Since the others are simpler and similar, we deal only with the case $r = 2$; we prove that

$$F_{n+4,r} - F_{n,r} \equiv 2\left\{{n+4+r \atop r}\right\}_r + 6\left\{{n+4+r \atop r+1}\right\}_r + 24\left\{{n+4+r \atop r+2}\right\}_r$$
$$-2\left\{{n+r \atop r}\right\}_r - 6\left\{{n+r \atop r+1}\right\}_r - 24\left\{{n+r \atop r+2}\right\}_r \equiv 0 \pmod{10}.$$

It is enough to prove that the paired terms with the same lower parameter are divisible by five. Our special $r$-Stirling number values imply that

$$\left\{{n+4+r \atop r}\right\}_r - \left\{{n+r \atop r}\right\}_r = 2^n(2^4-1),$$
$$\left\{{n+4+r \atop r+1}\right\}_r - \left\{{n+r \atop r+1}\right\}_r = 3^n(3^4-1) - 2^n(2^4-1),$$
$$\left\{{n+4+r \atop r+2}\right\}_r - \left\{{n+r \atop r+2}\right\}_r = \frac{1}{2}4^n(4^4-1) - 3^n(3^4-1) + \frac{1}{2}2^n(2^4-1).$$

For any $n > 0$, these values are all divisible by five, so we are done.

### 5.3 The restricted Bell numbers – Congruence 3

Let us prove Congruence 3. Miksa, Moser and Wyman [30] proved that

*Congruence* 8.

$$B_{n+p,\leq m} \equiv B_{n,\leq m} \pmod{p} \quad (m < p)$$

holds for any prime $p$. In particular, if $p = 5$, we have that

*Congruence* 9.

$$B_{n+5,\leq m} \equiv B_{n,\leq m} \pmod{5} \quad (m = 2, 3, 4).$$

Utilizing (9), we can prove the periodicity of the last digits if we prove that $B_{n,\leq m}$ is even. This will hold just when $m = 2$ or 3.

A theorem of Miksa, Moser, and Wyman [30, Theorem 2.] says that

$$B_{n+1,\leq m} = B_{n,\leq m} + \binom{n}{1} B_{n-1,\leq m} + \binom{n}{2} B_{n-2,\leq m} + \cdots + \binom{n}{m-1} B_{n-m+1,\leq m}. \tag{9}$$

In particular, if $m = 2$, we have

$$B_{n+1,\leq 2} = B_{n,\leq 2} + \binom{n}{1} B_{n-1,\leq 2},$$

and if $m = 3$, then

$$B_{n+1,\leq 3} = B_{n,\leq 3} + \binom{n}{1} B_{n-1,\leq 3} + \binom{n}{2} B_{n-2,\leq 3}.$$

These show that if two (resp. three) consecutive terms are even, then all the rest are also even. Checking the tables given above, we have, in particular, that $B_{n,\leq 2}$ is even for all $n \geq 2$ and $B_{n,\leq 3}$ is even for all $n \geq 4$.

Taking the above considerations, Congruence 3 is proven.

Although $B_{n,\leq 4}$ does not share this property for small $n$, it can happen that for a large $n$ three consecutive terms are even. This would imply the periodicity from that index. For larger $m$ this is not enough, because Congruence 9 does not hold for these $m$.

#### 5.3.1 A formula for the restricted Bell numbers

Although we do not need them, we also can prove easily the following properties:

$$\begin{aligned} B_{n,\leq m} &= B_n \quad (n \leq m), \\ B_{n,\leq m} &= B_n - \sum_{k=1}^{n-m} \binom{n}{m+k} B_{n-m-k} \quad (m < n \leq 2m). \end{aligned}$$

Here $B_n$ is the $n$th Bell number, defined by

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

The case $n \leq m$ is trivial, since $n < m$ means that there is no restriction, so we get back the Bell numbers indeed. If $m < n \leq 2m$, from the number of all the partitions on $n$ elements (which is $B_n$), we have to exclude the tilted cases, i.e., when one of the blocks contains more than $m$ elements. There can be only one such block, say $A$. If $A$ contains $m + k$ elements ($0 < k \leq n - m$), we have to choose these elements coming to $A$; there are $\binom{n}{m+k}$ cases. In the other blocks there are $n - (m + k)$ elements, which can be partitioned in $B_{n-m-k}$ ways. If we substract all of these possibilities parametrized by $k$, we are done.

In particular, we have that

$$\begin{aligned} B_{m+1,\leq m} &= B_{m+1} - 1 \quad (m > 1), \\ B_{m+2,\leq m} &= B_{m+2} - 1 - (m + 2) \quad (m > 1). \end{aligned}$$

### 5.4 The restricted Fubini numbers – Congruence 4

The number $F_{n,\leq m}$ (see (3)) counts the ordered partitions on $n$ elements, where the blocks in each partition cannot contain more than $m$ elements. We can give the next interpretation as well. There are $F_{n,\leq m}$ ways to classify $n$ persons in a competition where draws are allowed but no more than $m$ persons can have the same position. This helps us to find a simple recursion for $F_{n,\leq m}$: from the $n$ persons $k$ will go to the first place ($1 \leq k \leq m$) and the remaining competitors will classified to the rest of the positions in $F_{n-k,\leq m}$ ways. In formula,

$$F_{n,\leq m} = \binom{n}{1} F_{n-1,\leq m} + \binom{n}{2} F_{n-2,\leq m} + \cdots + \binom{n}{m} F_{n-m,\leq m}. \tag{10}$$

Hence we see that to determine $F_{n,\leq m}$ we need the previous $m$ members of the sequence. In particular, when $m = 2, 3, 4$, to get divisibility by 10, it is enough to see that there are two, three, four consecutive terms divisible by 10. This is satisfied, thus we have justified (4).

Proving (5), we use the definition of the restricted Fubini numbers directly:

$$F_{n,\leq m} = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\leq m} \equiv \left\{ {n \atop 1} \right\}_{\leq m} \pmod 2 \quad (n > 0).$$

The value $\left\{ {n \atop 1} \right\}_{\leq m}$ is zero if $n > m$.

### 5.5 The restricted factorials – Congruence 5

To prove Congruence 5, we begin with the dual of the identity (9):

$$A_{n+1,\leq m} = \tag{11}$$

$$A_{n,\leq m} + nA_{n-1,\leq m} + n(n-1)A_{n-2,\leq m} + \cdots + n(n-1)\cdots(n-m)A_{n-m+1,\leq m}.$$

The initial values are $A_{0,\leq m} = A_{1,\leq m} = 1$.

The proof is as follows. The number $A_{n+1,\leq m}$ gives the number of permutations on the set $\{1, 2, \ldots, n+1\}$ such that none of the cycles in the permutation contains more than $m$ elements. We pick up the last element, say. This can go to a one element cycle, and the remaining $n$ elements go to a restricted permutations in $A_{n,\leq m}$ ways. If the cycle of the last element contains $k \geq 1$ additional elements ($k < m$), we have to form a cycle with these elements, and the remaining elements go to a restricted permutation in $A_{n-k,\leq m}$ ways. We choose these elements in $\binom{n}{k}$ ways. But, since the order of the elements in the cycle counts, we multiply $\binom{n}{k}$ with $k!$. Summing over $k = 1, 2, \ldots, m-1$, we are done.

This identity helps us to prove the next congruence, which is the dual of Congruence 8:

*Congruence* 10.

$$A_{n+p,\leq m} \equiv A_{n,\leq m} \pmod p \quad (m < p),$$

where $p$ is a prime number.

First we prove that

*Congruence* 11.
$$A_{p,\le m} \equiv 1 \pmod p \quad (m < p).$$

This can be seen via the following representation:

$$A_{n,\le m} = \sum_{1a_1+2a_2+\cdots+ma_m=n} \frac{n!}{1^{a_1}a_1!2^{a_2}a_2!\cdots m^{a_m}a_m!}.$$

The validity of this representation can be seen easily: if we construct a partition on $n$ elements with cycles containing at most $m$ elements, then we can have, say, $a_1$ cycles with one element, $a_2$ cycles with two elements, and finally $a_m$ cycles with $m$ elements. The order $n!$ of the elements must be divided by the order of the cycles with the same elements ($a_1!\cdots a_m!$) and with the identical arrangements in the separate cycles ($1^{a_1}\cdots m^{a_m}$).

Now Congruence 11 can be justified easily: if $m < p$, in the denominator $p$ does not appear as a factor, except when $a_1 = p$. Therefore in the sum all the terms will congruent to 0 modulo $p$, and the term corresponding to $a_1 = p$, $a_2 = \cdots = a_m = 0$ is congruent to 1 modulo $p$.

Having these results, we can give the proof of Congruence 10 by induction. If $n = 0$ the result is just Congruence 11, since $A_{0,\le m} = 1$. Let us suppose that the result holds true for all $k \le n$. By (11) we have that

$$\begin{gathered} A_{n+1+p,\le m} = \\ A_{n+p,\le m} + (n+p)A_{n+p-1,\le m} + (n+p)(n+p-1)A_{n+p-2,\le m} + \cdots + \\ (n+p)(n+p-1)\cdots(n+p-m)A_{n+p-m+1,\le m}. \end{gathered}$$

In the factors of the restricted factorials $p$ can be deleted modulo $p$. This and the induction hypothesis yield that

$$\begin{gathered} A_{n+1+p,\le m} \equiv \\ A_{n,\le m} + nA_{n-1,\le m} + n(n-1)A_{n-2,\le m} + \cdots + \\ (n)(n-1)\cdots(n-m)A_{n-m+1,\le m} \pmod p. \end{gathered}$$

The sum equals to $A_{n+1,\le m}$, and Congruence 10 has been proven.

Finishing up the proof of Congruence 5, we apply (11) when $m = 3, 4$ (the $m = 2$ case is done by the $A_{n,\le 2} = B_{n,\le 2}$ correspondence) to get

$$\begin{aligned} A_{n+1,\le 3} &= A_{n,\le 3} + nA_{n-1,\le 3} + n(n-1)A_{n-2,\le 3}, \\ A_{n+1,\le 4} &= A_{n,\le 4} + nA_{n-1,\le 4} + n(n-1)A_{n-2,\le 4} + n(n-1)(n-2)A_{n-3,\le 4}. \end{aligned}$$

These show that if respectively three and four consecutive terms are even, the rest are even as well. Checking the given tables for these sequences this is justified. Specializing Congruence 10 with $p = 5$, Congruence 5 is proven.

To terminate this section, we additionally prove the next congruence.

*Congruence* 12.
$$A_{n,\leq m} \equiv 0 \pmod{10} \quad (n > m > 4).$$

Identity (11) shows that to determine $A_{n,\leq m}$ we need $m$ consecutive terms terminating with $A_{n-1,\leq m}$. Moreover, $A_{n,\leq m} = n!$ if $n \leq m$ which is congruent to 0 modulo 10 whenever $m > 4$. Hence, again by the recursion, it is enough to prove that

$$A_{m+1,\leq m} \equiv A_{m+2,\leq m} \equiv A_{m+3,\leq m} \equiv A_{m+4,\leq m} \equiv 0 \pmod{10}. \tag{12}$$

First, we consider the case

$$A_{m+1,\leq m} =$$
$$A_{m,\leq m} + mA_{m-1\leq m} + m(m-1)A_{m-2,\leq m} + \cdots + m(m-1)\cdots 2A_{1,\leq m} =$$
$$m! + m(m-1)! + m(m-1)(m-2)! + \cdots + m(m-1)\cdots 2\cdot 1! = m\cdot m!.$$

If $m > 4$, this is already congruent to 0 modulo 10. Let us continue with

$$A_{m+2,\leq m} =$$
$$A_{m+1,\leq m} + (m+1)A_{m\leq m} + (m+1)mA_{m-1,\leq m} + \cdots + (m+1)m\cdots 3A_{2,\leq m} =$$
$$m\cdot m! + (m+1)m! + (m+1)m(m-1)! + \cdots + (m+1)m\cdots 3\cdot 2!$$

All the factors are divisible by 10, because $m > 4$. Hence, together with the last point, we proved the first two congruence of (12). The remaining cases can be treated similarly.

## 5.6 Associated Fubini numbers – Congruences 6 and 7

Now we turn to the parity of the sequences $F_{n,\geq m}$, that is, we prove Congruence 6:

$$F_{n,\geq m} \equiv \left\{ {n \atop 0} \right\}_{\geq m} + \left\{ {n \atop 1} \right\}_{\geq m} \pmod{2}.$$

The first Stirling number term is zero if $n > 0$, and the second one is 1 if $n \geq m$, otherwise it is zero as well.

The proof of Congruence 7 needs to be separated into the different cases when $m = 2, 3, 4, 5$.

First, let $m = 2$. Then we can prove the following special values for associated Stirling numbers of the second kind:

$$\left\{ {n \atop 2} \right\}_{\geq 2} = \frac{1}{2}(2^n - 2n - 2), \tag{13}$$
$$\left\{ {n \atop 3} \right\}_{\geq 2} = \frac{1}{6}(3^n - 3\cdot 2^n) - \frac{1}{2}n(2^{n-1} - 1) + \frac{1}{2}(n^2 + 1), \tag{14}$$
$$\left\{ {n \atop 4} \right\}_{\geq 2} = \frac{4^n}{24} - \frac{3^n}{18}(n+3) - \frac{1}{6}(n^3 + 2n + 1) + \frac{2^n}{16}(n^2 + 3n + 4). \tag{15}$$

These relations hold for $n \geq 4, 6, 8$, respectively. To see why the first relation holds, let us form a partition of $n$ elements into two blocks such that these blocks contain at least two elements. We can sort our elements into the two blocks in $2^n$ ways, and then sort out the tilted partitions. A partition is tilted if one of the blocks does not contain any element (two possibilities), or one of the blocks contains just one element; which is $2n$ possibilities. Since the order of the blocks does not matter, we divide by two. This consideration gives (13).

Relation (14) can be proven as follows. We pick one of the blocks from the three and select at least two elements putting them into this block, but we cannot select more than $n-4$ items to assure us that in the other blocks remain at least $2+2$ elements. Let $k$ be the number of the selected items. The two unpicked blocks can be considered as a partition on $n-k$ elements with two blocks: $\left\{ {n-k \atop 2} \right\}_{\geq 2}$ possibilities. Hence we get our intermediate relation:

$$\left\{ {n \atop 3} \right\}_{\geq 2} = \frac{1}{3} \sum_{k=2}^{n-4} \binom{n}{k} \left\{ {n-k \atop 2} \right\}_{\geq 2}.$$

We divided by 3, because to pick the initial block we had three equivalent possibilities. The order of the blocks does not count. Finally, substituting the special value (13), after some sum binomial manipulations we are done.

We note that the above identity can be generalized as follows:

$$\left\{ {n \atop k} \right\}_{\geq m} = \frac{1}{k} \sum_{k=m}^{n-(k-1)m} \binom{n}{k} \left\{ {n-k \atop k-1} \right\}_{\geq m} \quad (n \geq km). \tag{16}$$

The just presented generalization is applicable to prove (15), too.

Now let us go back to the proof of Congruence 7 with $m = 2$.

$$F_{n+20,\geq 2} - F_{n,\geq 2} = \sum_{k=0}^{n+20} k! \left\{ {n+20 \atop k} \right\}_{\geq 2} - \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\geq 2} \equiv$$

$$\equiv \left\{ {n+20 \atop 0} \right\}_{\geq 2} + \left\{ {n+20 \atop 1} \right\}_{\geq 2} + 2\left\{ {n+20 \atop 2} \right\}_{\geq 2} + 6\left\{ {n+20 \atop 3} \right\}_{\geq 2} + 24\left\{ {n+20 \atop 4} \right\}_{\geq 2}$$
$$-\left\{ {n \atop 0} \right\}_{\geq 2} - \left\{ {n \atop 1} \right\}_{\geq 2} - 2\left\{ {n \atop 2} \right\}_{\geq 2} - 6\left\{ {n \atop 3} \right\}_{\geq 2} - 24\left\{ {n \atop 4} \right\}_{\geq 2} \pmod{10}.$$

The first terms with lower parameters 0 and 1 are cancelled by the respective terms in the second line. What remains is

$$F_{n+20,\geq 2} - F_{n,\geq 2} = \frac{1}{6}\big( -320 \cdot 3^n(1939523823 + 87169610n) +$$
$$15 \big(219902325555 \cdot 2^{2n+1} - 8(1547 + 6n(39 + 2n)) +$$
$$3 \cdot 2^n(92833928 + n(8808038 + 209715n))\big)\big).$$

An induction shows that this is always divisible by 5 if $n \geq 5$.

Now let us fix $m = 3$. The special values for the associated Stirling numbers with small lower parameters are

$$\begin{aligned}
\left\{ n \atop 2 \right\}_{\geq 3} &= \frac{1}{2}\left(2^n - 2 - 2n - 2\binom{n}{2}\right), \\
\left\{ n \atop 3 \right\}_{\geq 3} &= \frac{1}{16}\big(24 - 3 \cdot 2^{3+n} + 8 \cdot 3^n + 12n - 9 \cdot 2^n \cdot n + \\
&\quad 42n^2 - 3 \cdot 2^n \cdot n^2 - 12n^3 + 6n^4\big), \\
\left\{ n \atop 4 \right\}_{\geq 3} &= -3^{-2+n}(n^2 + 5n + 18) + \\
&\quad \frac{1}{64}\big(2^{2n+5} + 3 \cdot 2^n(64 + 42n + 19n^2 + 2n^3 + n^4) - \\
&\quad 16(8 - 32n + 112n^2 - 91n^3 + 43n^4 - 9n^5 + n^6)\big).
\end{aligned}$$

The first identity can be proven by the following combinatorial argument: we separate the $n$ elements into two blocks in $2^n$ ways. The blocks have to contain at least two elements, so we substract the cases when one of the blocks is empty (2 possibilities), contains one element ($2n$ cases), or contains two elements ($2\binom{n}{2}$ cases). The order of the blocks is indifferent so we divide by 2.

The two remaining special values are consequences of (16).

Following the argument we presented above calculating $F_{n+20,\geq 2} - F_{n,\geq 2}$, one can prove Congruence 7 for $m = 3$.

The rest of the cases (when $m = 4, 5$) can be treated similarly, however, the calculations are more involved technically.

At the end we note that it is easy to prove the corresponding formula of (10) with respect to the associated Fubini numbers.

$$F_{n,\geq m} = \binom{n}{n} F_{0,\geq m} + \binom{n}{n-1} F_{1,\geq m} + \cdots + \binom{n}{m} F_{n-m,\geq m} \quad (n \geq m).$$

# 6 Closing remarks

In the present paper we discussed the modular properties (mostly in base 10) of several combinatorial numbers coming from restricted set partitions and cycle decomposition of permutations. In our case the restriction means that large or small blocks/cycles are not permitted (restricted and associated Stirling numbers, respectively) or the first elements are in different blocks/cycles ($r$-Stirlings).

## 6.1 Other bases

It is sure that the proven congruences are not the only ones of this style. For instance, the above used approach can also be applied to prove that the last digits of the Fubini numbers are periodic of length 4 not just in the decimal base but in bases 2, 3, 4, 5, 6, 8 as well:

$$F_{n+4} \equiv F_n \pmod{2, 3, 4, 5, 6, 8} \quad (n > 4),$$

from which Congruence 1 already follows.

What is more, in base 3 we have that

$$F_{n+3} \equiv F_n + \begin{cases} 1, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Such kind of results probably exists for the other sequences of this paper, like the $r$-Fubini numbers.

Other class of results was proved by Kauffman [23], who revealed periodicity of the Fubini numbers with higher modulus. Namely, she proved that

$$\begin{aligned} F_{n+20} &\equiv F_n \pmod{100} \quad (n > 1), \\ F_{n+100} &\equiv F_n \pmod{1\,000} \quad (n > 2), \\ F_{n+500} &\equiv F_n \pmod{10\,000} \quad (n > 3). \end{aligned}$$

These together with the Gross congruence

$$F_{n+4} \equiv F_n \pmod{10}$$

suggest that there might be a minimal period $m(k)$ for every $10^k$, that is,

$$F_{n+m(k)} \equiv F_n \pmod{10^k}$$

for some function $m$. Hence $m(1) = 4$, $m(2) = 20$, $m(3) = 100$ and $m(4) = 500$. It would be interesting to find the function $m$ in general.

## 6.2 Possible directions of investigation

In the literature there exist other generalizations of the "unrestricted" partitions and permutations (which are enumerated by the classical Stirling numbers of the first and second kind).

In a recent paper Mihoubi and Maamra [29] defined the $(r_1, \ldots, r_p)$-Stirling numbers which are extensions of the $r$-Stirling numbers. One could ask about the periodicity and other modular properties of the $(r_1, \ldots, r_p)$-Fubini numbers. Up to our knowledge, there is no investigation with respect to these numbers, however, there is a manuscript on $(r_1, \ldots, r_p)$-Bell numbers by Maamra [25].

Another direction of generalization comes from lattice theory. In 1973 Dowling [16] constructed a class of geometric lattices with fixed underlying finite groups. The Whitney numbers of these lattices are generalizations of the Stirling numbers of both kinds with an additional parameter (which is the order of the underlying group). There are several papers dealing with these numbers [2, 3, 7, 21, 27, 32, 33]. The periodicity of these sequences would be an interesting question to deal with. In particular, one could use the papers of Benoumhani [2, 3] who made some short remarks on the new Fubini numbers derived from Whitney numbers.

A relatively new direction of research is the graph theoretical extension of the Stirling and Bell numbers [17, 19, 24]. It could be an interesting question, how to define Fubini numbers in this setting, because there is no block order. However, if it is possible, one could investigate the modular properties of these numbers, too.

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