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Finite Reciprocal Sums Involving Summands That are Balanced Products of Generalized Fibonacci Numbers

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Abstract

In this paper we find closed forms, in terms of rational numbers, for certain finite sums. The denominator of each summand is a finite product of terms drawn from two sequences that are generalizations of the Fibonacci and Lucas numbers.

1 Introduction

In [1, 2] we considered certain types of finite reciprocal sums involving generalized Fibonacci numbers. Indeed we gave closed forms, in terms of rational numbers, for these sums. Our purpose here is to give closed forms for finite reciprocal sums that are of a different type than those considered in [1, 2], thereby extending the work in [1, 2]. As in [1, 2], our results can be used to produce finite reciprocal sums that involve the Fibonacci and Lucas numbers.

We begin by introducing the three pairs of integer sequences that feature in this paper. Let $a \geq 0$ and $b \geq 0$ be integers with $(a, b) \neq (0, 0)$. For p a positive integer we define, for all integers n, the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b,$$

and

$$\overline{W}_n = W_{n-1} + W_{n+1}.$$

For $(a,b,p)=(0,1,1)$ we have $\{W_n\}$=$\{F_n\}$, and $\{\overline{W}_n\}$=$\{L_n\}$, which are the Fibonacci and Lucas numbers, respectively. Retaining the parameter p, and taking $(a,b)=(0,1)$, we write $\{W_n\}$=$\{U_n\}$, and $\{\overline{W}_n\}$=$\{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas numbers, respectively.

Let α and β denote the two distinct real roots of $x^2-px-1=0$. Set $A=b-a\beta$ and $B=b-a\alpha$. Then the closed forms (the Binet forms) for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta},$$

and

$$\overline{W}_n=A\alpha^n+B\beta^n.$$

We require also the constants $e_W=AB=b^2-pab-a^2$, and $\Delta=p^2+4$.

Throughout this paper we take $k\geq 1$, $m\geq 0$, and $n\geq 2$ to be integers. Let $m_1<m_2$ and $m_3<m_4$ be non-negative integers with $m_1+m_2=m_3+m_4$. We begin by giving a closed form for the finite sum

$$S_4(k,m,n,m_1,\ldots,m_4)=\sum_{i=1}^{n-1}\frac{1}{W_{k(i+m_1)+m}W_{k(i+m_2)+m}\overline{W}_{k(i+m_3)+m}\overline{W}_{k(i+m_4)+m}}.$$

Because of the conditions on the m_i we consider S_4 to be the most intriguing sum that we present in this paper. We also give closed forms for similar sums that have longer products in the denominator, and to this end we introduce some notation. For integers $0<m_1<m_2$ write

$$\begin{aligned}&P_6(W,\overline{W},k,m,i,0,m_1,m_2)\\&=W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}\overline{W}_{ki+m}\overline{W}_{k(i+m_1)+m}\overline{W}_{k(i+m_2)+m}.\end{aligned}$$

Likewise, for integers $0<m_1<m_2<m_3$ write

$$\begin{aligned}&P_8(W,\overline{W},k,m,i,0,m_1,m_2,m_3)\\&=W_{ki+m}W_{k(i+m_1)+m}\cdots W_{k(i+m_3)+m}\overline{W}_{ki+m}\overline{W}_{k(i+m_1)+m}\cdots\overline{W}_{k(i+m_3)+m}.\end{aligned}$$

We say that P_8 consists of a *balanced* product of eight terms that are drawn from the sequences $\{W_n\}$ and $\{\overline{W}_n\}$. We also consider such products where the terms are drawn from the sequences $\{U_n\}$ and $\{V_n\}$, and where the product $U_{ki+m}V_{ki+m}$ is not included. For instance $P_4(U,V,k,m,i,m_1,m_2)$ denotes the product of $U_{k(i+m_1)+m}U_{k(i+m_2)+m}$ and $V_{k(i+m_1)+m}V_{k(i+m_2)+m}$. Since $U_nV_n=U_{2n}$, P_4 can be shortened to a product of two terms from the sequence $\{U_n\}$. However, we choose to retain the longer form for P_4 (and for expressions analogous P_4) in order to highlight the relationship between the numerator and the denominator of the summand. Later, however, when giving examples of our results that involve F_n and L_n, we present these examples in simplified form.

Using the notation that we have just introduced, we now define three finite sums whose closed forms we give in this paper. Let $0 < m_1 < m_2$ be integers. Define

$$S_6^0(k,m,n,m_1,m_2) = \sum_{i=1}^{n-1} \frac{P_2(U,V,k,m,i,0)}{P_6(W,\overline{W},k,m,i,0,m_1,m_2)},$$

$$S_6^1(k,m,n,m_1,m_2) = \sum_{i=1}^{n-1} \frac{P_2(U,V,k,m,i,m_1)}{P_6(W,\overline{W},k,m,i,0,m_1,m_2)},$$

and

$$S_6^2(k,m,n,m_1,m_2) = \sum_{i=1}^{n-1} \frac{P_2(U,V,k,m,i,m_2)}{P_6(W,\overline{W},k,m,i,0,m_1,m_2)}.$$

In each of these three cases the numerator of the summand consists of a product of two terms, and the denominator of the summand consists of a product of six terms.

We evaluate each of the finite sums that we consider in this paper in terms of rational numbers. In addition to the four finite sums defined above, we consider analogous finite sums where the denominator of the summand consists of a product of eight, or ten, or twelve terms. In each case the numerator of the summand is either unity, or is a product of terms drawn from the sequences $\{U_n\}$ and $\{V_n\}$ and is defined in terms of the P notation that we have introduced. Furthermore, we consider only finite sums where *the number of terms that constitute the product in the denominator of the summand exceeds the number of terms that constitute the product in the numerator of the summand by a multiple of four.* Indeed these are the only types of sums, with the structure described earlier in this paragraph, for which we have been able to find closed forms.

In Section 2 we present one result, namely the closed form for S_4, and give a proof. In Section 3 we present the closed forms for S_6^0, S_6^1, and S_6^2. In subsequent sections we present a selection of the results that we have found that involve longer products in the denominator of the summand. Indeed, we limit the scope of this paper to finite sums that have four, or six, or eight, or ten, or twelve products in the denominator of the summand.

There are two finite sums that feature throughout. For integers $0 \le l_1 < l_2$ these finite sums are

$$\Omega_W(k,m,n,l_1,l_2) = \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{W_{k(i+2)+m}W_{k(i+n)+m}},$$

and

$$\Omega_{\overline{W}}(k,m,n,l_1,l_2) = \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{\overline{W}_{k(i+2)+m}\overline{W}_{k(i+n)+m}}.$$

To prevent the presentation from becoming too unwieldy, we suppress certain arguments from quantities when there is no danger of confusion. For instance $S_4(n)$ will denote $S_4(k,m,n,m_1,m_2,m_3,m_4)$ when we want n to vary and the other parameters to remain fixed. Likewise $\Omega_W(k,m,n,l_1,l_2)$ will be denoted by $\Omega_W(l_1,l_2)$ when l_1 and l_2 vary and the other parameters remain fixed.

We now give two identities involving Ω_W and $\Omega_{\overline{W}}$ that are required for the proofs of all the theorems in this paper. We state these identities as lemmas.

Lemma 1. *With Ω_W as defined above, we have*

$$U_{k(n-1)}\Omega_W(n+1) - U_{k(n-2)}\Omega_W(n) = \frac{(-1)^{k(n+l_1)}U_{k(l_2-l_1)}}{W_{k(n+l_1)+m}W_{k(n+l_2)+m}}.$$

Lemma 2. *With $\Omega_{\overline{W}}$ as defined above, we have*

$$U_{k(n-1)}\Omega_{\overline{W}}(n+1) - U_{k(n-2)}\Omega_{\overline{W}}(n) = \frac{(-1)^{k(n+l_1)}U_{k(l_2-l_1)}}{\overline{W}_{k(n+l_1)+m}\overline{W}_{k(n+l_2)+m}}.$$

The proof of Lemma 1 was given in [2], and since the proof of Lemma 2 is similar we refrain from giving it here.

2 A closed form for S_4

Let k, m, n, m_1, m_2, m_3, and m_4 satisfy the constraints given earlier in the definition of S_4. Set

$$a_0 = a_0(k,m_1,m_2,m_3,m_4) = e_W U_{(m_4-m_3)k}U_{(m_2-m_1)k}V_{(m_4-m_1)k}V_{(m_4-m_2)k}.$$

Then

Theorem 3. *With S_4 as defined in Section 1,*

$$\begin{aligned} a_0\left(S_4(n) - S_4(2)\right) &= (-1)^m U_{k(n-2)}\big((-1)^{k(m_1+m_3)}U_{(m_4-m_3)k}\Omega_W(m_1,m_2) \\ &\quad - \Delta U_{(m_2-m_1)k}\Omega_{\overline{W}}(m_3,m_4)\big). \end{aligned}$$

Proof. In [2] we demonstrated two methods of proof, and both methods apply here. The first method required quite a lot to set up, relying heavily upon generalized Fibonacci identities. The second method was more direct, and mechanical, relying upon the closed forms of the relevant sequences. We use the second method here since it is transparent and can be used to effectively prove all the theorems in this paper. To this end, with $\alpha = \left(p+\sqrt{\Delta}\right)/2$, it is advantageous to write the closed forms of the sequences in question as

$$\begin{aligned} U_n &= \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{\Delta}, \\ V_n &= \alpha^n + (-1)^n\alpha^{-n}, \\ W_n &= \left(\left(b+a\alpha^{-1}\right)\alpha^n + (-1)^{n+1}\left(b-a\alpha\right)\alpha^{-n}\right)/\sqrt{\Delta}, \\ \overline{W}_n &= \left(b+a\alpha^{-1}\right)\alpha^n + (-1)^n\left(b-a\alpha\right)\alpha^{-n}, \end{aligned}$$

where these closed forms are valid for all integers n. Furthermore we set $p = \alpha - \alpha^{-1}$, and $e_W = b^2 - pab - a^2$.

We remind the reader that all the finite sums in this paper are defined for $n \geq 2$, and so it is for these values of n that the following argument holds. In the statement of Theorem 3, denote the quantity on the left side by $L(n)$ and the quantity on the right side by $R(n)$. We first prove that

$$L(n+1) - L(n) = R(n+1) - R(n). \tag{1}$$

With the previously stated restrictions on the relevant parameters, we have

$$L(n+1) - L(n) = \frac{e_W U_{(m_4-m_3)k} U_{(m_2-m_1)k} V_{(m_4-m_1)k} V_{(m_4-m_2)k}}{W_{k(n+m_1)+m} W_{k(n+m_2)+m} \overline{W}_{k(n+m_3)+m} \overline{W}_{k(n+m_4)+m}}. \tag{2}$$

With the use of Lemma 1 and Lemma 2 we can write down, after some straightforward algebra, the expression for the numerator of $R(n+1) - R(n)$. This expression is

$$\begin{aligned} (-1)^{k(n+m_3)+m} U_{(m_4-m_3)k} U_{(m_2-m_1)k} \big(&\overline{W}_{k(n+m_3)+m} \overline{W}_{k(n+m_4)+m} \\ &-\Delta W_{k(n+m_1)+m} W_{k(n+m_2)+m}\big). \end{aligned} \tag{3}$$

Furthermore, $L(n+1) - L(n)$ and $R(n+1) - R(n)$ have identical denominators, so to prove (1) it is enough to prove that

$$\begin{aligned} e_W V_{(m_4-m_1)k} V_{(m_4-m_2)k} \quad = \quad (-1)^{k(n+m_3)+m} \big(&\overline{W}_{k(n+m_3)+m} \overline{W}_{k(n+m_4)+m} \\ &-\Delta W_{k(n+m_1)+m} W_{k(n+m_2)+m}\big). \end{aligned} \tag{4}$$

To this end we consider the difference of the expressions on the left and right sides of (4), replace m_4 by $m_1 + m_2 - m_3$, and express everything in terms of the closed forms. With the use of the computer algebra system *Mathematica 8* we find that a factor of the resulting expression is $(-1)^{2(k(n+m_3)+m)} - 1$, and this proves (4). This, together with the fact that $L(2) = R(2) = 0$, establishes Theorem 3. $\square$

In the proof above the key identity is (4). Likewise, the proof of each theorem in this paper hinges around the proof of a key identity that is analogous to (4), and each such identity follows immediately by substitution of the appropriate closed forms. The method is mechanical and is not dependent upon and special identities. However, the use of a computer algebra system (in our case *Mathematica 8*) is essential. The proof above serves as a template for the proof of each theorem in this paper, and so we state the theorems in the sections that follow without proof.

We pause to give two examples. Let $k = 1$, $m = 0$, and $(m_1, m_2, m_3, m_4) = (0, 3, 1, 2)$. Then for $W_n = F_n$ the result in Theorem 3 becomes

$$36 \sum_{i=1}^{n-1} \frac{1}{F_i F_{i+3} L_{i+1} L_{i+2}} = 1 + F_{n-2} \left(\frac{6}{F_n} - \frac{3}{F_{n+1}} + \frac{2}{F_{n+2}} - \frac{15}{L_{n+1}} \right).$$

Next let $k=2$, $m=0$, and $(m_1,m_2,m_3,m_4)=(2,3,1,4)$. Then for $W_n=F_n$ the result in Theorem 3 becomes

$$\begin{aligned}
&40790736\sum_{i=1}^{n-1}\frac{1}{F_{2(i+2)}F_{2(i+3)}L_{2(i+1)}L_{2(i+4)}}\\
&=282+F_{2(n-2)}\left(\frac{92496}{F_{2(n+2)}}-\frac{67445}{L_{2(n+1)}}-\frac{25830}{L_{2(n+2)}}-\frac{9870}{L_{2(n+3)}}\right).
\end{aligned}$$

3 Closed forms for S_6^0, S_6^1, and S_6^2

As stated in the introduction, here, and in the sequel we take $k\geq 1$, $m\geq 0$, and $n\geq 2$ to be integers. In this section we take $0<m_1<m_2$ to be integers.

For $0\leq i\leq 2$ define $a_i=a_i(k,m,m_1,m_2)$ as

$$\begin{aligned}
a_0 &= 2(-1)^{m+1}e_W^3U_{2m_1k}U_{2m_2k}U_{2(m_2-m_1)k},\\
a_1 &= U_{2(m_2-m_1)k}W_0\overline{W}_0,\\
a_2 &= -U_{2m_1k}W_{m_2k}\overline{W}_{m_2k}.
\end{aligned}$$

We then have

Theorem 4. *With S_6^0 as defined in Section 1,*

$$\begin{aligned}
a_0\left(S_6^0(n)-S_6^0(2)\right) &= U_{k(n-2)}\left(a_1\Omega_W(0,m_1)+a_2\Omega_W(m_1,m_2)\right.\\
&\quad \left.-\Delta a_1\Omega_{\overline{W}}(0,m_1)-\Delta a_2\Omega_{\overline{W}}(m_1,m_2)\right).
\end{aligned}$$

Next, for $1\leq i\leq 2$ define $b_i=b_i(k,m_1,m_2)$ by

$$\begin{aligned}
b_1 &= U_{2(m_2-m_1)k}W_{-m_1k}\overline{W}_{-m_1k},\\
b_2 &= -U_{2m_1k}W_{(m_2-m_1)k}\overline{W}_{(m_2-m_1)k}.
\end{aligned}$$

With a_0 as for Theorem 4, we have

Theorem 5. *Let S_6^1 be as defined in Section 1. Then*

$$\begin{aligned}
a_0\left(S_6^1(n)-S_6^1(2)\right) &= U_{k(n-2)}\left(b_1\Omega_W(0,m_1)+b_2\Omega_W(m_1,m_2)\right.\\
&\quad \left.-\Delta b_1\Omega_{\overline{W}}(0,m_1)-\Delta b_2\Omega_{\overline{W}}(m_1,m_2)\right).
\end{aligned}$$

For $1\leq i\leq 2$ define $c_i=c_i(k,m_1,m_2)$ as

$$\begin{aligned}
c_1 &= U_{2(m_2-m_1)k}W_{-m_2k}\overline{W}_{-m_2k},\\
c_2 &= -U_{2m_1k}W_0\overline{W}_0.
\end{aligned}$$

Then, with a_0 as for Theorem 4, we have

Theorem 6. *Let S_6^2 be as defined in Section 1. Then*

$$\begin{aligned} a_0\left(S_6^2(n) - S_6^2(2)\right) &= U_{k(n-2)}\left(c_1\Omega_W(0,m_1) + c_2\Omega_W(m_1,m_2)\right. \\ &\quad \left. - \Delta c_1\Omega_{\overline{W}}(0,m_1) - \Delta c_2\Omega_{\overline{W}}(m_1,m_2)\right). \end{aligned}$$

Let $k = 1$, $m = 0$, and $(m_1, m_2) = (1, 2)$. Take $W_n = F_{n+2}$. Then with the use of the identity $F_nL_n = F_{2n}$ to simplify the summand, and also the right side, the result in Theorem 5 becomes

$$27720\sum_{i=1}^{n-1}\frac{F_{2(i+1)}}{F_{2(i+2)}F_{2(i+3)}F_{2(i+4)}} = 9 + 8F_{2(n-2)}\left(\frac{-55}{F_{2(n+2)}} + \frac{168}{F_{2(n+3)}}\right).$$

4 The summand has eight factors in the denominator

In this section we take $0 < m_1 < m_2 < m_3$ to be integers.

Define the finite sum

$$S_8(k,m,n,m_1,m_2,m_3) = \sum_{i=1}^{n-1}\frac{1}{P_8(W,\overline{W},k,m,i,0,m_1,m_2,m_3)}.$$

For $0 \le i \le 3$ define $a_i = a_i(k,m,m_1,m_2,m_3)$ by

$$\begin{aligned} a_0 &= 2e_W^3U_{2m_1k}U_{2m_2k}U_{2m_3k}U_{2(m_3-m_2)k}U_{2(m_3-m_1)k}U_{2(m_2-m_1)k}, \\ a_1 &= (-1)^mU_{2(m_3-m_2)k}U_{2(m_3-m_1)k}U_{2(m_2-m_1)k}, \\ a_2 &= (-1)^{m+1}U_{2m_1k}U_{2(m_3-m_2)k}U_{2(m_3+m_2-m_1)k}, \\ a_3 &= (-1)^mU_{2m_1k}U_{2m_2k}U_{2(m_2-m_1)k}. \end{aligned}$$

Then

Theorem 7. *We have*

$$\begin{aligned} a_0\left(S_8(n) - S_8(2)\right) &= U_{k(n-2)}\left(a_1\Omega_W(0,m_1) + a_2\Omega_W(m_1,m_2)\right. \\ &\quad + a_3\Omega_W(m_2,m_3) - \Delta a_1\Omega_{\overline{W}}(0,m_1) \\ &\quad \left. - \Delta a_2\Omega_{\overline{W}}(m_1,m_2) - \Delta a_3\Omega_{\overline{W}}(m_2,m_3)\right). \end{aligned}$$

We have found that as the number of products in the denominator of the summand increases it becomes more difficult to write down the closed form of the corresponding finite sum. The same is true as the number of products in the numerator of the summand increases. Indeed, to find the closed form for the finite sum T_8, defined below, we needed to specialize the values of m_1, m_2, and m_3 in a manner that we soon make clear.

Define the sum

$$T_8(k,m,n,m_1,m_2,m_3) = \sum_{i=1}^{n-1}\frac{P_4\left(U,V,k,m,i,m_1,m_2\right)}{P_8(W,\overline{W},k,m,i,0,m_1,m_2,m_3)}.$$

Here, and in the sequel, we take $g \geq 1$ to be an integer. For $0 \leq i \leq 3$ define the quantities $b_i = b_i(g, k, m)$ as follows:

$$\begin{aligned}
b_0 &= 2e_W^5 U_{2gk} U_{4gk} U_{6gk}, \\
b_1 &= (-1)^m W_{-2gk} W_{-gk} \overline{W}_{-2gk} \overline{W}_{-gk}, \\
b_2 &= (-1)^{m+1} V_{2gk} W_{-gk} W_{gk} \overline{W}_{-gk} \overline{W}_{gk}, \\
b_3 &= (-1)^m W_{gk} W_{2gk} \overline{W}_{gk} \overline{W}_{2gk}.
\end{aligned}$$

We now state our next theorem.

Theorem 8. *Let $(m_1, m_2, m_3) = (g, 2g, 3g)$, so that 0, m_1, m_2, and m_3 form an arithmetic progression. Then*

$$\begin{aligned}
b_0 \left(T_8(n) - T_8(2)\right) = {} & U_{k(n-2)} \left(b_1 \Omega_W(0, g) + b_2 \Omega_W(g, 2g)\right. \\
& + b_3 \Omega_W(2g, 3g) - \Delta b_1 \Omega_{\overline{W}}(0, g) \\
& \left. - \Delta b_2 \Omega_{\overline{W}}(g, 2g) - \Delta b_3 \Omega_{\overline{W}}(2g, 3g)\right).
\end{aligned}$$

Let $k = 1$, $m = 0$, and $g = 1$. Then for $W_n = F_{n+3}$ the result in Theorem 8 becomes

$$\begin{aligned}
& 167207040 \sum_{i=1}^{n-1} \frac{F_{2(i+1)} F_{2(i+2)}}{F_{2(i+3)} F_{2(i+4)} F_{2(i+5)} F_{2(i+6)}} \\
& = 64 + 63 F_{2(n-2)} \left(\frac{6032}{F_{2(n+3)}} - \frac{145145}{F_{2(n+4)}} + \frac{338800}{F_{2(n+5)}} \right).
\end{aligned}$$

We attempted to find the closed forms for certain finite sums analogous to T_8, but without success. Firstly, we considered T_8 as defined above but with the m_i defined as different multiples of g, such as $(m_1, m_2, m_3) = (2g, 3g, 5g)$. Secondly, in the definition of T_8, we replaced $P_4(U, V, k, m, m_1, m_2)$ by $P_4(U, V, k, m, 0, m_2)$, and by $P_4(U, V, k, m, m_1, m_3)$. In attempting to find the corresponding closed forms in each of these two cases we set $(m_1, m_2, m_3) = (g, 2g, 3g)$. Put simply, there seems to be a fine line between success and failure.

5 The summand has ten factors in the denominator

In this section we take $0 < m_1 < m_2 < m_3 < m_4$ to be integers.

Define the sum

$$S_{10}(k, m, n, m_1, m_2, m_3, m_4) = \sum_{i=1}^{n-1} \frac{P_6(U, V, k, m, i, m_1, m_2, m_3)}{P_{10}(W, \overline{W}, k, m, i, 0, m_1, m_2, m_3, m_4)}.$$

For $0 \le i \le 4$ define the quantities $a_i = a_i(g,k,m)$ as follows:

$$\begin{aligned}
a_0 &= 2e_W^7 U_{2gk} U_{4gk} U_{6gk} U_{8gk},\\
a_1 &= (-1)^{m+1} W_{-3gk} W_{-2gk} W_{-gk} \overline{W}_{-3gk} \overline{W}_{-2gk} \overline{W}_{-gk},\\
a_2 &= (-1)^{m} \left(V_{4gk}+1\right) W_{-2gk} W_{-gk} W_{gk} \overline{W}_{-2gk} \overline{W}_{-gk} \overline{W}_{gk},\\
a_3 &= (-1)^{m+1} \left(V_{4gk}+1\right) W_{-gk} W_{gk} W_{2gk} \overline{W}_{-gk} \overline{W}_{gk} \overline{W}_{2gk},\\
a_4 &= (-1)^{m} W_{gk} W_{2gk} W_{3gk} \overline{W}_{gk} \overline{W}_{2gk} \overline{W}_{3gk}.
\end{aligned}$$

Once again, to discover our next result we needed the m_i to take on special values.

Theorem 9. *Let $(m_1, m_2, m_3, m_4) = (g, 2g, 3g, 4g)$, so that 0, m_1, m_2, m_3, and m_4 form an arithmetic progression. Then*

$$\begin{aligned}
a_0\left(S_{10}(n) - S_{10}(2)\right) &= U_{k(n-2)}\left(a_1\Omega_W(0,g) + a_2\Omega_W(g,2g)\right.\\
&\quad + a_3\Omega_W(2g,3g) + a_4\Omega_W(3g,4g)\\
&\quad - \Delta a_1\Omega_{\overline{W}}(0,g) - \Delta a_2\Omega_{\overline{W}}(g,2g)\\
&\quad \left. - \Delta a_3\Omega_{\overline{W}}(2g,3g) - \Delta a_4\Omega_{\overline{W}}(3g,4g)\right).
\end{aligned}$$

We have found that the most interesting examples of our results occur when we take $W_n = F_{n+c}$ for some non-negative integer c. Accordingly, as an instance of Theorem 9 let $k = 1$, $m = 0$, and $g = 1$. Then for $W_n = F_{n+5}$ Theorem 9 yields

$$\begin{aligned}
&b_0 \sum_{i=1}^{n-1} \frac{F_{2(i+1)}F_{2(i+2)}F_{2(i+3)}}{F_{2(i+5)}F_{2(i+6)}F_{2(i+7)}F_{2(i+8)}F_{2(i+9)}}\\
&= 7 + 144F_{2(n-2)}\left(\frac{b_1}{F_{2(n+5)}} + \frac{b_2}{F_{2(n+6)}} + \frac{b_3}{F_{2(n+7)}} + \frac{b_4}{F_{2(n+8)}}\right),
\end{aligned}$$

where

$$\begin{aligned}
b_0 &= 13009146630480,\\
b_1 &= -239632085,\\
b_2 &= 35147981440,\\
b_3 &= -632668766730,\\
b_4 &= 1419740509642.
\end{aligned}$$

We also considered variants of S_{10} that we obtained in ways similar to those described in the paragraph at the end of Section 4. However, in each case we were unable to find the closed form of the corresponding finite sum. Furthermore, we considered summands with only two factors in the numerator. Once again, in each case, we were unable to find the closed form of the corresponding finite sum. Theorem 9 is the only result of its kind (i.e., where the summand has ten factors in the denominator, and where the m_i are multiples of a positive integer parameter g) that we could find. We have discovered closed forms for such sums where the m_i are *specific* integers, but we refrain from giving these sums.

6 The summand has twelve factors in the denominator

In this section we take $0 < m_1 < m_2 < m_3 < m_4 < m_5$ to be integers.

Define the sum

$$S_{12}(k, m, n, m_1, m_2, m_3, m_4, m_5) = \sum_{i=1}^{n-1} \frac{1}{P_{12}(W, \overline{W}, k, m, i, 0, m_1, m_2, m_3, m_4, m_5)}.$$

For $0 \le i \le 5$ we define the quantities $a_i = a_i(g, k, m)$ as

$$\begin{aligned} a_0 &= 2e_W^5 U_{2gk} U_{4gk} U_{8gk} U_{10gk} U_{12gk}, \\ a_1 &= a_5 = (-1)^m, \\ a_2 &= a_4 = (-1)^{m+1} \left(V_{8gk} + V_{4gk} - 1\right), \\ a_3 &= (-1)^m \left(V_{12gk} - V_{8gk} + 2\right). \end{aligned}$$

In our next theorem 0, m_1, m_2, m_3, m_4, and m_5 are not required to form an arithmetic progression.

Theorem 10. *Setting* $(m_1, m_2, m_3, m_4, m_5) = (g, 2g, 4g, 5g, 6g)$ *we have*

$$\begin{aligned} a_0 \left(S_{12}(n) - S_{12}(2)\right) = \; & U_{k(n-2)} \left(a_1 \Omega_W(0, g) + a_2 \Omega_W(g, 2g)\right. \\ & + a_3 \Omega_W(2g, 4g) + a_4 \Omega_W(4g, 5g) \\ & + a_5 \Omega_W(5g, 6g) - \Delta a_1 \Omega_{\overline{W}}(0, g) \\ & - \Delta a_2 \Omega_{\overline{W}}(g, 2g) - \Delta a_3 \Omega_{\overline{W}}(2g, 4g) \\ & \left. - \Delta a_4 \Omega_{\overline{W}}(4g, 5g) - \Delta a_5 \Omega_{\overline{W}}(5g, 6g)\right). \end{aligned}$$

We also managed to find a closed form for S_{12} if $(m_1, m_2, m_3, m_4, m_5) = (g, 2g, 3g, 4g, 5g)$, but we refrain from giving this result here.

To indicate other types of results that are possible, define the sum

$$T_{12}(k, m, n, m_1, m_2, m_3, m_4, m_5) = \sum_{i=1}^{n-1} \frac{P_8\left(U, V, k, m, i, m_1, m_2, m_3, m_4\right)}{P_{12}(W, \overline{W}, k, m, i, 0, m_1, m_2, m_3, m_4, m_5)}.$$

We managed to find a closed form for T_{12} under the assumption that the m_i are certain multiples of g. One such instance is for $(m_1, m_2, m_3, m_4, m_5)$=$(g, 2g, 3g, 4g, 5g)$. Another instance is for $(m_1, m_2, m_3, m_4, m_5)$=$(2g, 3g, 4g, 5g, 7g)$. We have also discovered other results of a similar nature that we do not present here.

7 Concluding comments

Earlier we stated that we chose to limit the scope of this paper to finite sums that have four, or six, or eight, or ten, or twelve products in the denominator of the summand. We have, however, discovered closed forms for finite sums (with the structure described in the introduction) that have fourteen, or sixteen, or eighteen, or twenty products in the denominator of the summand. The possibilities seem limitless.

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