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Lagrange's Algorithm Revisited: Solving $at^2 + btu + cu^2 = n$ in the Case of Negative Discriminant

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Abstract

We make more accessible a neglected continued fraction algorithm of Lagrange for solving the equation $at^2 + btu + cu^2 = n$ in relatively prime integers t, u, where $a > 0$, $\gcd(a, n) = 1$, and $D = b^2 - 4ac < 0$. The cases $D = -4$ and $D = -3$ present a consecutive convergents phenomenon which aids the search for solutions.

1 Introduction

At the end of a memoir in 1770, Lagrange [8, pp. 717–726] gave a method for finding the solutions of a positive definite binary form equation

$$at^2 + btu + cu^2 = n, \tag{1}$$

where $\gcd(t,u)=1$, $\gcd(a,n)=1$, $b^2-4ac<0$, $a>0$, and $n>0$. For such a solution, $\gcd(u,n)=1$ and hence the congruence $\theta u \equiv t \pmod{n}$ has a unique solution θ in the range $-n/2<\theta\leq n/2$. Then

$$\begin{aligned} at^2+btu+cu^2 &\equiv 0 \pmod{n} \\ a(\theta u)^2+b(\theta u)u+cu^2 &\equiv 0 \pmod{n} \\ a\theta^2+b\theta+c &\equiv 0 \pmod{n}. \end{aligned} \tag{2}$$

Lagrange [8, p. 700] used the transformation

$$t=\theta u-ny \tag{3}$$

to convert equation (1) to

$$Pu^2+Quy+Ry^2=1, \tag{4}$$

where $P=(a\theta^2+b\theta+c)/n, Q=-(2a\theta+b), R=na$.

(We remark that, conversely, if (u,y) is a solution of (4), then $(t,u)=(\theta u-ny,u)$ is a solution of (1) with $\gcd(t,u)=1$.)

We note that $D=b^2-4ac=Q^2-4PR$. Also if (u,y) is a solution of (4), so is $(-u,-y)$.

Lagrange [8] proved in Sections $20, 35$ and 39 of his paper that $u/y, y>0$ is a convergent to $-Q/2P$. His proof was long and hard to follow. The aim of this paper is to give a short proof that Lagrange's assertion holds, apart from certain exceptional cases. If $D\neq -3$, this is done in Section 3, where we use the following standard test due to Lagrange [7, Satz 2.11, p. 39]:

Lemma 1. *If a rational x/y with $\gcd(x,y)=1$ and $y\geq 1$ has the property that $|\omega-x/y|<1/2y^2$, then x/y is a convergent of the continued fraction expansion of ω.*

If $D=-3$, more care is needed. In Section 5, we use the following criterion from [3, Theorem 172, p. 140]:

Lemma 2. *If $\omega=\frac{P\zeta+R}{Q\zeta+S}$, where $\zeta>1$ and P,Q,R,S are integers such that $Q>S>0$ and $PS-QR=\pm1$, then $R/S=A_{n-1}/B_{n-1}$ and $P/Q=A_n/B_n$ are consecutive convergents of ω. Also ζ is the $(n+1)$th complete convergent of ω.*

(The author [5] used this approach successfully in an earlier paper [5] on Lagrange's work, when $D>0$.)

Lagrange gave solution bounds

$$u\leq\sqrt{4R/(4PR-Q^2)},\quad y\leq\sqrt{4P/(4PR-Q^2)}, \tag{5}$$

which are easy to derive by completion of the square in equation (4).

We note that in a series of papers, K. S. Williams [9] also considered congruence (2), but did not consider equation (4) and instead examined the continued fraction of θ/n, thereby generalizing a method of Hermite and Cornacchia. The algorithm presented in Section 6 of the present paper is quite different and is also easy to program.

We use the continued fraction notation $[a_0, \ldots, a_n]$.

It is well-known (see [1, Theorem 59, p. 75]) that equation (4), when soluble, has two solutions if $D < -4$, four solutions if $D = -4$ and six solutions if $D = -3$.

It is easy to show that if $D = -4, Q = 2N$ and (u, y) is a solution of (4), then $(-(Nu + Ry), Pu + Ny)$ is also a solution. Whereas if $D = -3, Q = 2N + 1$ and (u, y) is a solution of (4), then $(-(Nu+Ry), Pu+(N+1)y)$ and $(-(u(N+1)+yR), Pu+Ny)$ are also solutions. In sections 4 and 5, if $D = -4$ or -3, it is shown that apart from certain exceptional cases, these solutions of (4), apart from sign, arise from consecutive convergents to $-Q/2P$. This important fact is used in the algorithm of Section 6 and was not mentioned by Lagrange.

The algorithm is available for online experimentation at [6].

Remark 3. The assumption that $\gcd(a, n) > 1$ involves no loss of generality. For we can assume that $\gcd(a, b, c) = 1$. Then as pointed out by Gauss [2, p. 221] (also see [4, Lemma 2, pp. 311–312]), there exist relatively prime integers α, γ such that $a\alpha^2 + b\alpha\gamma + c\gamma^2 = A$, where $\gcd(A, n) = 1$. The construction uses the factorization of n. Then if $\alpha\delta - \beta\gamma = 1$, the transformation $t = \alpha T + \beta U, u = \gamma T + \delta U$ converts $at^2 + btu + cu^2$ to $AT^2 + BTU + CU^2$, with the two forms representing the same integers.

2 Exceptional cases

We first list some exceptional cases where the solutions (u, y) of (4) are easily found.

(a) $D < -4$ and $P = 1$. Then the solutions are $(u, y) = \pm(1, 0)$.

(b) $D = -4$. Then $Q = 2N$.

  (i) If $P = 1$, then $R = N^2 + 1$ and the solutions (u, y) are $\pm(1, 0)$ and $\pm(-N, 1)$. Here $(-N, 1) = (A_0, B_0)$.

  (ii) If $P = 2$, then $R = (N^2 + 1)/2$, where N is odd and the solutions (u, y) are $\pm(\frac{(-N+1)}{2}, 1)$ and $\pm(\frac{-(N+1)}{2}, 1)$. Here $(-(N + 1)/2, 1) = (A_1, B_1)$.

(c) $D = -3$. Then $Q = 2N + 1$.

  (i) If $P = 1$, then $R = N^2 + N + 1$ and the solutions (u, y) are $\pm(1, 0), \pm(-N, 1)$ and $\pm(-(N + 1), 1)$. Here $(-(N + 1), 1) = (A_0, B_0)$.

  (ii) If $P = 3$, then $R = (N^2 + N + 1)/3$, where $N \equiv 1 \pmod 3$ and solutions (u, y) are $\pm(\frac{(-N+1)}{3}, 1), \pm(\frac{-(2N+1)}{3}, 2)$ and $\pm(\frac{-(N+2)}{3}, 1)$. Here $(\frac{-(2N+1)}{3}, 2) = (A_1, B_1)$ and $(\frac{-(N+2)}{3}, 1) = (A_0, B_0)$.

From now on, we exclude these cases.

3 The case $D \neq -3$

Theorem 4. *Let u and $y > 0$ be integers satisfying (4), where $D = Q^2 - 4PR < 0$ and P, Q, R are integers, $P > 0, D \neq -3$ and $P \neq 2$ if $D = -4$. Then u/y is a convergent to $\omega = -Q/2P$.*

Proof. We derive the inequality

$$\left|\omega - \frac{u}{y}\right| < \frac{1}{2y^2}. \tag{6}$$

Then Lemma 1 shows that u/y is a convergent to $\omega = -Q/2P$.

(a) Let $Q = 2N$. Then $\omega = -N/P$ and

$$\begin{aligned}\left|\omega - \frac{u}{y}\right| = \left|-\frac{N}{P} - \frac{u}{y}\right| &< \frac{1}{2y^2}\\ \iff |Pu + Ny| &< \frac{P}{2y}.\end{aligned} \tag{7}$$

From (4), with $\Delta = -D/4 = PR - N^2$, we have

$$u = \frac{-Ny \pm \sqrt{P - \Delta y^2}}{P} \tag{8}$$

and hence

$$Pu + Ny = \pm\sqrt{P - \Delta y^2}. \tag{9}$$

Then (7) becomes

$$\sqrt{P - \Delta y^2} < \frac{P}{2y}, \tag{10}$$

which reduces, on cross-multiplying, to

$$(P - 2y^2)^2 + 4(\Delta - 1)y^4 > 0. \tag{11}$$

However (11) holds if $\Delta > 1$ or if $\Delta = 1$ and $P \neq 2y^2$. But if $\Delta = 1$ and $P = 2y^2$, then $y = 1$, as $\gcd(P, y) = 1$. Hence $P = 2$ and this case was excluded.

(b) Let $Q = 2N + 1$ and $\Delta = -D = 4PR - (2N + 1)^2 > 0$. Then

$$\begin{aligned}\left|\omega - \frac{u}{y}\right| = \left|-\frac{2N+1}{2P} - \frac{u}{y}\right| &< \frac{1}{2y^2}\\ \iff |2Pu + (2N + 1)y| &< \frac{P}{y}.\end{aligned} \tag{12}$$

From (4), we have

$$u = \frac{-(2N + 1)y \pm \sqrt{4P - \Delta y^2}}{2P} \tag{13}$$

and hence

$$2Pu + (2N+1)y = \pm\sqrt{4P - \Delta y^2}. \tag{14}$$

Then (12) becomes

$$\sqrt{4P - \Delta y^2} < \frac{P}{y}, \tag{15}$$

which reduces, on cross-multiplying, to

$$(P - 2y^2)^2 + (\Delta - 4)y^4 > 0. \tag{16}$$

This inequality holds, as $\Delta \equiv -1 \pmod 4$ and so $\Delta > 4$ if $\Delta \neq 3$. $\square$

4 The case $D = -4$: Finer detail

The next result has the useful computational aspect that once we find a convergent that gives a solution of (4), we know that the next convergent will also give a solution and complete the search for that value of θ.

Theorem 5. *Let $D = -4, P \neq 1, 2$ and $(u, y), y > 0$ be a solution of (4). Then $Q = 2N$ and $\frac{u}{y}$ and $\frac{-t(Nu+Ry)}{t(Pu+Ny)}$ are consecutive convergents to $-Q/2P$, where $t = sgn(Pu + Ny)$.*

Proof. We have the identity

$$\frac{-Q}{2P} = \frac{-N}{P} = \frac{u\xi - Nu - Ry}{y\xi + Pu + Ny}, \tag{17}$$

where $\xi = \frac{y}{Pu+Ny}$. From equation (8) with $\Delta = 1$, we have

$$Pu + Ny = \pm\sqrt{P - y^2}, \tag{18}$$

where $P > y^2$. (We note that $P = y^2$ would imply $y = 1 = P$, as $\gcd(P, y) = 1$ and this is excluded.)

Case 1. Assume $Pu+Ny = \sqrt{P - y^2}$. Then $\xi = y/\sqrt{P - y^2} > 0$. Then $\xi \neq 1$, as otherwise $\sqrt{P - y^2} = y, P = 2y^2$ and $y = 1, P = 2$, as $\gcd(P, y) = 1$; however this case was excluded.

(i) Assume $2y^2 > P$. Then $\xi > 1$. For

$$\begin{aligned} \xi > 1 &\iff y > \sqrt{P - y^2} \\ &\iff 2y^2 > P. \end{aligned}$$

Also

$$\frac{-N}{P} = \frac{u\xi + (-Nu - Ry)}{y\xi + (Pu + Ny)}.$$

Then as $y > Pu + Ny > 0$, Lemma 2 implies that $\frac{u}{y} = \frac{A_m}{B_m}$ and $\frac{-Nu-Ry}{Pu+Ny} = \frac{A_{m-1}}{B_{m-1}}$ are consecutive convergents to $-N/P$.

(ii) Assume $2y^2 < P$. Then $0 < \xi < 1$. Also

$$\frac{-N}{P} = \frac{u - (Nu - Ry)(\xi^{-1})}{y + (Pu + Ny)(\xi^{-1})}.$$

Then as $y < Pu + Ny$, Lemma 2 implies that $\frac{u}{y} = \frac{A_{m-1}}{B_{m-1}}$ and $\frac{-Nu-Ry}{Pu+Ny} = \frac{A_m}{B_m}$ are consecutive convergents to $-N/P$.

Case 2. Assume $Pu + Ny = -\sqrt{P - y^2}$. Then $\xi = y/(-\sqrt{P - y^2}) < 0$. We cannot have $\xi = -1$ as otherwise $P = 2$.

(i) Assume $2y^2 > P$. Then $|\xi| > 1$ and hence $-\xi > 1$. Also

$$\frac{-N}{P} = \frac{u(-\xi) + Nu + Ry}{y(-\xi) - (Pu + Ny)},$$

and $y > -(Pu + Ny) > 0$. Hence $\frac{u}{y} = \frac{A_m}{B_m}$ and $\frac{Nu+Ry}{-(Pu+Ny)} = \frac{A_{m-1}}{B_{m-1}}$ are consecutive convergents to $-N/P$.

(ii) Assume $2y^2 < P$. Then $|\xi| < 1$ and hence $-\xi^{-1} > 1$. Also

$$\frac{-N}{P} = \frac{u + (Nu + Ry)(-\xi^{-1})}{y - (Pu + Ny)(-\xi^{-1})},$$

where $y < -(Pu + Ny)$. Hence $\frac{u}{y} = \frac{A_{m-1}}{B_{m-1}}$ and $\frac{Nu+Ry}{-(Pu+Ny)} = \frac{A_m}{B_m}$ are consecutive convergents to $-N/P$.

$\square$

5 The case $D = -3$

The case $D = -3$ was excluded from Theorem 4 and we discuss it now.

The next result has the useful computational aspect that once we find a convergent that gives a solution of (4), we know that the next two convergents will give two further solutions and complete the search for that value of θ.

Theorem 6. *Let $D = -3, P \neq 1, 3$ and $(u, y), y > 0$ be a solution of (4). Then $Q = 2N + 1$ and the rational numbers*

$$\frac{u}{y}, \quad \frac{-s(Nu + Ry)}{s(Pu + (N + 1)y)}, \quad \frac{t(u(N + 1) + yR)}{t(Pu + Ny)}$$

are consecutive convergents in some order to $-Q/2P$, where $s = sgn(Pu + (N + 1)y$ and $t = sgn(Pu + Ny)$.

Proof. We have the identity

$$\frac{-Q}{2P} = \frac{-(2N+1)}{2P} = \frac{u\xi - (N+1)u - Ry}{y\xi + Pu + Ny}, \tag{19}$$

where $\xi = \frac{Pu+(N+2)y}{2Pu+(2N+1)y}$. From equation (14) with $\Delta = 3$, we have

$$2Pu + (2N+1)y = \pm\sqrt{4P - 3y^2}, \tag{20}$$

where $4P > 3y^2$. (We have $4P - 3y^2 \neq 0$, as otherwise $4P = 3y^2$ and hence $y = 2, P = 3$, which was excluded.)

Case 1. Assume $2Pu + (2N+1)y = \sqrt{4P - 3y^2}$. Then

$$Pu + (N+2)y = \frac{\sqrt{4P - 3y^2} + 3y}{2} > 0$$

and hence $\xi > 0$. Also $\xi \neq 1$. For

$$\begin{aligned}
\xi = 1 \implies \frac{\sqrt{4P-3y^2}+3y}{2} &= \sqrt{4P-3y^2} \\
\implies 3y &= \sqrt{4P-3y^2} \\
\implies 3y^2 &= P \\
\implies y &= 1, P = 3,
\end{aligned}$$

which was excluded.

We note that $Pu + Ny > 0 \iff \sqrt{4P - 3y^2} > y \iff P > y^2$.

(i) Assume $3y^2 < P$. Then $0 < \xi < 1$. For

$$\begin{aligned}
\xi < 1 &\iff Pu + (N+2)y < 2Pu + (2N+1)y \\
&\iff y < Pu + Ny \\
&\iff y < \frac{\sqrt{4P-3y^2} - y}{2} \\
&\iff 3y < \sqrt{4P-3y^2} \\
&\iff 3y^2 < P.
\end{aligned}$$

Then (19) gives

$$\frac{-Q}{2P} = \frac{-(2N+1)}{2P} = \frac{u - ((N+1)u + Ry)\xi^{-1}}{y + (Pu+Ny)\xi^{-1}}. \tag{21}$$

Also we have $y < Pu + Ny$ and $\xi^{-1} > 1$. Then Lemma 2 applied to (21) implies that $\frac{u}{y} = \frac{A_{m-1}}{B_{m-1}}$ and $\frac{-(N+1)u-Ry}{Pu+Ny} = \frac{A_m}{B_m}$ are consecutive convergents of $-Q/2P$.

(ii) Assume $3y^2 > P > y^2$. Then we have $\xi > 1, y > Pu + Ny > 0$ and by Lemma 2 applied to equation (21), it follows that $\frac{u}{y} = \frac{A_r}{B_r}$ and $\frac{-(N+1)u-Ry}{Pu+Ny} = \frac{A_{r-1}}{B_{r-1}}$ are consecutive convergents to $-Q/2P$.

(iii) Assume $P < y^2$. Then we have $\xi > 2$. For

$$\begin{aligned}\xi > 2 &\iff Pu + (N+2)y > 2(2Pu + (2N+1)y)\\ &\iff 0 > Pu + Ny\\ &\iff P < y^2.\end{aligned}$$

We rewrite equation (19) as

$$\frac{-(2N+1)}{2P} = \frac{u(\xi-1) - (Nu+Ry)}{y(\xi-1) + Pu + (N+1)y}. \tag{22}$$

Then we have $\xi - 1 > 1, y > Pu + (N+1)y > 0$ and by Lemma 2 applied to equation (22), it follows that $\frac{u}{y} = \frac{A_s}{B_s}$ and $\frac{-Nu-Ry}{Pu+(N+1)y} = \frac{A_{s-1}}{B_{s-1}}$ are consecutive convergents to $-Q/2P$.

We now link up each pair of solution convergents found in Cases 1(i)–(iii) with a third solution convergent. We start by employing the equations

$$\xi_1 = \frac{-Pu-(N-1)y}{2Pu+(2N+1)y}, \tag{23}$$

$$\frac{-(2N+1)}{2P} = \frac{u\xi_1 - Nu - Ry}{y\xi_1 + Pu + (N+1)y}. \tag{24}$$

We find a pair of convergents which we list corresponding to Cases 1(i) and (ii):

$$\text{(i)}\quad \frac{u}{y} = \frac{A_{m-1}}{B_{m-1}},\ \frac{-(N+1)u-Ry}{Pu+Ny} = \frac{A_m}{B_m}\ \frac{-Nu-Ry}{Pu+(N+1)y} = \frac{A_{m+1}}{B_{m+1}},$$

$$\text{(ii)}\quad \frac{-(N+1)u-Ry}{Pu+Ny} = \frac{A_{r-1}}{B_{r-1}},\ \frac{u}{y} = \frac{A_r}{B_r},\ \frac{-Nu-Ry}{Pu+(N+1)y} = \frac{A_{r+1}}{B_{r+1}}.$$

For Case 1(iii), we employ the equations

$$\xi_2 = \frac{-Pu-(N-1)y}{Pu+(N+2)y}, \tag{25}$$

$$\frac{-(2N+1)}{2P} = \frac{((N+1)u+Ry)\xi_2 - Nu - Ry}{-(Pu+Ny)\xi_2 + Pu + (N+1)y}. \tag{26}$$

We then find a pair of convergents that is listed with the pair found in Case 1(iii):

$$\text{(iii)}\quad \frac{(N+1)u+Ry}{-(Pu+Ny)} = \frac{A_{s-1}}{B_{s-1}} \text{ and } \frac{-Nu-Ry}{Pu+(N+1)y} = \frac{A_s}{B_s},\ \frac{u}{y} = \frac{A_{s+1}}{B_{s+1}}.$$

This finishes Case 1.

Case 2. Assume $2(Pu + Ny) + y = -\sqrt{4P - 3y^2}$. Summarising, we find after tedious calculation, the following three results:

(i) $P > 3y^2$:

$$\frac{u}{y} = \frac{A_{m-1}}{B_{m-1}}, \quad \frac{Nu + Ry}{-Pu - (N+1)y} = \frac{A_m}{B_m}, \quad \frac{(N+1)u + Ry}{-Pu - Ny} = \frac{A_m}{B_{m+1}};$$

(ii) $3y^2 > P > y^2$:

$$\frac{Nu + Ry}{-Pu - (N+1)y} = \frac{A_{m-1}}{B_{m-1}}, \quad \frac{u}{y} = \frac{A_m}{B_m}, \quad \frac{(N+1)u + Ry}{-Pu - Ny} = \frac{A_{m+1}}{B_{m+1}};$$

(iii) $y^2 > P$:

$$\frac{-Nu - Ry}{Pu + (N+1)y} = \frac{A_{m-1}}{B_{m-1}}, \quad \frac{(N+1)u + Ry}{-Pu - Ny} = \frac{A_m}{B_m}, \quad \frac{u}{y} = \frac{A_{m+1}}{B_{m+1}}.$$

□

6 The continued fraction based algorithm

The following algorithm finds all solutions (t, u) with $\gcd(t, u) = 1$, of equation (1), when $\gcd(a, n) = 1$. The most time-consuming part involves solving the quadratic congruence (2), as this depends on finding the prime factorization of n. This is also a feature of Williams' algorithm, as well as Gauss' algorithm in [1, p. 75]. This dependence means that for practial purposes, n is restricted to less than 200 digits. The present algorithm, like that of Williams, also uses Euclid's algorithm, whereas Gauss' algorithm uses reduction of positive definite forms, and these have similar running times.

Input: Integers $a, b, c, n, b^2 - 4ac < 0, n > 0, \gcd(a, n) = 1$.
Output: All solutions, if any, of $at^2 + btu + cu^2 = n, \gcd(t, u) = 1$.
Solve $a\theta^2 + b\theta + c \equiv 0 \pmod{n}$, $-n/2 < \theta \leq n/2$.
If there are no solutions, **exit**.
Let $\theta_0, \ldots, \theta_{s-1}$ be the congruence solutions in the range $(-n/2, n/2]$.
$D := b^2 - 4ac$.
for $k = 0, \ldots, s-1$ **do**
  $P := (a\theta_k^2 + b\theta_k + c)/n, Q := 2a\theta_k + b$;
  if $D < -4$ **and** $P = 1$ **then** $(u, y) := \pm(1, 0)$;
  if $D = -4$, **then** $N := Q/2$;
    if $P = 1$, **then** $(u, y) := \pm(1, 0), \pm(-N, 1)$;

if $P = 2$, **then** $(u, y) := \pm(-N + 1)/2, 1), \pm(-(N + 1)/2, 1)$.
if $D = -3$, **then** $N := (Q - 1)/2$.
if $P = 1$, **then** $(u, y) := \pm(1, 0), \pm(-N, 1), \pm(-(N + 1), 1)$.
if $P = 3$, **then**
$(u, y) := \pm((-N + 1)/3, 1), \pm(-(2N + 1)/3, 2), \pm(-(N + 2)/3, 1)$.
print exceptional solutions $(t, u) := (\theta_k u - ny, u)$;
continue to next k;
$i := 0$;
bound $:= \sqrt{4P/(-D)}$;
calculate convergent A_0/B_0 of $-Q/2P$;
while ($B_i \leq$ bound) **do**
if $aA_i^2 + bA_iB_i + cB_i^2 = 1$ **then**
print solutions $(t, u) := \pm(\theta_k A_i - nB_i, A_i)$;
if $D < -4$, **continue** to next k;
if $D = -4$ **or** -3 **then**
calculate convergent A_{i+1}/B_{i+1} of $-Q/2P$;
print solutions $(t, u) := \pm(\theta_k A_{i+1} - nB_{i+1}, A_{i+1})$;
if $D = -4$, **continue** to next k;
if $D = -3$, **then**
calculate convergent A_{i+2}/B_{i+2} of $-Q/2P$;
print solutions $(t, u) := \pm(\theta_k A_{i+2} - nB_{i+2}, A_{i+2})$;
continue to next k;
$i := i + 1$;
calculate convergent A_i/B_i of $-Q/2P$;
end while loop;
end for loop.

Example 7. Find all solutions of $7t^2 - 9tu + 3u^2 = 19, \gcd(t, u) = 1$. Here $D = -3$.

The congruence $7\theta^2 - 9\theta + 3 \equiv 0 \pmod{19}$ has solutions $\theta = -1$ and 5.

$\theta = -1$: The transformation $t = -u - 19y$ converts $7t^2 - 9tu + 3u^2 = 19$ to $u^2 + 23uy + 133y^2 = 1$. This is one of the exceptional cases from Section 2, with solutions $(u, y) = (\pm 1, 0), \pm(-11, 1), \pm(-12, 1)$.

These produce primitive solutions $(t, u) = \pm(1, -1), \pm(8, -11), \pm(7, -12)$.

$\theta = 5$: The transformation $t = 5u - 19y$ converts $7t^2 - 9tu + 3u^2 = 19$ to $7u^2 - 61uy + 133y^2 = 1$.

Then $-Q/2P = 61/14 = [4, 2, 1, 4]$ and we find that convergents $A_0/B_0 = 4/1, A_1/B_1 = 9/2, A_2/B_2 = 13/3$ give solutions (u, y). These in turn give $(t, u) = (1, 4), (7, 9), (8, 13)$. Hence we get primitive solutions $\pm(1, 4), \pm(7, 9), \pm(8, 13)$ of $7t^2 - 9tu + 3u^2 = 19$.

Hence the equation $7t^2 - 9tu + 3u^2 = 19$ has 12 primitive solutions.

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