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Counting Miura-ori Foldings

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Abstract

We consider the problem of enumerating the different ways in which the classic Miura map fold crease pattern can be folded flat. Specifically, we aim to count the number $M(n,m)$ of ways to assign mountains and valleys to the creases so that each vertex in a m by n Miura map fold will be able to fold flat. Recurrence relations and closed formulas are found for small n and arbitrary m. We also prove that the array of numbers generated by $M(n,m)$ is equivalent to the number of ways to properly 3-vertex-color a $m \times n$ grid graph with one vertex pre-colored.

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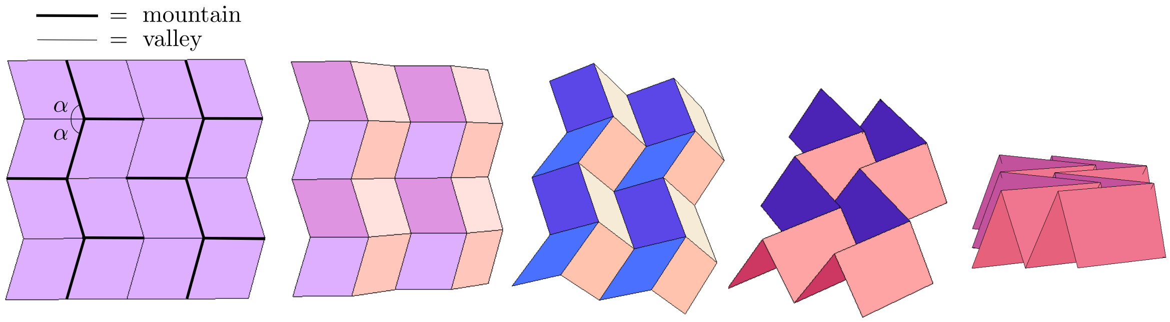

Figure 1: A 4×4 Miura-ori with the standard MV assignment. Bold creases are mountains and non-bold creases are valleys.

1 Introduction

In the mathematics of origami (paper folding), enumerating the number of ways in which a crease pattern can fold up is often difficult. Even the seemingly simple postage-stamp folding problem, where the crease pattern is a grid of orthogonal lines, is unsolved [7, 9, 11]. Furthermore, origami enumeration problems emerge naturally from the study of crumpling polymer membranes, where asymptotic results are obtained for restricted families of crease patterns [1]. Also, in the growing field of self-folding structures knowing the possible folded configurations of a given crease pattern is essential for ensuring that a structure folds into the desired shape.

To be more specific, there are different ways one can count origami foldings of a given crease pattern. To use the postage-stamp folding problem as an example, we may try to count the number of different ways we can assign the creases between vertices in the grid to be *mountains* (convex) or *valleys* (concave) while being able to fold into a flat square without the paper self-intersecting. This problem is unsolved [7]. Alternatively, we can also count the number of ways the layers of paper can be re-arranged in the final, folded state. This version of the question had been explored more; e.g., Koehler [9] wrote an early paper while Jensen and Guttmann [5] improved algorithms for the one-dimensional stamp folding with layers problem from the point of view of meanders. Lunnon (among others) found algorithms for two-dimensional stamp folding with layers counted [11, 12]. In this paper we focus on the former version of the counting foldings problem and ignore variable layer orders of the origami model.

A general theory for enumerating mountain-valley foldings remains elusive except in the single-vertex case [3]. Therefore finding families of crease patterns for which these problems are tractable is of value. One family that has attracted considerable interest among origami applications in engineering and nature is the *Miura map fold*, also known as the *Miura-ori* [13, 14, 16, 19]. Its crease pattern is made of an $m \times n$ array of congruent parallelograms in a herringbone pattern. Figure 1 shows this crease pattern; each vertex consists of two congruent acute angles α and two congruent obtuse angles $\pi - \alpha$, where $0 < \alpha < \pi/2$.

We can determine how a crease pattern folds up from its *mountain-valley (MV) assignment*. If we think of our crease pattern C as a connected, embedded graph in the plane, with vertex set $V(C)$ and edge set $E(C)$, where each edge is a line segment of our crease pattern, then a MV assignment is a function $\mu : E(C) \to \{-1, 1\}$, where -1 indicates a valley and $+1$ a mountain crease. In order for the crease pattern to fold, the function μ must obey certain rules, as will be described in Section 2.

The MV assignment shown in Figure 1 is the standard way in which the Miura-ori is folded flat since it allows the paper to fold and unfold smoothly and rigidly [14]. Many other MV assignments are possible, however, that guarantee each vertex will fold flat. Unfortunately, as we will discuss in Section 5, not every Miura-ori MV assignment that *locally* folds flat will *globally* fold flat.

In this paper we discuss enumerating the number of MV assignments for an $m \times n$ Miura-ori crease pattern that locally fold flat. Specifically, we obtain closed formulas for small values of m and prove a bijection between the general case and the problem of counting the number of ways to 3-vertex color a $m \times n$ grid graph with one vertex pre-colored.

2 Local flat folding conditions

In order for a vertex in an origami crease pattern to fold flat (i.e., can be pressed in a book without crumpling or adding new creases), certain geometric and combinatorial rules must be followed [2].

Kawasaki's theorem states that a vertex v in C will fold flat if and only if the alternating sum of the angles around v, in order, equals zero. In the Miura-ori, we have that this alternating sum about each vertex is

$$\alpha - \alpha + (\pi - \alpha) - (\pi - \alpha) = 0,$$

and therefore each vertex will fold flat.

Maekawa's theorem states that if vertex v in C folds flat, then the difference between the number of mountain and the number of valley creases adjacent to v must be two. Note that this holds at every vertex of the example in Figure 1.

Maekawa's theorem implies that the degree four vertices in the Miura-ori must have three mountain creases and one valley, or vice versa, emanating from them. But not all combinations of three mountains and one valley will work. Using the labeling shown in Figure 2, we cannot have $\mu(e_4) = V$ and $\mu(e_i) = M$ for $i = 1, 2, 3$. Readers are encouraged to try this for themselves and discover how the two acute angles α cannot contain the obtuse angles if e_1, e_2, and e_3 creases are all mountains. We refer to this phenomenon as the *bird's foot forcing*, since these Miura-ori vertices look like a bird's foot. The same thing happens if we reverse all the creases (so that $\mu(e_4) = M$ and the others are V), and so there are only six different MV assignments that can be assigned to a Miura-ori vertex. These are shown in Figure 2.

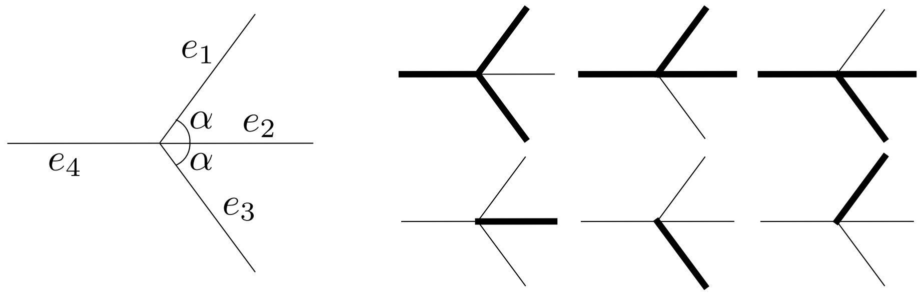

Figure 2: A Miura-ori vertex and the six MV assignments it can have, following the bird's foot forcing.

We say that a MV assignment $\mu : E(C) \to \{\pm 1\}$ is *locally flat foldable* if μ matches one of the MV assignments shown in Figure 2 for every vertex of C. In other words, each vertex will fold flat on its own. We then let $M(m,n)$ denote the number of different locally flat foldable MV assignments μ that can be made for a $m \times n$ Miura-ori (with m rows and n columns of parallelograms). Figure 2 shows that $M(2,2) = 6$. In the next section we develop some closed formulas for cases of $M(m,n)$.

3 Recursions and formulas

First let us calculate $M(2,n)$. This is easily done, since we know that there are six ways to fold a single vertex, so let this vertex be one of our end vertices. Then the vertex adjacent to it has one line already assigned as a mountain or valley, so this vertex is restricted to only having three possible assignments. We then continue on to each adjacent vertex until we get to the $(n-1)$-th vertex. The recursion is $M(2,n) = 3M(2,n-1)$, $M(2,2) = 6$. Therefore

$$M(2,n) = 2 \cdot 3^{n-1}.$$

Notice that this same argument works to show that $M(m,2) = 2 \cdot 3^{m-1}$. In fact, $M(m,n) = M(n,m)$ always holds, but we will delay a proof of this until Section 5.

Looking at a larger case than $m \times 2$ presents new problems. For this, a Lemma will prove useful. We say that a Miura-ori vertex *points left* if the "heel" of the bird's foot points left, as they do in Figure 2. *Points right* is defined similarly.

Lemma 1. *Let C be the crease pattern for a $2 \times n$ Miura-ori with all the vertices pointing left, and let μ be a MV assignment for C. Let c be the left-most crease in C and let γ be a simple closed curve around all the vertices in C (i.e., crossing all the non-internal creases). Let M_γ (resp., V_γ) be the number of mountain (resp., valley) creases that γ crosses. Then we have: If μ is locally flat-foldable then $\mu(c) = (M_\gamma - V_\gamma)/2$.*

As will be seen in the proof, what Lemma 1 is really saying is that $M_\gamma - V_\gamma$ will always equal ± 2, just as in Maekawa's theorem, and that the MV parity of the crease c will control

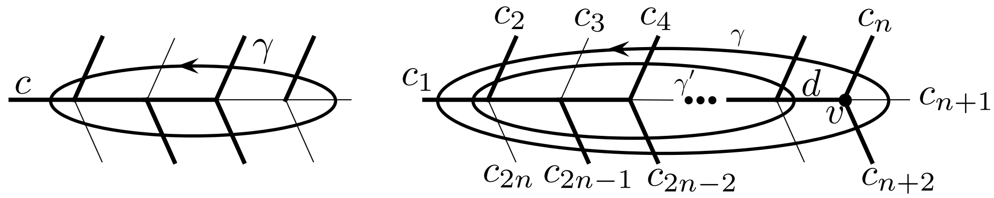

Figure 3: As per Lemma 1, a 2×5 Miura-ori with $M_\gamma = 6$, $V_\gamma = 4$, and $\mu(c) = 1$ (left) and a $2 \times n$ Miura-ori as used in the proof (right).

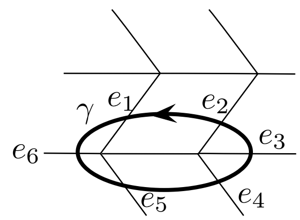

Figure 4: The last row of vertices in an $m \times 3$ Miura-ori crease pattern.

the plus/minus parity of $M_\gamma - V_\gamma$. See Figure 3 for an example. (Note that these properties are not typically present in multiple-vertex flat origami crease patterns. See [4, Activity 23] for examples.)

Proof. We proceed by induction on n. When $n = 2$ we have only one vertex. If the vertex folds flat, then Maekawa's theorem gives us that $M_\gamma - V_\gamma = \pm 2$ and consulting the six possible MV assignments in Figure 2 confirms the result.

Now let C be a $2 \times n$ Miura-ori with MV assignment μ as in the statement of the lemma and label the external creases $c_1, \ldots, c_{2n}$ going clockwise around the crease pattern, with c_1 the left-most crease. Then the right-most crease is c_{n+1}. Let d be the remaining unlabeled crease of the right-most vertex v, so that v has creases d, c_n, c_{n+1} and c_{n+2} going clockwise, as seen in Figure 3.

Suppose that μ is locally flat-foldable in C. Then μ is also locally flat-foldable on the crease pattern $C - v$, which is a $2 \times (n-1)$ Miura-ori. Using the induction hypotheses, we may assume without loss of generality that $M_{\gamma'} - V_{\gamma'} = 2$ for γ' surrounding this $2 \times (n-1)$ Miura-ori, so that c_1 is a M. (The $M_{\gamma'} - V_{\gamma'} = -2$ case is attained by simply switching all the Ms to Vs and vice-versa.) We now look at the crease d.

If d is a M then c_n, c_{n+1} and c_{n+2} must have 2 Ms and 1 V in order for v to be flat foldable. So if we extend γ' to γ which includes v then we still have $M_\gamma - V_\gamma = 2$ since γ no longer crosses d but now crosses c_n, c_{n+1}, and c_{n+2} (we subtract an M add 2 Ms and a V). The case where d is a V is similar, and in both cases, we conclude that $\mu(c_1) = (M_\gamma - V_\gamma)/2$. This completes the proof of the lemma. □

We will now develop recursive equations for $M(m, 3)$ by looking at cases. For an $m \times 3$ crease pattern C, denote the creases adjacent to the bottom-most row of two vertices as in

Figure 4. Then let

$$\begin{aligned}
A_m &= \text{the number of } m \times 3 \text{ MV assignments with } \mu(e_4) = \mu(e_5), \text{ and} \\
B_m &= \text{the number of } m \times 3 \text{ MV assignments with } \mu(e_4) \neq \mu(e_5).
\end{aligned}$$

We start with the initial conditions $A_1 = 2$ and $B_1 = 2$, which makes sense if we think of the 1×3 case as having two parallel creases and zero vertices. Then consider A_m where we have e_4 and e_5 are both Ms and e_6 is also a M. Then drawing a curve γ as in Figure 4, Lemma 1 tells us that $M_\gamma - V_\gamma = 2$ and thus exactly one of e_1, e_2, e_3 is a M. All three of these cases give us flat-foldable vertices for the bottom row. Two of them have $\mu(e_1) \neq \mu(e_2)$ and thus have B_{m-1} ways to assign Ms and Vs to the remaining creases, while one case has e_1 and e_2 being both V, and so has A_{m-1} ways to finish the rest of the crease pattern.

However, if e_6 is a V, then we have $M_\gamma - V_\gamma = -2$ and then all three of e_1, e_2, e_3 are valleys. This gives us A_{m-1} ways to assign Ms and Vs to the remaining creases. Furthermore, the same division of cases would happen if we had chosen e_4 and e_5 to both be Vs. Thus,

$$A_m = 2A_{m-1} + 2B_{m-1}.$$

For B_m, let e_4 be a V and e_5 be a M. Then if e_6 is a M we have $M_\gamma - V_\gamma = 2$, implying that exactly one of e_1, e_2, e_3 is a V. All of these cases are locally flat-foldable for these two vertices, giving is two B_{m-1} cases and one A_{m-1} case. If e_6 is a valley, then, interestingly, e_1 is forced to be a V by the bird's foot forcing. Since $M_\gamma - V_\gamma = -2$, we have that exactly one of e_2 and e_3 is a M, giving us one B_{m-1} and one A_{m-1} case. Thus

$$B_m = 2A_{m-1} + 3B_{m-1}.$$

Then $M(m, 3) = A_m + B_m$. Using the standard techniques of solving a system of first-order linear recurrences (e.g., [18, Sec. 7.5]), we obtain the generating function $(1-x)/(1-5x+2x^2)$ for $M(m, 3)$. The zeros of the denominator are $\alpha_\pm = (5 \pm \sqrt{17})/4$, giving us the closed form

$$M(m, 3) = c_+(1/\alpha_+)^m + c_-(1/\alpha_-)^m$$

where $c_\pm = (17 \pm 3\sqrt{17})/34$.

The $m \times 4$ and $m \times 5$ cases are handled similarly, but with many more cases and details to check, which we skip here for brevity. For $m \times 4$ we let the bottom-most creases be e_1 e_2, and e_3 (playing the same role as e_4 and e_5 in the previous case) and let

$$\begin{aligned}
A_m &= \text{the number of MV assignments with } \mu(e_1) = \mu(e_2) = \mu(e_3), \\
B_m &= \text{the number of MV assignments with } \mu(e_1) = \mu(e_2) \neq \mu(e_3), \\
C_m &= \text{the number of MV assignments with } \mu(e_1) = \mu(e_3) \neq \mu(e_2), \text{ and} \\
D_m &= \text{the number of MV assignments with } \mu e_1 \neq \mu(e_2) = \mu(e_3).
\end{aligned}$$

Examining these cases carefully results in the following first-order recurrences:

$A_m = 2A_{m-1} + B_{m-1} + C_{m-1} + D_{m-1}$
$B_m = A_{m-1} + 2B_{m-1} + 2C_{m-1} + 2D_{m-1}$
$C_m = A_{m-1} + 3B_{m-1} + 3C_{m-1} + D_{m-1}$
$D_m = A_{m-1} + 2B_{m-1} + 2C_{m-1} + 2D_{m-1}$, and $A_1 = B_1 = C_1 = D_1 = 2$.

Once again, we obtain a generating function for $M(m, 4) = A_m + B_m + C_m + D_m$:

$$\frac{4 - 12x + 6x^2}{3 - 27x + 45x^2 - 18x^3}.$$

The denominator of this has zeros $\alpha_1 \approx 1.65406$, $\alpha_2 \approx 0.702511$, and $\alpha_3 \approx 0.143432$, but their expression using radicals is quite messy. The approximate formula is

$$M(m, 4) = c_1(1/\alpha_1)^m + c_2(1/\alpha_2)^m + c_3(1/\alpha_3)^m$$

where $c_1 \approx 0.0132428$, $c_2 \approx 0.21837$, and $c_3 \approx 1.10172$.

The recurrence equations obtained for the $m \times 5$ case using these same methods are:
$A_m = 2A_{m-1} + B_{m-1} + C_{m-1} + D_{m-1} + E_{m-1}$
$B_m = A_{m-1} + 2B_{m-1} + C_{m-1} + D_{m-1} + E_{m-1} + F_{m-1} + G_{m-1} + H_{m-1}$
$C_m = A_{m-1} + B_{m-1} + 2C_{m-1} + D_{m-1} + E_{m-1} + F_{m-1} + 2G_{m-1} + 2H_{m-1}$
$D_m = A_{m-1} + B_{m-1} + C_{m-1} + 2D_{m-1} + E_{m-1} + F_{m-1} + 2G_{m-1} + 2H_{m-1}$
$E_m = A_{m-1} + B_{m-1} + C_{m-1} + D_{m-1} + 2E_{m-1} + F_{m-1} + G_{m-1} + H_{m-1}$
$F_m = B_{m-1} + C_{m-1} + D_{m-1} + E_{m-1} + 2F_{m-1} + 2G_{m-1} + 2H_{m-1}$
$G_m = B_{m-1} + 2C_{m-1} + 2D_{m-1} + E_{m-1} + 2F_{m-1} + 2G_{m-1} + 2H_{m-1}$
$H_m = B_{m-1} + 2C_{m-1} + 2D_{m-1} + E_{m-1} + 2F_{m-1} + 2G_{m-1} + 3H_{m-1}$.

Here the initial $m = 1$ terms are all 2 and $M(m, 5) = A_m + B_m + \cdots + H_m$. Once again we obtain a generating function:

$$\frac{2 - 16x + 36x^2 - 30x^3 + 8x^4}{1 - 16x + 65x^2 - 92x^3 + 48x^4 - 8x^5}.$$

An approximate closed formula can be created from this, although note that the denominator is an irreducible quintic. A closed formula for this system evaded the authors.

4 Coloring bijection

The formulas and recurrences generated thus far can be used to produce the following table of data for $M(m, n)$:

$n \backslash m$	2	3	4	5	6	7	8
2	6	18	54	162	486	1458	4374
3	18	82	374	1706	7782	35498	161926
4	54	374	2604	18150	126534	882180	6150510
5	162	1706	18150	193662	2068146	22091514	235994086

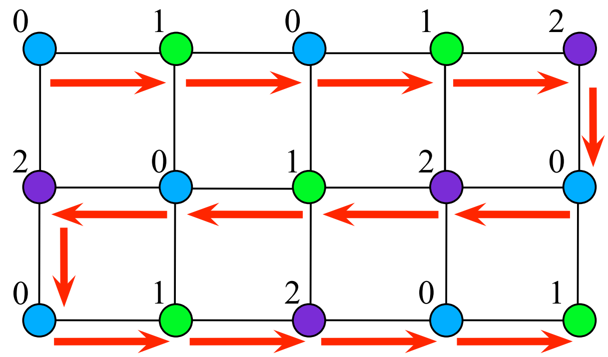

Figure 5: A 3×5 grid graph with proper 3-vertex coloring and arrows showing the order of constructing the coloring sequence s_k.

This table of numbers fits exactly the sequence A078099 in the On-Line Encyclopedia of Integer Sequences [20], which encodes $T(m,n)$ = the number of ways of 3-coloring the vertices of a grid graph (with m rows and n columns of vertices) with one vertex pre-colored. This cannot be a mere coincidence.

Our aim in this section is to construct a bijection between Miura-ori locally flat-foldable MV assignments and such grid graph colorings, i.e., that $M(m,n) = T(m,n)$.

Define an $m \times n$ *grid graph* to be a planar grid graph with m rows and n columns of vertices. We will establish a bijection between the number of proper 3-vertex colorings of $m \times n$ grid graphs with one vertex pre-colored and the number of locally flat-foldable $m \times n$ Miura-ori crease patterns by defining a finite sequence s_k for both of them.

Given a properly 3-vertex colored $m \times n$ grid graph, let the colors be the elements of $\mathbb{Z}_3$ and let the first element of our sequence, s_0 be the color of the upper-left vertex, which we assume to be zero (our pre-colored vertex). Then proceed to the right along the first row of vertices, assigning the colors to the sequence terms s_1 thru s_n. Then let s_{n+1} be the color of the right-most vertex in the second row of vertices (directly under the vertex with s_n's color) and proceed to the left. Continue in this way, snaking back-and-forth across the rows of the grid graph to generate the *grid graph coloring sequence* s_k. For example, the grid graph coloring shown in Figure 5 is

$$s_k : 0, 1, 0, 1, 2, 0, 2, 1, 0, 2, 0, 1, 2, 0, 1$$

Let $v(s_i)$ denote the vertex whose color was assigned to s_i. Note that $v(s_i)$ is adjacent to $v(s_{i+1})$ for $i = 1$ to $mn - 1$, so therefore $s_i \neq s_{i+1}$. Also, vertices directly above and below each other must have different colors. This can be captured in the following relation:

$$v(s_i) \text{ is adjacent to } v(s_j) \text{ if } j - i = 2n - 1 - (2i \bmod 2n)$$

where $j > i$.

We now describe a way to generate a sequence s_k from an $m \times n$ locally flat-foldable Miura-ori crease pattern with MV assignment $\mu : C \to \{-1, 1\}$. We will call such a sequence

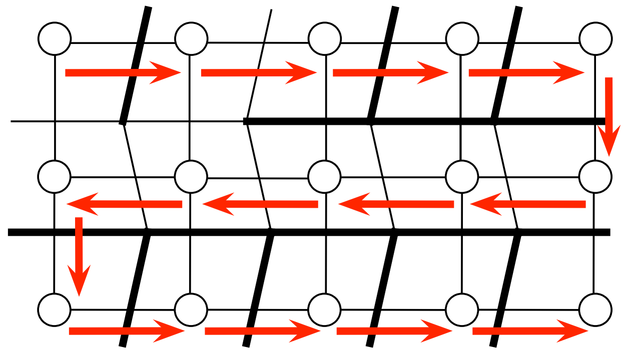

Figure 6: A 3×5 Miura-ori crease pattern with MV assignment generates a sequence. Bold creases are mountains and non-bold creases are valleys. This crease pattern generates the Miura-ori sequence $s_k : 0, 1, 0, 1, 2, 0, 2, 1, 0, 2, 0, 1, 2, 0, 1$.

a *Miura-ori sequence*. First orient our crease pattern so that the top row of vertices all point left. Then imagine an $m \times n$ grid graph overlaid on our crease pattern so that every parallelogram of our crease pattern has a vertex of the grid graph inside of it, as in a planar dual graph. We then use the same ordering of the vertices as if we were making a coloring sequence for the grid graph. The vertex $v(s_0)$ will be in the upper-left face of the crease pattern, and we set $s_0 = 0$. Let us denote c_i to be the crease between vertices $v(s_{i-1})$ and $v(s_i)$. We then assign the remaining terms of the sequence recursively as follows:

$$s_i \equiv s_{i-1} + \mu(c_i) \pmod 3.$$

See Figure 6 for an example of creating a Miura-ori sequence.

In the following Lemma we prove that our recursion rule for generating the Miura-ori sequence holds for horizontal creases as well.

Lemma 2. *Given a Miura-ori sequence s_k generated by an $m \times n$ Miura-ori crease pattern, consider non-negative integers i, j with $j - i = 2n - 1 - (2i \bmod 2n)$ and let d_i be the crease between $v(s_i)$ and $v(s_j)$. Then $s_j \equiv s_i + \mu(d_i) \pmod 3$.*

Proof. If $j - i = 2n - 1 - (2i \bmod 2n)$ then $j > i$ and $v(s_i)$ is directly above $v(s_j)$ in the grid graph that is superimposed on our crease pattern. Telescoping the recursion for s_j, we obtain

$$s_j \equiv s_i + \sum_{k=i+1}^{j} \mu(c_k) \pmod 3.$$

Therefore we want to show that $\mu(d_i) = \sum_{k=i+1}^{j} \mu(c_k)$. For this, we apply Lemma 1. Let γ be a simple closed curve through the creases, in order, $c_{i+1}, \ldots, c_j, d_i$. Then, by Lemma 1 we have

$$2\mu(d_i) = M_\gamma - V_\gamma = \mu(d_i) + \sum_{k=i+1}^{j} \mu(c_k),$$

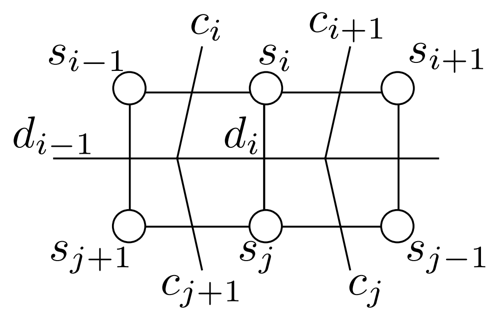

Figure 7: Two Miura-ori vertices that point left with the creases labeled and superimposed grid graph section.

where the second equality is obtained by simply adding all the values of μ along γ. Thus $\mu(d_i) = \sum_{k=i+1}^{j} \mu(c_k)$ and we are done. □

Lemma 2 shows that every $m \times n$ Miura-ori sequence also gives an $m \times n$ grid graph coloring sequence with $s_0 = 0$, and this correspondence is injective; a given Miura-ori crease pattern with MV assignment will generate a grid graph sequence with a well-defined, proper coloring.

Conversely, given a proper 3-vertex coloring of an $m \times n$ grid graph whose upper-left vertex is colored 0 and its coloring sequence s_k (where $s_0 = 0$), we can overlay the grid onto an $m \times n$ Miura-ori crease pattern (with the top row vertices pointing left) and generate a MV assignment from the recurrence $s_i \equiv s_{i-1} + \mu(c_i) \pmod 3$. That is,

$$\mu(c_i) = \begin{cases} 1, & \text{if } s_i - s_{i-1} \equiv 1 \pmod 3; \\ -1, & \text{if } s_i - s_{i-1} \equiv 2 \pmod 3. \end{cases}$$

Note, however, that this does not define μ for all the creases in the Miura-ori, but only creases c_i between vertices $v(s_{i-1})$ and $v(s_i)$. Nonetheless, the MV assignment μ defined thus far will generate the same sequence s_k that the grid graph coloring generated. Therefore we can use the result of Lemma 2 to assign μ for the remaining creases as follows: If crease d_i is the crease directly below vertex $v(s_i)$ and directly above $v(s_j)$ (so that $v(s_i)$ and $v(s_j)$ are adjacent), then let

$$\mu(d_i) = \begin{cases} 1, & \text{if } s_j - s_i \equiv 1 \pmod 3; \\ -1, & \text{if } s_j - s_i \equiv 2 \pmod 3. \end{cases}$$

Lemma 3. *The $m \times n$ Miura-ori MV assignment μ created by the above procedure is locally flat-foldable.*

Proof. All that needs to be shown is that at each bird's foot vertex we have that the "heel" crease has the same MV parity as the majority of the creases around the vertex.

Take an arbitrary vertex in the crease pattern, and suppose it points left. Label the creases and consider the section of the superimposed grid graph around the vertex, as in

Figure 7. From the construction of μ, we have

$$\begin{aligned} s_i &\equiv s_{i-1} + \mu(c_i) \pmod 3 \\ s_j &\equiv s_i + \mu(d_i) \pmod 3 \\ s_{j+1} &\equiv s_j + \mu(c_{j+1}) \pmod 3 \\ s_{j+1} &\equiv s_{i-1} + \mu(d_{i-1}) \pmod 3 \end{aligned}$$

Substituting the first three equations into one another, we obtain $s_{j+1} \equiv s_{i-1} + \mu(c_i) + \mu(d_i) + \mu(c_{j+1}) \pmod 3$, and since each of the μ values are only ± 1, we have

$$\mu(d_{i-1}) = \mu(c_i) + \mu(d_i) + \mu(c_{j+1}).$$

Therefore, crease d_{i-1}, which is the bird's foot heel, must have the same MV parity as the majority of the creases at this vertex. The case where the vertex points right is similar. □

Therefore we have that the $m \times n$ Miura-ori MV assignment generated by a coloring sequence s_k is locally flat foldable, and this generated MV assignment is well-defined. That is, this correspondence between proper 3-vertex colorings of an $m \times n$ grid graph where the upper-left vertex is colored 0 and $m \times n$ Miura-ori locally flat-foldable MV assignments is injective.

We conclude that the correspondence between grid graph colorings with one vertex pre-colored and Miura-ori locally flat-foldable crease pattern MV assignments is a bijection. Therefore, if we let $T(m, n) =$ the number of proper 3-vertex colorings of an $m \times n$ grid graph (m rows and n columns of squares) with the upper-left vertex pre-colored, then we have proven the following:

Theorem 4. *$T(m, n) = M(m, n)$ for all $m, n \geq 1$.*

Unfortunately, there are no known formulas for 3-coloring grid graphs, e.g. the problem of finding the chromatic polynomial of an $n \times m$ grid graph is unsolved [6, Problem 14.7]. However, in the OEIS listing for A078099 a recursive transfer matrix for $T(m, n)$ is given. Specifically, define matrices $M(1) = [1]$,

$$M(m+1) = \begin{bmatrix} M(m) & M(m)^T \\ 0 & M(m) \end{bmatrix},$$

and $W(m) = M(m) + M(m)^T$ (where A^T denotes the transpose of A). Then $T(m, n)$ equals the sum of the entries of $W(m)^{n-1}$. One way to see this is to first note that

$$W(m+1) = \begin{bmatrix} W(m) & M(m)^T \\ M(m) & W(m) \end{bmatrix}. \tag{1}$$

Then think of the rows and columns of $W(m)$ as representing the different equivalence classes C_i of ways to vertex-color an $m \times 1$ grid graph with colors in $\mathbb{Z}_3$, where two such colorings

$\vec{x}, \vec{y} \in \mathbb{Z}_3^m$ are equivalent if either $\vec{x} + (1)^m$ or $\vec{x} + (2)^m$ is equivalent to $\vec{y}$ (mod 3) (where $(i)^m$ denotes the m-dimensional vector with i in every coordinate). Then each equivalence class C_i contains three vectors from $\mathbb{Z}_3^m$ and $W(m)$ is the adjacency matrix for the graph whose vertices are the classes C_i where we connect C_i and C_j by an edge for every pair of vectors $\vec{x} \in C_i$ and $\vec{y} \in C_j$ that are different in every coordinate. Then finding $T(m, n)$ is a matter of counting the walks of length $n-1$ on this graph, i.e., summing the entries of $W(m)^{n-1}$. Examining the entries of the classes C_i reveals a recursive structure in m that explains equation (1).

5 Further observations and questions

One immediate consequence of Theorem 4 is the following:

Corollary 5. *$M(m, n) = M(n, m)$ for all $m, n \geq 1$.*

This simply follows from the symmetry of the grid graph colorings, giving us $T(n, m) = T(m, n)$. Corollary 5 is surprising because the Miura-ori crease pattern itself does not follow this symmetry; an $m \times n$ Miura-ori does not look the same as an $n \times m$ Miura-ori.

We are fortunate that statistical mechanics results exist for enumerating our $m \times n$ grid graph coloring problem for very large m and n. The problem is the same as counting the number of Eulerian orientations on an $m \times n$ grid graph on a torus, which is exactly the antiferroelectric model for two-dimensional ice lattices, called the *square ice* model [10, 15]. In 1967, Lieb used a transfer matrix method to show that a grid graph with N vertices (where N is very large, say 10^{23}) will have

$$(4/3)^{3N/2}$$

ways to be properly vertex colored using three colors with one vertex pre-colored. Therefore this gives the number of locally flat-foldable MV assignments for a Miura-ori crease pattern with N parallelograms, where N is very large.

The transfer matrix in equation (1) can be used to find more generating functions for $M(m, n)$ than were shown in Section 3. That is, following Stanley [17, Sec. 14.7], the generating function $G_m(x)$ for $M(m, n)$ for a fixed m will be

$$G_m(x) = x \sum_{i,j} \frac{(-1)^{i+j} \det(I - xW(m) : j, i)}{\det(I - xW(m))},$$

where $(A : j, i)$ denotes the matrix A with the j-th row and i-th column removed, and the sum ranges over all rows and columns of $W(m)$, which is a $2^{m-1} \times 2^{m-1}$ matrix. Thus calculating $G_m(x)$ is computationally expensive. We used *Mathematica* to find $G_m(x)$ up to $m = 7$ and record the approximate constant and exponential terms (that do not tend to zero) of the generating function sum in Table 1.

	constant term	exponential term
$m=2$	2/3	3
$m=3$	0.8638	4.56155
$m=4$	0.21837	1.42347
	1.10172	6.97196
$m=5$	0.0308918	1.32083
	0.323552	3.11838
	1.39119	10.6829
$m=6$	0.0405757	1.20258
	0.359996	1.3816
	0.00056118	1.68048
	0.0523544	2.68963
	0.454567	5.94907
	1.74463	16.392

	constant term	exponential term
$m=7$	0.0409272	1.42028
	0.00119768	1.75517
	0.0751216	2.48635
	0.00137438	3.12982
	0.526551	3.47859
	0.0792136	5.14803
	0.617056	10.6046
	2.17679	25.1741

Table 1: The approximate terms of $G_m(x) = \sum(\text{constant})(\text{exponent})^n$, the generating function for $M(m,n)$, up to $m=7$.

As mentioned previously, not all of the locally flat-foldable Miura-ori MV assignments are globally flat-foldable. An example is shown in Figure 8. It is simply one of the non-foldable counterexamples to the postage-stamp folding problem reported by Justin [7] modified to be a Miura-ori. To see why it is impossible to fold, note that the vertices labeled v_1 and v_2 are Miura-ori vertices with the standard MV assignment, as we saw in Figure 1. On one hand, the mountain creases along the line l_1 will force the parallelogram faces labeled A and B to be folded inside the Miura-ori vertex v_1. On the other hand, the mountain creases along the line l_2 will force the faces A and B to be folded inside the Miura-ori vertex v_2. A and B cannot be folded into both of these two vertices, and thus this MV assignment is impossible to globally fold flat, even though each vertex is flat-foldable.

Such counterexamples are the result of layers of the paper being forced in incompatible orders by the MV assignment across multiple vertices. As in the postage-stamp folding problem, keeping track of such impossible layer orderings is notoriously difficult. One would need to use completely different (and as yet, undiscovered) techniques to modify our $M(m,n)$ formulas to guarantee global as well as local flat-foldability.

Nonetheless, the fact that enumerating locally flat-foldable MV assignments is be equivalent to a graph coloring problem seems natural. Perhaps the methods presented in this paper can be generalized to other families or even to arbitrary flat-foldable crease patterns.

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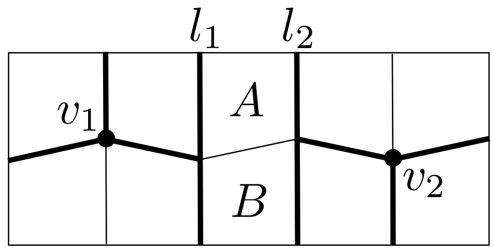

Figure 8: A locally flat-foldable 5×1 Miura-ori MV assignment that is not globally flat-foldable (rotated to conserve space).

the square ice model and reference [10], and to Crystal Wang and an anonymous referee for useful comments and corrections.

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