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n-Color Odd Self-Inverse Compositions

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Abstract

An n-color odd self-inverse composition is an n-color self-inverse composition with odd parts. In this paper, we obtain generating functions, explicit formulas, and recurrence formulas for n-color odd self-inverse compositions.

1 Introduction

In the classical theory of partitions, compositions were first defined by MacMahon [9] as ordered partitions. For example, there are 5 partitions and 8 compositions of 4. The partitions are 4, 31, 22, 211, 1111 and the compositions are 4, 31, 13, 22, 211, 121, 112, 1111.

Agarwal and Andrews [1] defined an n-color partition as a partition in which a part of size n can come in n different colors. They denoted different colors by subscripts: n_1, n_2, ..., n_n. In analogy with MacMahon's ordinary compositions, Agarwal [2] defined an n-color composition as an n-color ordered partition. Thus, for example, there are 8 n-color compositions of 3, viz.,

$$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 2_1, 1_1 2_2, 1_1 1_1 1_1.$$

More properties of n-color compositions were given in [3, 5].

Definition 1. ([9]) A composition is said to be self-inverse when the parts of the composition read from left to right are identical with the parts when read from right to left.

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In analogy with the definition above for classical self-inverse compositions, Narang and Agarwal [10] defined an n-color self-inverse composition and gave some properties of them.

Definition 2. ([10]) An n-color odd composition is an n-color composition with odd parts.

For example there are 8 n-color self-inverse compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4, 2_1 2_1, 2_2 2_2, 1_1 2_1 1_1, 1_1 2_2 1_1.$$

In 2010, the author [6] also defined an n-color even self-inverse composition and gave some properties.

Definition 3. ([6]) An n-color even composition is an n-color composition whose parts are even.

Definition 4. ([6]) An n-color even composition whose parts read from left to right are identical with when read from right to left is called an n-color even self-inverse composition.

Thus, for example, there are 8 n-color even self-inverse compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4, 2_1 2_1, 2_1 2_2, 2_2 2_1, 2_2 2_2.$$

And there are 6 n-color even self-inverse compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4, 2_1 2_1, 2_2 2_2.$$

Recently, the author [7] studied n-color odd compositions.

Definition 5. ([7]) An n-color odd composition is an n-color composition whose parts are odd.

Thus, for example, there are 7 n-color odd compositions of 4, viz.,

$$3_1 1_1, 3_2 1_1, 3_3 1_1, 1_1 3_1, 1_1 3_2, 1_1 3_3, 1_1 1_1 1_1 1_1.$$

In this paper, we shall study n-color odd self-inverse compositions.

Definition 6. An n-color odd composition whose parts read from left to right are identical with when read from right to left is called an n-color odd self-inverse composition.

Thus, for example, there are 4 n-color odd self-inverse compositions of 6, viz.,

$$3_1 3_1, 3_2 3_2, 3_3 3_3, 1_1 1_1 1_1 1_1 1_1 1_1.$$

In section 2 we shall give explicit formulas, recurrence formulas, generating functions for n-color odd self-inverse compositions.

The author [7] proved the following theorems.

Theorem 7. ([7]) *Let $C_o(m,q)$ and $C_o(q)$ denote the enumerative generating functions for $C_o(m,\nu)$ and $C_o(\nu)$, respectively, where $C_o(m,\nu)$ is the number of n-color odd compositions of ν into m parts and $C_o(\nu)$ is the number of n-color odd compositions of ν. Then*

$$C_o(m,q) = \frac{q^m(1+q^2)^m}{(1-q^2)^{2m}}, \tag{1}$$

$$C_o(q) = \frac{q+q^3}{1-q-2q^2-q^3+q^4}, \tag{2}$$

$$C_o(m,\nu) = \sum_{i+j=\frac{\nu-m}{2}} \binom{2m+i-1}{2m-1}\binom{m}{j}, \tag{3}$$

$$C_o(\nu) = \sum_{m\le\nu} \sum_{i+j=\frac{\nu-m}{2}} \binom{2m+i-1}{2m-1}\binom{m}{j}. \tag{4}$$

where $(\nu-m)$ is even, and $(\nu-m)\geq 0$; $0\leq i,j$ are integers.

Theorem 8. ([7]) *Let $C_o(\nu)$ denote the number of n-color odd compositions of ν. Then*

$$C_o(1)=1, C_o(2)=1, C_o(3)=4, C_o(4)=7 \text{ and}$$
$$C_o(\nu)=C_o(\nu-1)+2C_o(\nu-2)+C_o(\nu-3)-C_o(\nu-4) \ for \ \nu>4.$$

2 Main results

In this section, we first prove the following explicit formulas for the number of n-color odd self-inverse compositions.

Theorem 9. *Let $S(O,\nu)$ denote the number of n-color odd self-inverse compositions of ν. Then*

$$(1)\quad S(O,2\nu+1) = (2\nu+1) + \sum_{t=1}^{2\nu-1} \sum_{m\le\frac{2\nu+1-t}{2}} \sum_{i+j=\frac{2\nu+1-t-2m}{4}} t\binom{2m+i-1}{2m-1}\binom{m}{j},$$

where $\nu=0,1,2,\ldots$; $t=2k+1, k=0,1,2,\ldots,(\nu-1)$; $0\leq\frac{2\nu+1-t-2m}{2}$ is even; $0\leq i,j$ are integers.

$$(2)\quad S(O,2\nu) = \sum_{m\le\nu} \sum_{i+j=\frac{\nu-m}{2}} \binom{2m+i-1}{2m-1}\binom{m}{j},$$

where $0\leq\nu-m$ is even, and $0\leq i,j$ are integers.

Proof. (1) Obviously, an odd number which is $2\nu+1$ ($\nu=0,1,2,\ldots$) can have odd self-inverse n-color compositions only when the number of parts is odd. There are $2\nu+1$ n-color odd self-inverse compositions when the number of parts is only one. An odd self-inverse compositions of $2\nu+1$ into $2m+1$ ($m\geq 1$) parts can be read as a central part, say, t (where t is odd) and two identical odd n-color compositions of $\frac{2\nu+1-t}{2}$ into m parts on each side of the central part. The number of odd n-color compositions of $\frac{2\nu+1-t}{2}$ into m parts is $C_o(m,\frac{2\nu+1-t}{2})$ by equation (3). Now the central part can appear in t ways. Therefore, the number of n-color odd self-inverse compositions of $2\nu+1$ is

$$\begin{aligned} S(O,2\nu+1) &= (2\nu+1)+\sum_{t=1}^{2\nu-1}\sum_{m\leq\frac{2\nu+1-t}{2}} tC_o\left(m,\frac{2\nu+1-t}{2}\right) \\ &= (2\nu+1)+\sum_{t=1}^{2\nu-1}\sum_{m\leq\frac{2\nu+1-t}{2}}\sum_{i+j=\frac{2\nu+1-t-2m}{4}} t\binom{2m+i-1}{2m-1}\binom{m}{j}. \end{aligned}$$

(2) For even numbers 2ν ($\nu=1,2,\ldots$), we can have odd self-inverse n-color compositions only when the number of parts is even, and the two identical odd n-color compositions are exactly odd n-color compositions of ν, from equation (4) we see that the number of these is

$$\sum_{m\leq\nu}\sum_{i+j=\frac{\nu-m}{2}}\binom{2m+i-1}{2m-1}\binom{m}{j}.$$

Hence, the number of n-color odd self-inverse compositions of 2ν is

$$S(O,2\nu)=\sum_{m\leq\nu}\sum_{i+j=\frac{\nu-m}{2}}\binom{2m+i-1}{2m-1}\binom{m}{j}.$$

We complete the proof of this theorem. □

From the proof of this theorem we can see that odd n have n-color odd self-inverse compositions where the number of parts is odd. And even n have n-color odd self-inverse compositions where the number of parts is even. Let $S_o(\nu,m)$ denote the number of n-color odd self-inverse compositions of ν into m parts. Then we can get the following formula easily.

$$S_o(2k+1,2l+1)=\sum_{t=1}^{2k-1}\sum_{i+j=\frac{2k+1-t-2l}{4}}\binom{2l+i-1}{2l-1}\binom{l}{j}.$$

where t is odd, k,l are integers and $k,l\geq 0$.

$$S_o(2k,2l)=\sum_{i+j=\frac{k-l}{2}}\binom{2l+i-1}{2l-1}\binom{l}{j}.$$

Table 1: $S_o(\nu, m)$ when both ν and m are odd

$\nu \backslash m$	1	3	5	7	9	11	13	15	17	19	s_ν
1	1	0	0	0	0	0	0	0	0	0	1
3	3	1	0	0	0	0	0	0	0	0	4
5	5	3	1	0	0	0	0	0	0	0	9
7	7	8	3	1	0	0	0	0	0	0	19
9	9	16	11	3	1	0	0	0	0	0	40
11	11	29	25	16	3	1	0	0	0	0	83
13	13	49	56	34	17	3	1	0	0	0	173
15	15	72	110	96	43	20	3	1	0	0	360
17	17	104	206	200	143	52	23	3	1	0	749
19	19	145	346	442	317	199	61	26	3	1	1559

where k, l are integers and $k, l \geq 0$.

Now $S_o(\nu, m)$ with $\nu, m = 1, 2, \ldots, 20$ is given in Tables 1 and 2.

From Tables 1 and 2 we can see the recurrence formulas for the number of the n-color odd self-inverse compositions of ν. So we prove the following recurrence relations.

Table 2: $S_o(\nu, m)$ when both ν and m are even

$\nu \backslash m$	2	4	6	8	10	12	14	16	18	20	t_ν
2	1	0	0	0	0	0	0	0	0	0	1
4	0	1	0	0	0	0	0	0	0	0	1
6	3	0	1	0	0	0	0	0	0	0	4
8	0	6	0	1	0	0	0	0	0	0	7
10	5	0	9	0	1	0	0	0	0	0	15
12	0	19	0	12	0	1	0	0	0	0	32
14	7	0	42	0	15	0	1	0	0	0	65
16	0	44	0	74	0	18	0	1	0	0	137
18	9	0	138	0	115	0	21	0	1	0	284
20	0	85	0	316	0	165	0	24	0	1	591

Theorem 10. *Let s_ν and t_ν denote the number of n-color odd self-inverse compositions for $2\nu + 1$ and 2ν, respectively. Then*

$$
\begin{aligned}
(1) \qquad s_0 &= 1,\ s_1 = 4,\ s_2 = 9,\ s_3 = 19 \ \ \textit{and} \\
s_\nu &= s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3} - s_{\nu-4} \quad \textit{for} \quad \nu > 3 \\
(2) \qquad t_1 &= 1,\ t_2 = 1,\ t_3 = 4,\ t_4 = 7 \ \ \textit{and} \\
t_\nu &= t_{\nu-1} + 2t_{\nu-2} + t_{\nu-3} - t_{\nu-4} \quad \textit{for} \quad \nu > 4.
\end{aligned}
$$

Proof. (Combinatorial) (1) To prove that $s_\nu = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3} - s_{\nu-4}$, we split the n-color odd self-inverse compositions enumerated by $s_\nu + s_{\nu-4}$ into four classes:

(A) s_ν with 1_1 on both ends.
(B) s_ν with 3_3 on both ends.
(C) s_ν with h_t on both ends, $h > 1$, $1 \le t \le h-2$ and n-color odd self-inverse compositions of $2\nu+1$ of form $(2\nu+1)_u$, $1 \le u \le 2\nu-3$.
(D) s_ν with h_t on both ends except 3_3, $h > 1$, $h-1 \le t \le h$, $(2\nu+1)_u$, $2\nu-2 \le u \le 2\nu+1$ and those enumerated by $s_{\nu-4}$.

We transform the n-color odd self-inverse compositions in class (A) by deleting 1_1 on both ends. This produces n-color odd self-inverse compositions enumerated by $s_{\nu-1}$. Conversely, for any n-color odd composition enumerated by $s_{\nu-1}$ we add 1_1 on both ends to produce the elements of the class (A). In this way we establish that there are exactly $s_{\nu-1}$ elements in the class (A).

Similarly, we can produce $s_{\nu-3}$ n-color odd self-inverse compositions in class (B) by deleting 3_3 on both ends.

Next, we transform the n-color odd self-inverse compositions in class (C) by subtracting 2 from h, that is, replacing h_t by $(h-2)_t$ and subtracting 4 from $2\nu+1$ of $(2\nu+1)_u$, $1 \le u \le 2\nu-3$. This transformation also establishes the fact that there are exactly $s_{\nu-2}$ elements in class (C).

Finally, we transform the elements in class (D) as follows: Subtract 2_2 from h_t on both ends, that is, replace h_t by $(h-2)_{(t-2)}$, $h > 3$, $h-1 \le t \le h$, while replace h_t by $(h-2)_{(t-1)}$ when $h = 3$, $t = 2$. We will get those n-color odd self-inverse compositions of $2\nu-3$ with h_t on both ends, $h-1 \le t \le h$ except self-inverse odd compositions in one part. We also replace $(2\nu+1)_u$ by $(2\nu-3)_{u-4}$, $2\nu-2 \le u \le 2\nu+1$. To get the remaining n-color odd compositions from $s_{\nu-4}$ we add 2 to both ends, that is, replace h_t by $(h+2)_t$. For n-color odd self-inverse compositions into one part we add 4 ,that is, replace $(2\nu-7)_t$ by $(2\nu-3)_t, 1 \le t \le 2\nu-7$. We see that the number of n-color odd self-inverse compositions in class (D) is also equal to $s_{\nu-2}$. Hence, we have $s_\nu + s_{\nu-4} = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3}$. viz., $s_\nu = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3} - s_{\nu-4}$.

(2) From Theorem 8 and Theorem 9, we obtain the recurrence formula of t_ν easily. Thus, we complete the proof. $\square$

We easily get the following generating functions by the recurrence relations.

Corollary 11.

$$(1) \qquad \sum_{\nu=0}^{\infty} s_\nu q^\nu = \frac{(1+q)^3}{1-q-2q^2-q^3+q^4}.$$

$$(2) \qquad \sum_{\nu=1}^{\infty} t_\nu q^\nu = \frac{q+q^3}{1-q-2q^2-q^3+q^4}.$$

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