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Enumeration of Edges in Some Lattices of Paths

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Abstract

We enumerate the edges in the Hasse diagram of several lattices arising in the combinatorial context of lattice paths. Specifically, we consider the case of Dyck, Grand Dyck, Motzkin, Grand Motzkin, Schröder and Grand Schröder lattices. Finally, we give a general formula for the number of edges in an arbitrary Young lattice (which can be interpreted in a natural way as a lattice of paths).

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1 Introduction

Fixing a Cartesian coordinate system in the discrete plane $\mathbb{Z} \times \mathbb{Z}$, for any class of lattice paths starting from the origin and ending at the same point on the x-axis, we can define a partial order by declaring that a path γ_1 is less than or equal to another path γ_2 whenever γ_1 lies weakly below γ_2. In some cases, the resulting poset has a structure of distributive lattice [10]. This is true, for instance, for the more common classes of paths: *Dyck* and *Grand Dyck paths*, *Motzkin* and *Grand Motzkin paths*, *Schröder* and *Grand Schröder paths*, in correspondence of which we have *Dyck* and *Grand Dyck lattices*, *Motzkin* and *Grand Motzkin lattices*, *Schröder* and *Grand Schröder lattices*. In all these cases, considering the paths according to their length or semi-length, we have a sequence of distributive lattices. Another important class is given by the (finite) *Young lattices*, which can be interpreted in a very natural way as lattices of paths. In particular, the Dyck lattices are isomorphic to the Young lattices associated with a staircase shape, and the Grand Dyck lattices are isomorphic to the Young lattices associated with a rectangular shape.

In this paper, we enumerate the edges of the Hasse diagram for all distributive lattices recalled above. For the classical paths, using the method of path decomposition, we obtain the generating series with respect to certain parameters and then (applying Proposition 1) we obtain the edge series. For the Young lattices, we obtain a general formula for the number of edges in the Hasse diagram of $\mathcal{Y}_\lambda$ which is valid for an arbitrary partition λ.

We also introduce the *Hasse index* of a poset as the density index of the associated Hasse diagram (the quotient between the number of edges and the number of vertices) and we prove that the Hasse index of all sequences of lattices of classical paths is always related (equal, asymptotically equivalent, or asymptotically quasi-equivalent) to the Hasse index of Boolean lattices.

The enumeration of the edges in the Hasse diagram of a poset P can be considered as a specialization of the more general case of enumerating all saturated chains in P, which is clearly much more complicated. In a forthcoming article [9] we obtain a general formula for counting saturated chains of any finite length k in any Dyck lattice.

2 Background

2.1 Lattice paths

In this paper, we will consider the following classes of paths.

1. The class $\mathcal{GD}$ of *Grand Dyck paths*, i.e., the class of all lattice paths starting from the origin, ending on the x-axis, and consisting of *up steps* $U = (1,1)$ and *down steps* $D = (1,-1)$. The class $\mathcal{D}$ of *Dyck paths* [7, 21], consisting of all Grand Dyck paths never going below the x-axis.

2. The class $\mathcal{GM}$ of *Grand Motzkin paths*, i.e., the class of all lattice paths starting from the origin, ending on the x-axis, and consisting of *up steps* $U = (1,1)$, *down steps*

$D = (1, -1)$ and *horizontal steps* $H = (1, 0)$. The class $\mathcal{M}$ of *Motzkin paths* [21], consisting of all Grand Motzkin paths never going below the x-axis.

3. The class $\mathcal{GS}$ of *Grand Schröder paths* (or *central Delannoy paths*), i.e., the class of all lattice paths starting from the origin, ending on the x-axis, and consisting of *up steps* $U = (1, 1)$, *down steps* $D = (1, -1)$ and *double horizontal steps* $H = (2, 0)$. The class $\mathcal{S}$ of *Schröder paths*, consisting of all Grand Schröder paths never going below the x-axis.

The *length* of a path is the number of its steps, and the *semi-length* of a path of even length is half of the number of its steps. In general, for a class $\mathcal{P}$ containing paths of every length (as in the case of Grand Motzkin paths and Motzkin paths) we denote by $\mathcal{P}_n$ the set of all paths in $\mathcal{P}$ having length n. If the class $\mathcal{P}$ contains only paths of even length (as in the case of Grand Dyck paths, Dyck paths, Grand Schröder paths and Schröder paths) we denote by $\mathcal{P}_n$ the set of all paths in $\mathcal{P}$ having semi-length n. Sometimes, we write $\bullet$ for the *empty path* (consisting of zero steps).

For a class $\mathcal{P}$ containing paths never going below the x-axis, we define the class $\overline{\mathcal{P}}$ of *reflected paths* as the set of all paths obtained by reflecting about the x-axis the paths in $\mathcal{P}$. In particular, we will consider the class $\overline{\mathcal{D}}$ of *reflected Dyck paths*, the class $\overline{\mathcal{M}}$ of *reflected Motzkin paths*, and the class $\overline{\mathcal{S}}$ of *reflected Schröder paths*. Moreover, we say that a *reflected path* $\gamma \in \overline{\mathcal{P}}$ is *subelevated* when $\gamma = D\gamma' U$, for any $\gamma' \in \overline{\mathcal{P}}$.

A path can always be considered as a word on the alphabet given by the set of possible steps. A factor of a path γ (considered as a word) is a word α such that $\gamma = \gamma' \alpha \gamma''$. We write $\omega_\alpha(\gamma)$ for the number of all occurrences of the word α as a factor of γ. In particular, if α is a path starting and ending at the same level, then we write $\omega^*_\alpha(\gamma)$ for the number of all occurrences of the word α as a factor of γ not on the x-axis. For instance, if $\gamma = UUDHDHUHD$, then $\omega_H(\gamma) = 3$ and $\omega^*_H(\gamma) = 2$ (since the second horizontal step lies on the x-axis).

2.2 Enumeration and asymptotics

For simplicity, we recall some well known enumerative properties of the paths considered in Subsection 2.1 that will be used in the rest of the paper.

The Grand Dyck paths are enumerated by the *central binomial coefficients* $\binom{2n}{n}$ [19, A000984], and the Dyck paths are enumerated by the *Catalan numbers* $C_n = \binom{2n}{n}\frac{1}{n+1}$ [19, A000108]. These numbers have generating series

$$B(x) = \sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}, \qquad C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

The *trinomial coefficient* $\binom{n;3}{k}$ is defined as the coefficient of x^k in the expansion of $(1 + x + x^2)^n$ [6] [19, A027907]. The Grand Motzkin paths are enumerated by the *central* *trinomial coefficients* $\binom{n;3}{n}$ [19, A002426], and the Motzkin paths are enumerated by the *Motzkin numbers* M_n [19, A001006]. These numbers have generating series

$$T(x) = \sum_{n\geq 0} \binom{n;3}{n} x^n = \frac{1}{\sqrt{1-2x-3x^2}}, \qquad M(x) = \sum_{n\geq 0} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

The Grand Schröder paths are enumerated by the *central Delannoy numbers* d_n [19, A001850], and the Schröder paths are enumerated by the *large Schröder numbers* r_n [19, A006318]. These numbers have generating series

$$d(x) = \sum_{n\geq 0} d_n x^n = \frac{1}{\sqrt{1-6x+x^2}}, \qquad r(x) = \sum_{n\geq 0} r_n x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x}.$$

Given two sequences a_n and b_n (where b_n is definitively non zero), we recall that the notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to +\infty$. Moreover, we recall that the *Darboux theorem* [3, p. 252] says that: given a complex number $\xi \neq 0$ and a complex function $f(x)$ analytic at the origin, if $f(x) = (1 - x/\xi)^{-\alpha}\psi(x)$ where $\psi(x)$ is a series with radius of convergence $R > |\xi|$ and $\alpha \notin \{0, -1, -2, \dots\}$, then

$$[x^n]f(x) \sim \frac{\psi(\xi)}{\xi^n} \frac{n^{\alpha-1}}{\Gamma(\alpha)},$$

where $\Gamma(z)$ is Euler's Gamma function. Using this theorem, it is possible to obtain the following asymptotic expansions:

$$\binom{n;3}{n} \sim \frac{3^n}{2}\sqrt{\frac{3}{n\pi}}, \quad M_n \sim \frac{3^{n+1}}{2n}\sqrt{\frac{3}{n\pi}}, \quad d_n \sim \frac{(1+\sqrt{2})^{2n+1}}{2\sqrt{\sqrt{2}n\pi}}, \quad r_n \sim \frac{(1+\sqrt{2})^{2n+1}}{n\sqrt{2\sqrt{2}n\pi}}. \tag{1}$$

Finally, we recall the following elementary useful expansion

$$\frac{x^r}{(1-x)^{s+1}} = \sum_{n\geq 0} \binom{n-r+s}{s} x^n.$$

2.3 Edge enumeration

The *Hasse diagram* $\mathcal{H}(P)$ of a finite poset P is a (directed) graph representing the poset, where the vertices are the elements of P and the adjacency relation is the cover relation. The number of vertices is $|P|$. We will denote with $\ell(P)$ the number of edges in $\mathcal{H}(P)$. If Δx is the set of all elements covering x and ∇x is the set of all elements covered by x, then we have

$$\ell(P) = \sum_{x\in P} |\Delta x| = \sum_{x\in P} |\nabla x|.$$

Moreover, if we consider the polynomials

$$\Delta(P;q) = \sum_{x\in P} q^{|\Delta x|} \qquad \text{and} \qquad \nabla(P;q) = \sum_{x\in P} q^{|\nabla x|},$$

then we have at once the identities

$$\ell(P) = [\partial_q \Delta(P;q)]_{q=1} \qquad \text{and} \qquad \ell(P) = [\partial_q \nabla(P;q)]_{q=1}\,, \tag{2}$$

where ∂_q denotes the partial derivative with respect to q. The *edge generating series* of a sequence of posets $\mathscr{P} = \{P_0, P_1, P_2, \ldots\}$ is the ordinary generating series $\ell_{\mathscr{P}}(x)$ of the numbers $\ell(P_n)$. Similarly, the Δ*-series* and ∇*-series* associated with the sequence $\mathscr{P}$ are the generating series $\Delta_{\mathscr{P}}(q;x)$ and $\nabla_{\mathscr{P}}(q;x)$ of the polynomials $\Delta(P_n;q)$ and $\nabla(P_n;q)$, respectively. From identities (2), we have at once

Proposition 1. *The edge generating series for the sequence of posets $\mathscr{P} = \{P_0, P_1, P_2, \ldots\}$ can be obtained from the associated Δ-series and ∇-series as follows:*

$$\ell_{\mathscr{P}}(x) = [\partial_q \Delta_{\mathcal{P}}(q;x)]_{q=1} \qquad \textit{and} \qquad \ell_{\mathscr{P}}(x) = [\partial_q \nabla_{\mathcal{P}}(q;x)]_{q=1}\,.$$

The *density index* $i(G)$ of a graph G is the quotient between the number of edges and the number of vertices, i.e., $i(G) = |E(G)|/|V(G)|$. This index has been considered in the study of topologies for the interconnection of parallel multicomputers, especially in the attempt of finding alternative topologies to the classical one given by the Boolean cube [13, 16], and in many other circumstances [5, 15]). Here, we define the *Hasse index* $i(P)$ of a poset P as the density index of its Hasse diagram, i.e., $i(P) = \ell(P)/|P|$. For instance, the Hasse index of a *Boolean lattice* B_n is $i(B_n) = \ell(B_n)/|B_n| = n/2$, since $|B_n| = 2^n$ and $\ell(B_n) = n2^{n-1}$. We say that the Hasse index of a sequence of posets $\mathscr{P} = \{P_0, P_1, P_2, \ldots\}$ is *Boolean* when $i(P_n) = n/2$, is *asymptotically Boolean* when $i(P_n) \sim n/2$ as $n \to +\infty$, and is *asymptotically quasi Boolean* when there exists a small non-negative constant c such that $i(P_n) \sim (1/2 \pm c)n$ as $n \to +\infty$. Here, we can assume $c \leq 1/10$.

Let $\mathcal{GP}$ be a class of lattice paths (the *Grand paths*) starting from the origin, ending on the x-axis consisting of steps of some kind (and respecting possible restrictions). Then let $\mathcal{P}$ be the class of all paths in $\mathcal{GP}$ never going below the x-axis. We say that the class $\mathcal{P}$ is *Hasse-tamed* if the Hasse index of the associated posets of paths is asymptotically equivalent to the Hasse index of the associated posets of Grand paths, i.e., $i(\mathcal{GP}_n) \sim i(\mathcal{P}_n)$ as $n \to +\infty$. In all main examples we will consider, the property of being Hasse-tamed is true. However, there are also classes of paths without such a property, as in the case of the Fibonacci paths, considered in Section 6.

In the rest of the paper, given a class $\mathcal{P}$ of paths, we write $\ell_{\mathcal{P}}(x)$ for the edge generating series $\ell_{\mathscr{P}}(x)$ associated with the sequence $\mathscr{P}$ of posets generated by all paths in $\mathcal{P}$.

For convenience, we report in Table 1 the first few values of the number of edges for the various lattices we will consider in the paper. Moreover, we observe that they appear in the On-Line Encyclopedia of Integer Sequenecs [19] as follows: $\ell(\mathcal{F}_n)$ form sequence A001629, $\ell(\mathcal{GF}_n)$ form sequence A095977, $\ell(\mathcal{D}_n)$ form sequence A002054, $\ell(\mathcal{GD}_n)$

form sequence A002457, $\ell(\mathcal{M}_n)$ form sequence A025567, $\ell(\mathcal{GM}_n)/2$ form sequence A132894, $\ell(\mathcal{GS}_n)/2$ form sequence A108666. Finally, notice that the partial sums for the sequence $\ell(\mathcal{S}_n)$ form sequence A026002.

n	0	1	2	3	4	5	6	7	8	9	10
$\ell(\mathcal{F}_n)$	0	0	1	2	5	10	20	38	71	130	235
$\ell(\mathcal{GF}_n)$	0	0	2	4	14	32	82	188	438	984	2202
$\ell(\mathcal{D}_n)$	0	0	1	5	21	84	330	1287	5005	19448	75582
$\ell(\mathcal{GD}_n)$	0	1	6	30	140	630	2772	12012	51480	218790	923780
$\ell(\mathcal{M}_n)$	0	0	1	4	13	40	120	356	1050	3088	9069
$\ell(\mathcal{GM}_n)$	0	0	2	8	30	104	350	1152	3738	12000	38214
$\ell(\mathcal{S}_n)$	0	1	6	34	190	1058	5894	32898	184062	1032322	5803270
$\ell(\mathcal{GS}_n)$	0	2	16	114	768	5010	32016	201698	1257472	7777314	47800080

Table 1: Number of edges in some lattices of paths.

3 Dyck and Grand Dyck lattices

Proposition 2. *For any Dyck path γ, we have $|\Delta\gamma| = \omega_{DU}(\gamma)$ and $|\nabla\gamma| = \omega^*_{UD}(\gamma)$. Similarly, for any Grand Dyck path γ, we have $|\Delta\gamma| = \omega_{DU}(\gamma)$ and $|\nabla\gamma| = \omega_{UD}(\gamma)$.*

Proof. In a Dyck lattice, a path γ is covered by all paths that can be obtained from γ by replacing a valley DU with a peak UD, and covers all paths that can be obtained from γ by replacing a peak UD (not at level 0) with a valley DU. In a Grand Dyck lattice, the situation is similar. □

Proposition 3. *The generating series for the class of Dyck paths with respect to semi-length (marked by x) and valleys (marked by q) is*

$$f(q;x) = \frac{1-(1-q)x-\sqrt{1-2(1+q)x+(1-q)^2x^2}}{2qx}. \tag{3}$$

Proof. Any non-empty Dyck path γ decomposes uniquely as $\gamma = U\gamma' D$ (with $\gamma' \in \mathcal{D}$) or as $\gamma = U\gamma' D\gamma''$ (with $\gamma', \gamma'' \in \mathcal{D}$, $\gamma'' \neq \bullet$). Hence, we have the identity $f(q;x) = 1 + xf(q;x) + qxf(q;x)(f(q;x)-1)$ whose solution is series (3). □

Notice that series (3) is essentially the generating series of *Narayana numbers* [19, A001263] and that this statistic is well known [7].

Theorem 4. *The edge generating series for Dyck lattices is*

$$\ell_{\mathcal{D}}(x) = \frac{1-3x-(1-x)\sqrt{1-4x}}{2x\sqrt{1-4x}}. \tag{4}$$

Moreover, for every $n \in \mathbb{N}$, $n \geq 2$, the number of edges in $\mathcal{D}_n$ is

$$\ell(\mathcal{D}_n) = \frac{1}{2}\binom{2n}{n}\frac{n-1}{n+1} = \binom{2n-1}{n-2}. \tag{5}$$

In particular, the Hasse index of Dyck lattices is asymptotically Boolean.

Proof. By Proposition 2, the Δ-series for Dyck lattices is series (3). So, by applying Proposition 1, we can obtain series (4). Moreover, since $\ell_{\mathcal{D}}(x) = (1+B(x))/2 - C(x)$, we obtain identity (5). Finally, since $|\mathcal{D}_n| = C_n$, we have $i(\mathcal{D}_n) = \ell(\mathcal{D}_n)/|\mathcal{D}_n| = (n-1)/2$ for every $n \geq 1$, and $i(\mathcal{D}_n) \sim n/2$. □

Proposition 5. *The generating series for the class of Grand Dyck paths with respect to semi-length (marked by x) and valleys (marked by q) is*

$$F(q;x) = \frac{1}{\sqrt{1-2(1+q)x+(1-q)^2x^2}}. \tag{6}$$

Proof. Let $\overline{\mathcal{D}}$ be the class of reflected Dyck paths (i.e., Grand Dyck paths never going above the x-axis), and let $G(q;x)$ be the corresponding generating series. Since every non empty path $\gamma \in \overline{\mathcal{D}}$ uniquely decomposes as $\gamma = DU\gamma'$ (with $\gamma' \in \overline{\mathcal{D}}$) or as $\gamma = D\gamma' U\gamma''$ (with $\gamma', \gamma'' \in \overline{\mathcal{D}}$, $\gamma' \neq \bullet$), we have the identity $G(q;x) = 1 + qxG(q;x) + x(G(q;x)-1)G(q;x)$, whose unique solution is

$$G(q;x) = \frac{1+(1-q)x-\sqrt{1-2(1+q)x+(1-q)^2x^2}}{2x}.$$

Now, let $U(q;x)$ be the generating series for the class of Grand Dyck paths starting with an up step and let $D(q;x)$ be the generating series for the class of Grand Dyck paths starting with a down step. Since every path $\gamma \in \mathcal{GD}$ uniquely decomposes as product of paths of the form $U\gamma D$ (with $\gamma \in \mathcal{D}$) and $D\gamma U$ (with $\gamma' \in \overline{\mathcal{D}}$), we have the linear system

$$\begin{cases} F(q;x) = 1 + U(q;x) + D(q;x) \\ U(q;x) = xf(q;x)(1 + qU(q;x) + D(q;x)) \\ D(q;x) = qxF(q;x) + x(G(q;x)-1)F(q;x) \end{cases}$$

from which it is straightforward to obtain identity (6). □

Notice that the generating series (6) appears in a more general form [14, Lemma 2.1].

Theorem 6. *The edge generating series for Grand Dyck lattices is*

$$\ell_{\mathcal{GD}}(x) = \frac{x}{(1-4x)^{3/2}}. \tag{7}$$

Moreover, the number of edges in $\mathcal{GD}_n$ is

$$\ell(\mathcal{GD}_n) = \binom{2n}{n}\frac{n}{2}. \tag{8}$$

In particular, the Hasse index of a Grand Dyck lattice is Boolean.

Proof. Proposition 2 implies that (6) is the Δ-series for Grand Dyck lattices. So, by applying Proposition 1, we obtain series (7). Then, by expanding this series, we have at once identity (8). Finally, since $|\mathcal{GD}_n| = \binom{2n}{n}$, we have $i(\mathcal{GD}_n) = \ell(\mathcal{GD}_n)/|\mathcal{GD}_n| = n/2$. □

Theorems 4 and 6 immediately imply

Proposition 7. *The class of Dyck lattices is Hasse-tamed: $i(\mathcal{D}_n) \sim i(\mathcal{GD}_n) \sim n/2$.*

4 Motzkin and Grand Motzkin lattices

Proposition 8. *For any Motzkin path γ, we have $|\Delta\gamma| = \omega_{HU}(\gamma) + \omega_{DH}(\gamma) + \omega_{DU}(\gamma) + \omega_{HH}(\gamma)$ and $|\nabla\gamma| = \omega_{UH}(\gamma) + \omega_{HD}(\gamma) + \omega_{UD}(\gamma) + \omega^*_{HH}(\gamma)$. Similarly, for any Grand Motzkin path γ, we have $|\Delta\gamma| = \omega_{HU}(\gamma) + \omega_{DH}(\gamma) + \omega_{DU}(\gamma) + \omega_{HH}(\gamma)$ and $|\nabla\gamma| = \omega_{UH}(\gamma) + \omega_{HD}(\gamma) + \omega_{UD}(\gamma) + \omega_{HH}(\gamma)$.*

Proof. In a Motzkin lattice, a path γ is covered by all paths that can be obtained from γ by replacing i) a factor HU with a factor UH, or ii) a factor DH with a factor HD, or iii) a valley DU with a double horizonal step HH, or iv) a double horizonal step HH with a peak UD. The number of paths covered by γ can be obtained in a similar way. In a Grand Motzkin lattice we have a similar situation. □

Proposition 9. *The generating series for the class of Motzkin paths with respect to length (marked by x) and to factors HU, DH, DU and HH (marked by q) is*

$$f(q;x) = \frac{1 - qx - (1-q)x^2 - \sqrt{(1+x)(1+(1-q)x)(1-(2+q)x+(1-q)x)}}{2qx^2}. \tag{9}$$

Proof. Let $h(q;x)$ be the generating series for the class of Motzkin paths starting with an horizontal step and let $u(q;x)$ be the generating series for the class of Motzkin paths starting with an up step. Since any non-empty Motzkin path γ decomposes uniquely as $\gamma = H\gamma'$ (with $\gamma' \in \mathcal{M}$), or as $\gamma = U\gamma' D\gamma''$ (with $\gamma', \gamma'' \in \mathcal{M}$), it is straightforward to obtain the linear system

$$\begin{cases} f(q;x) = 1 + h(q;x) + u(q;x) \\ h(q;x) = x(1 + qh(q;x) + qu(q;x)) \\ u(q;x) = x^2 f(q;x)(1 + qh(q;x) + qu(q;x)) \end{cases}$$

and consequently, solving such a system, to obtain identity (9). □

The numbers generated by series (9) are essentially sequence A110470 [19].

Theorem 10. *The edge generating series for Motzkin lattices is*

$$\ell_{\mathcal{M}}(x) = \frac{(1+x)(1-2x-x^2-(1-x)\sqrt{1-2x-3x^2})}{2x^2\sqrt{1-2x-3x^2}}. \tag{10}$$

Moreover, the number of edges in $\mathcal{M}_n$ can be expressed in one of the following ways

$$\ell(\mathcal{M}_n) = \binom{n;3}{n} - M_n + \binom{n-1;3}{n-1} - M_{n-1} \qquad (n \geq 1) \tag{11}$$

$$\ell(\mathcal{M}_n) = \binom{n;3}{n-2} + \binom{n-1;3}{n-3} \qquad (n \geq 3) \tag{12}$$

$$\ell(\mathcal{M}_n) = \frac{2}{n}\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{k}\frac{k(n-k)}{k+1} \qquad (n \geq 1). \tag{13}$$

In particular, we have the asymptotic expansions

$$\ell(\mathcal{M}_n) \sim \frac{2 \cdot 3^n}{\sqrt{3n\pi}} \qquad \text{and} \qquad i(\mathcal{M}_n) = \frac{\ell(\mathcal{M}_n)}{|\mathcal{M}_n|} \sim \frac{4}{9}n \tag{14}$$

and the Hasse index of the Motzkin lattices is asymptotically quasi Boolean.

Proof. By Proposition 8, the Δ-series for Motzkin lattices is series (9). So, by Proposition 1, we obtain series (10). It is easy to see that $\ell_{\mathcal{M}}(x) = (1+x)(T(x) - M(x))$, and consequently to obtain the first identity (11). Now, by Cauchy integral formula, we have

$$\ell(\mathcal{M}_n) = [x^n]\ell_{\mathcal{M}}(x) = \frac{1}{2\pi \mathrm{i}}\oint \ell_{\mathcal{M}}(z)\frac{\mathrm{d}z}{z^{n+1}}.$$

With the substitution $z = \frac{w}{1+w+w^2}$, we have $\mathrm{d}z = \frac{1-w^2}{(1+w+w^2)^2}\mathrm{d}w$ and

$$\begin{aligned}\ell(\mathcal{M}_n) &= \frac{1}{2\pi \mathrm{i}}\oint (1+w)^2(1+w+w^2)^{n-1}\frac{\mathrm{d}w}{w^{n-1}} = \\ &= [x^{n-2}](1+x)^2(1+x+x^2)^{n-1} = [x^{n-2}](1+x+x^2)^n + [x^{n-3}](1+x+x^2)^{n-1}\end{aligned}$$

from which we have identity (12).

From the identity

$$\ell_{\mathcal{M}}(x) = (1+x)(T(x) - M(x)) = \frac{1+x}{1-x}B\left(\frac{x^2}{(1-x)^2}\right) - \frac{1+x}{1-x}C\left(\frac{x^2}{(1-x)^2}\right)$$

we have the expansion

$$\begin{aligned}\ell_{\mathcal{M}}(x) &= (1+x)\sum_{k\geq 0}\binom{2k}{k}\frac{x^{2k}}{(1-x)^{2k+1}} - (1+x)\sum_{k\geq 0}\binom{2k}{k}\frac{1}{k+1}\frac{x^{2k}}{(1-x)^{2k+1}} \\ &= (1+x)\sum_{n\geq 0}\left[\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}\frac{k}{k+1}\right]x^n\end{aligned}$$

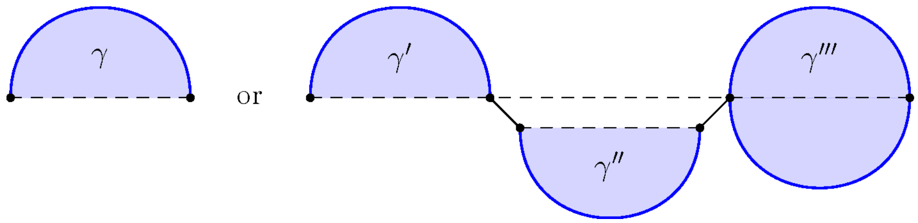

Figure 1: Decomposition of the Grand Motzkin paths used in the proof of Proposition 11 and of the Grand Shröder paths used in the proof of Proposition 17.

from which it is straightforward to obtain identity (13).

Finally, using identity (11) and the asymptotic expansions reported in (1) for the central trinomial coefficients and for the Motzkin numbers, we can obtain the first asymptotic equivalence in (14). Then, using once again the asymptotic expansion for the Motzkin numbers in (1), we also obtain the second asymptotic equivalence in (14). Since $4/9 \doteq 0.44$, the Hasse index is asymptotically quasi Boolean. $\square$

Proposition 11. *The generating series for the class of Grand Motzkin paths with respect to semi-length (marked by x) and to factors HU, DH, DU and HH (marked by q) is*

$$F(q;x) = \frac{1+(1-q)x}{\sqrt{(1+x)(1+(1-q)x)(1-(2+q)x+(1-q)x)}}. \tag{15}$$

Proof. Let $\overline{\mathcal{M}}$ be the class of reflected Motzkin paths (i.e., Grand Motzkin paths never going above the x-axis). Then, let $X(q;x)$ be the generating series for the class of Grand Motzkin paths starting with a step $X \in \{H, U, D\}$. Let $\overline{\mathcal{D}}$ be the class of subelevated reflected Motzkin paths, and let $\overline{D}(q;x)$ be its generating series.

Any Grand Motzkin path $\gamma \in \mathcal{GM}$ decomposes uniquely as follows (see Figure 1): γ is a Motzkin path, or otherwise $\gamma = \gamma'\gamma''\gamma''$ with $\gamma' \in \mathcal{M}$, $\gamma'' \in \overline{\mathcal{D}}$ and $\gamma''' \in \mathcal{GM}$. From this decomposition, we can obtain at once the equation

$$F(q;x) = f(q;x) + f(q;x)\overline{D}(q;x)F(q;x),$$

where $f(q;x)$ is series (9).

To obtain series $\overline{D}(q;x)$, let $\overline{f}(q;x)$, $\overline{h}(q;x)$, and $\overline{d}(q;x)$ be the generating series for the classes of reflected Motzkin paths with no restriction, starting with a horizonal step and starting with a down step, respectively. Moreover, let $\overline{H}(q;x)$ be the generating series for the class $\overline{\mathcal{H}}$ of reflected Motzkin paths of length at least two starting and ending with a horizontal step. Any non-empty reflected Motzkin path γ decomposes uniquely as $\gamma = H\gamma'$ (with $\gamma' \in \overline{\mathcal{M}}$), or as $\gamma = D\gamma'U\gamma''$ (with $\gamma', \gamma'' \in \overline{\mathcal{M}}$).

Let $\overline{\mathcal{H}_l}$ be the class of reflected Motzkin paths starting with a horizontal step, and let $\overline{\mathcal{H}_r}$ be the class of reflected Motzkin paths ending with a horizontal step. By reflecting paths along the vertical axis, we have a bijection between $\overline{\mathcal{H}_l}$ and $\overline{\mathcal{H}_r}$, and in both cases the generating series is $\overline{h}(q;x)$. Any subelevated reflected Motzkin path $\gamma \in \overline{\mathcal{D}}$ decomposes uniquely as follows (see Figure 2):

1. $\gamma = DU$
2. $\gamma = DHU$
3. $\gamma = D\gamma'\gamma''U$, with $\gamma' \in \overline{\mathcal{H}_l}$ and $\gamma'' \in \overline{\mathcal{D}}$
4. $\gamma = D\gamma'\gamma''U$, with $\gamma' \in \overline{\mathcal{D}}$ and $\gamma'' \in \overline{\mathcal{H}_r}$
5. $\gamma = D\gamma'U$, with $\gamma' \in \overline{\mathcal{H}}$
6. $\gamma = D\gamma'U$, with $\gamma' \in \overline{\mathcal{D}}$
7. $\gamma = D\gamma'\gamma''\gamma'''U$, with $\gamma', \gamma''' \in \overline{\mathcal{D}}$ and $\gamma'' \in \overline{\mathcal{M}}$

From these decompositions we can obtain the linear system

$$\begin{cases} \overline{f}(q;x) = 1 + \overline{h}(q;x) + \overline{d}(q;x) \\ \overline{h}(q;x) = x(1 + q\overline{h}(q;x) + \overline{d}(q;x)) \\ \overline{d}(q;x) = \overline{D}(q;x)\overline{f}(q;x) \\ \overline{D}(q;x) = x^2(q + q^2x + 2q\overline{h}(q;x)\overline{D}(q;x) + q^2\overline{H}(q;x) + \overline{D}(q;x) + \overline{D}(q;x)^2\overline{f}(q;x)) \\ \overline{H}(q;x) = \overline{D}(q;x). \end{cases}$$

The last identity comes from the simple bijection associating the path $D\gamma U \in \overline{\mathcal{D}}$ to the path $H\gamma H \in \overline{\mathcal{H}}$. By solving this system, we obtain the series

$$\overline{D}(q;x) = \frac{1 - qx - (1-q)x^2 - \sqrt{(1+x)(1+(1-q)x)(1-(2+q)x+(1-q)x)}}{2(1+(1-q)x)}.$$

Finally, by replacing this series in the first equation above, we can obtain series (15). $\square$

Theorem 12. *The edge generating series for Grand Motzkin lattices is*

$$\ell_{\mathcal{GM}}(x) = \frac{2x^2}{(1-3x)\sqrt{1-2x-3x^2}}. \tag{16}$$

Moreover, we have the identities

$$\ell(\mathcal{GM}_{n+2}) = 2\sum_{k=0}^{n} \binom{k;3}{k} 3^{n-k} \tag{17}$$

$$\ell(\mathcal{GM}_{n+2}) = \frac{2}{4^n}\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}(2k+1)3^k(-1)^{n-k} \tag{18}$$

$$\ell(\mathcal{GM}_{n+2}) = 2\sum_{k=0}^{n} \binom{n+1}{k+1}\binom{2k}{k}(-1)^k 3^{n-k} \tag{19}$$

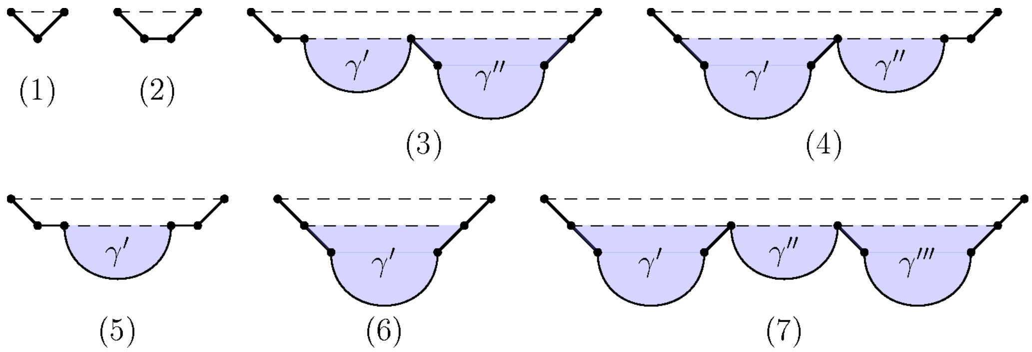

Figure 2: Decomposition of the subelevated reflected Motzkin paths used in the proof of Proposition 11.

and the asymptotic equivalences

$$\ell(\mathcal{GM}_n) \sim 2 \cdot 3^{n-2}\sqrt{\frac{3n}{\pi}} \qquad \text{and} \qquad i(\mathcal{GM}_n) \sim \frac{4}{9}n. \tag{20}$$

In particular, the Hasse index of Grand Motzkin lattices is asymptotically quasi Boolean.

Proof. Proposition 8 implies that (15) is the Δ-series for Grand Motzkin lattices. So, by applying Proposition 1, we obtain series (16). Since $\frac{\ell_{\mathcal{GM}}(x)}{x^2} = \frac{2}{1-3x}T(x)$, we have at once identity (17). Moreover, we also have

$$\frac{1}{2}\frac{\ell_{\mathcal{GM}}(x)}{x^2} = \frac{1}{(1-3x)^{3/2}}\frac{1}{\sqrt{1+x}} = \sum_{n\geq 0}\binom{2n}{n}\frac{3^n}{4^n}(2n+1)x^n \cdot \sum_{n\geq 0}\binom{2n}{n}\frac{(-1)^n}{4^n}x^n$$

from which we obtain identity (18). Finally, we have

$$\begin{aligned}\frac{1}{2}\frac{\ell_{\mathcal{GM}}(x)}{x^2} &= \frac{1}{(1-3x)\sqrt{(1-3x)^2+4x(1-3x)}} = \frac{1}{(1-3x)^2}\frac{1}{\sqrt{1+\frac{4x}{1-3x}}} = \\ &= \sum_{k\geq 0}\binom{2k}{k}(-1)^k\frac{x^k}{(1-3x)^{k+2}} = \sum_{k\geq 0}\binom{2k}{k}(-1)^k\sum_{n\geq 0}\binom{n+1}{k+1}3^{n-k}x^n\end{aligned}$$

from which we obtain identity (19).

From series (16), we have (for n sufficiently large) the identity

$$\ell(\mathcal{GM}_n) = 2[x^{n-2}](1+x)^{-1/2}\left(1-\frac{x}{1/3}\right)^{-3/2}.$$

Now, by applying the Darboux theorem, we have

$$\ell(\mathcal{GM}_n) \sim 2\frac{\psi(\xi)}{\xi^{n-2}}\frac{(n-2)^{\alpha-1}}{\Gamma(\alpha)}$$

where $\xi = 1/3$, $\psi(x) = (1+x)^{-1/2}$ and $\alpha = 3/2$. Since $\psi(\xi) = \sqrt{3}/2$ and $\Gamma(\alpha) = \Gamma(3/2) = \sqrt{\pi}/2$, we can obtain the asymptotic equivalence (20). Moreover, using the asymptotic expansion for the central trinomial coefficients given in (1), we have $i(\mathcal{GM}_n) \sim \frac{4}{9}n$. Since $4/9 \doteq 0.44$, the Hasse index is asymptotically quasi Boolean. $\square$

Theorems 10 and 12 immediately imply

Proposition 13. *The class of Motzkin lattices is Hasse-tamed:* $i(\mathcal{M}_n) \sim i(\mathcal{GM}_n) \sim \frac{4}{9}n$.

5 Schröder lattices and Grand Schröder lattices

Proposition 14. *For any Schröder path γ, we have $|\Delta\gamma| = \omega_H(\gamma) + \omega_{DU}(\gamma)$ and $|\nabla\gamma| = \omega_H^*(\gamma) + \omega_{UD}(\gamma)$. Similarly, for any Grand Schröder path γ, we have $|\Delta\gamma| = \omega_H(\gamma) + \omega_{DU}(\gamma)$ and $|\nabla\gamma| = \omega_H(\gamma) + \omega_{UD}(\gamma)$.*

Proof. In a Schröder lattice, a path γ is covered by all paths that can be obtained by replacing a horizontal step H with a peak UD, or a valley DU with a horizontal step H, and covers all paths that can be obtained by replacing a horizontal step H (not at level 0) with a valley DU or a peak UD with a horizontal step H. In a Grand Schröder lattice we have an analogous situation. $\square$

Proposition 15. *The generating series for the class of Schröder paths with respect to semi-length (marked by x) and horizontal steps H and valleys DU (marked by q) is*

$$f(q;x) = \frac{1 - x - \sqrt{1 - 2(1+2q)x + (1-2q)^2x^2}}{2qx(1+(1-q)x)}. \tag{21}$$

Proof. Let $h(q;x)$ be the generating series for the class $\mathcal{S}$ of all Schröder paths starting with a horizontal step and let $u(q;x)$ be the generating series for the class of all Schröder paths starting with an up step. Since any non-empty Schröder path γ decomposes uniquely as $\gamma = H\gamma'$ (with $\gamma' \in \mathcal{S}$) or $\gamma = U\gamma' D\gamma''$ (with $\gamma', \gamma'' \in \mathcal{S}$), we obtain the linear system

$$\begin{cases} f(q;x) = 1 + h(q;x) + u(q;x) \\ h(q;x) = qxf(q;x) \\ u(q;x) = xf(q;x)(1 + h(q;x) + qu(q;x)). \end{cases}$$

By solving for f, we can obtain series (21). $\square$

The mirror triangle generated by series (21) is sequence A090981 [19].

Theorem 16. *The edge generating series for Schröder lattices is*

$$\ell_{\mathcal{S}}(x) = \frac{(1-x)(1 - 4x + x^2 - (1-x)\sqrt{1-6x+x^2})}{2x\sqrt{1-6x+x^2}}. \tag{22}$$

Moreover, the number of edges in $\mathcal{S}_n$ can be expressed in one of the forms

$$\ell(\mathcal{S}_n) = d_n - r_n - d_{n-1} + r_{n-1} \tag{23}$$

$$\ell(\mathcal{S}_n) = \sum_{k=0}^{n} \binom{n+k-1}{n-k}\binom{2k}{k}\frac{k}{k+1}. \tag{24}$$

Finally, we have the asymptotic equivalences

$$\ell(\mathcal{S}_n) \sim \frac{(1+\sqrt{2})^{2n}}{\sqrt{\sqrt{2}n\pi}} \qquad \text{and} \qquad i(\mathcal{S}_n) = \frac{\ell(\mathcal{S}_n)}{|\mathcal{S}_n|} \sim (2-\sqrt{2})n. \tag{25}$$

In particular, the Hasse index of Schröder lattices is asymptotically quasi Boolean.

Proof. Proposition 14 implies that (21) is the Δ-series for Schröder lattices. So, by applying Proposition 1, we obtain series (22). It is easy to see that $\ell_{\mathcal{S}}(x) = (1-x)(d(x)-r(x))$. This implies at once identity (23). Moreover, expanding $\ell_{\mathcal{S}}(x) = (1-x)(d(x)-r(x))$ as follows

$$\ell_{\mathcal{S}}(x) = \frac{1}{\sqrt{1-\frac{4x}{(1-x)^2}}} - \frac{1-\sqrt{1-\frac{4x}{(1-x)^2}}}{2\frac{x}{(1-x)^2}} = \sum_{k\geq 0}\binom{2k}{k}\frac{x^k}{(1-x)^{2k}} - \sum_{k\geq 0}\binom{2k}{k}\frac{1}{k+1}\frac{x^k}{(1-x)^{2k}},$$

it is straightforward to obtain identity (24). Finally, using identity (23) and the asymptotics in (1), we obtain equivalences (25). Since $2-\sqrt{2} \doteq 0.58$, the Hasse index is asymptotically quasi Boolean. □

Proposition 17. *The generating series for the class of Grand Schröder paths with respect to semi-length (marked by x) and horizontal steps H and valleys DU (marked by q) is*

$$g(q;x) = \frac{1}{\sqrt{1-2(1+2q)x+(1-2q)^2x^2}}. \tag{26}$$

Proof. Let $\overline{\mathcal{S}}$ be the class of reflected Schröder paths (i.e., Grand Schröder paths never going above the x-axis), and let $\overline{f}(q;x)$ be the corresponding generating series.

Any Grand Schröder path $\gamma \in \mathcal{GM}$ decomposes uniquely (see again Figure 1) as follows: γ is a Schröder path, or otherwise $\gamma = \gamma'\gamma''\gamma''$ with $\gamma' \in \mathcal{S}$, $\gamma'' \in \overline{\mathcal{S}}$ and $\gamma''' \in \mathcal{GS}$. From this decomposition, we obtain at once the equation

$$g(q;x) = f(q;x) + xf(q;x)(q+\overline{f}(q;x)-1)g(q;x),$$

where $f(q;x)$ is series (21).

Since any non-empty reflected Schröder path γ decomposes uniquely as $\gamma = H\gamma'$ (with $\gamma' \in \overline{\mathcal{S}}$) or $\gamma = D\gamma'U\gamma''$ (with $\gamma',\gamma'' \in \overline{\mathcal{S}}$), we obtain the linear system

$$\begin{cases} \overline{f}(q;x) = 1 + \overline{h}(q;x) + \overline{d}(q;x) \\ \overline{h}(q;x) = qx\overline{f}(q;x) \\ \overline{d}(q;x) = x(q+\overline{f}(q;x)-1)\overline{f}(q;x), \end{cases}$$

where $\overline{h}(q;x)$ and $\overline{d}(q;x)$ are the generating series for the reflected Schröder paths starting with a horizontal step and with a down step, respectively. By solving for $\overline{f}(q;x)$, we obtain

$$\overline{f}(q;x) = \frac{1+x-2qx-\sqrt{(1+x-2qx)^2-4x}}{2x}.$$

Finally, by replacing this series in the above equation and by solving for $g(q;x)$, we obtain series (26). ◻

Theorem 18. *The edge generating series for Grand Schröder lattices is*

$$\ell_{\mathcal{GS}}(x) = \frac{2(x-x^2)}{(1-6x+x^2)^{3/2}}. \tag{27}$$

Moreover, we have the identities

$$\ell(\mathcal{GS}_n) = 2\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}(n-k) \tag{28}$$

$$\ell(\mathcal{GS}_n) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\frac{(n-2k)(n+k+2)}{k+1}2^{k+1}3^{n-2k-2}. \tag{29}$$

and the asymptotic equivalences

$$\ell(\mathcal{GS}_n) \sim \sqrt{\frac{n}{2\sqrt{2}\pi}}(1+\sqrt{2})^{2n} \qquad \text{and} \qquad i(\mathcal{GS}_n) = \frac{\ell(\mathcal{GS}_n)}{|\mathcal{GS}_n|} \sim (2-\sqrt{2})n. \tag{30}$$

In particular, the Hasse index of Grand Schröder lattices is asymptotically quasi Boolean.

Proof. By Proposition 14, the Δ-series for Grand Schröder lattices is series (26). So, by Proposition 1, we obtain series (27). By expanding this series as follows

$$\ell_{\mathcal{GS}}(x) = \frac{2x(1-x)}{((1-x)^2-4x)^{3/2}} = \frac{2x}{(1-x)^2}\frac{1}{\left(1-\frac{4x}{(1-x)^2}\right)^{3/2}} = 2\sum_{k\geq 0}\binom{2k}{k}(2k+1)\frac{x^{k+1}}{(1-x)^{2k+2}}$$

we can obtain identity (28). Formula (29) can be obtained in a similar way (just consider the identity $1-6x+x^2 = (1-3x)^2-8x^2$). Finally, using the Darboux theorem, we can obtain the first equivalence in (30). Then, using the asymptotics in (1), we can also obtain the second equivalence in (30). ◻

Theorems 16 and 18 immediately imply

Proposition 19. *The class of Schröder lattices is Hasse-tamed:* $i(\mathcal{S}_n) \sim i(\mathcal{GS}_n) \sim (2-\sqrt{2})n$.

6 Fibonacci and Grand Fibonacci posets

All classes of paths considered in the previous sections have the property of being Hasse-tamed. Here, we will consider a simple class of paths for which such a property is not true.

A *Fibonacci path*[3] is a Motzkin path confined in the horizontal strip $[0, 1]$ with horizontal steps only on the x-axis. A *Grand Fibonacci path* is a Grand Motzkin path confined in the horizontal strip $[-1, 1]$ with horizontal steps only on the x-axis. Let $\mathcal{F}$ be the class of *Fibonacci posets* (i.e., *Fibonacci semilattices* [11]), and let $\mathcal{GF}$ be the class of *Grand Fibonacci posets*.

Let F_n be the *Fibonacci numbers* (sequence A000045) and let L_n be the *Lucas numbers* (sequence A000204). By Binet formulas, we have the identities $F_n = (\varphi^n - \widehat{\varphi}^n)/\sqrt{5}$ and $L_n = \varphi^n + \widehat{\varphi}^n$, and the asymptotics $F_n \sim \varphi^n/\sqrt{5}$ and $L_n \sim \varphi^n$, where $\varphi = (1 + \sqrt{5})/2$ and $\widehat{\varphi} = (1 - \sqrt{5})/2$.

Theorem 20. *The edge generating series for the Fibonacci posets is*

$$\ell_{\mathcal{F}}(x) = \frac{x^2}{(1 - x - x^2)^2}. \tag{31}$$

Moreover, we have the explicit formula

$$\ell(\mathcal{F}_n) = \frac{nL_n - F_n}{5} \tag{32}$$

and the asymptotic equivalences

$$\ell(\mathcal{F}_n) \sim \frac{n}{5}\varphi^n \qquad \text{and} \qquad i(\mathcal{F}_n) \sim \frac{n}{\sqrt{5}\varphi}. \tag{33}$$

In particular, the Fibonacci posets are not asymptotically quasi Boolean.

Proof. In a Fibonacci poset, a path γ is covered by all paths that can be obtained from γ by replacing a horizontal double step HH (necessarily on the x-axis) with a peak UD. So, we have $|\Delta\gamma| = \omega_{HH}(\gamma)$. Let $f(q; x)$ be the generating series for Fibonacci paths with respect to horizontal double steps and length. Similarly, let $h(q; x)$ and $u(q; x)$ be the generating series for Fibonacci paths starting with a horizontal step or with an up step, respectively. Since any non-empty Fibonacci path γ decomposes uniquely as $\gamma = H\gamma'$ or as $\gamma = UD\gamma'$, where γ' is an arbitrary Fibonacci path in both cases, it is easy to obtain the identities: $f(q; x) = 1 + h(q; x) + u(q; x)$, $h(q; x) = x(1 + qh(q; x) + u(q; x))$, $u(q; x) = x^2 f(q; x)$. The solution of this linear system is straightforward and yields the series

$$f(q; x) = \frac{1 + (1 - q)x}{1 - qx - x^2 - (1 - q)x^3}.$$

[3] An equivalent definition of Fibonacci paths is as product of pyramids of semilength no more than 2 [2]

Now, by applying Proposition 1, we can obtain series (31). Expanding this series we obtain identity (32). From this identity and the identity $i(\mathcal{F}_n) = \frac{nL_n - F_n}{5F_{n+1}}$, we obtain at once asymptotics (33). Finally, since $1/(\sqrt{5}\varphi) \doteq 0.276$, the Fibonacci posets are not asymptotically quasi Boolean. $\square$

Theorem 21. *The edge generating series for the Grand Fibonacci posets is*

$$\ell_{\mathcal{GF}}(x) = \frac{2x^2}{(1-x-2x^2)^2}. \tag{34}$$

Moreover, we have the explicit formula

$$\ell(\mathcal{GF}_n) = \frac{(3n-1)2^{n+1} + (-1)^n 2(3n+1)}{27} \tag{35}$$

and the asymptotic equivalences

$$\ell(\mathcal{GF}_n) \sim \frac{n}{9}2^{n+1} \qquad \text{and} \qquad i(\mathcal{GF}_n) \sim \frac{n}{3}. \tag{36}$$

In particular, the Grand Fibonacci posets are not asymptotically quasi Boolean.

Proof. In a Grand Fibonacci poset, a path γ is covered by all paths that can be obtained from γ by replacing a horizontal double step HH (necessarily on the x-axis) with a peak UD, or by replacing a valley DU touching the line $y = -1$ with a horizontal double step HH on the x-axis. Then, let $F(q;x)$ be the generating series for Grand Fibonacci paths with respect to horizontal double steps and valleys touching the line $y = -1$ (marked by q) and length (marked by x). Similarly, let $H(q;x)$, $U(q;x)$ and $D(q;x)$ be the generating series for these paths starting with a horizontal step, with an up step or with a down step, respectively. Since any non-empty Grand Fibonacci path γ decomposes uniquely as $\gamma = H\gamma'$, as $\gamma = UD\gamma'$, or as $\gamma = DU\gamma'$, where γ' is an arbitrary Grand Fibonacci path in each case, it is easy to obtain the identities: $F(q;x) = 1 + H(q;x) + U(q;x) + D(q;x)$, $H(q;x) = x(1 + qH(q;x) + U(q;x) + D(q;x))$, $U(q;x) = x^2F(q;x)$, $D(q;x) = qx^2F(q;x)$. The solution of this linear system is straightforward and yields the series

$$F(q;x) = \frac{1 + (1-q)x}{1 - qx - (1+q)x^2 - (1-q^2)x^3}.$$

Now, by applying Proposition 1, we can obtain series (34). Decomposing this series as sum of partial fractions, it is easy to obtain identity (35). From this identity, we can obtain the first relation in (36). Finally, notice that the generating series for the vertices of the Grand Fibonacci posets is $g(x) = F(1;x) = 1/(1-x-2x^2)$ and that consequently the number of vertices is $g_n = (2^{n+1} + (-1)^n)/3 \sim 2^{n+1}/3$. So, we also have the second relation in (36). $\square$

Theorems 20 and 21 immediately imply

Proposition 22. *The class of Fibonacci lattices is not Hasse-tamed.*

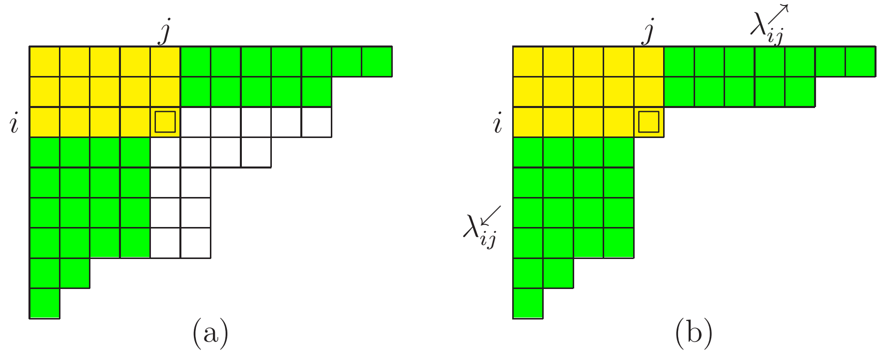

Figure 3: (a) Diagram of the partition $\lambda = (12, 10, 10, 8, 6, 6, 6, 2, 1)$, with the cell $(i, j) = (3, 5)$ marked by a square. (b) Diagram of the partition $\lambda^{\searrow}_{ij} = (12, 10, 5, 4, 4, 4, 4, 2, 1)$. In both pictures, the diagrams of the partitions $\lambda^{\nearrow}_{ij} = (7, 5)$ and $\lambda^{\swarrow}_{ij} = (4, 4, 4, 4, 2, 1)$ are in dark color.

7 Young lattices

A *partition* of a non-negative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The λ_i's are the *parts* of λ. The (*Ferrers*) *diagram* of λ is a left-justified array of squares (or dots) with exactly λ_i squares in the i-th row. Partitions can be ordered by magnitude of parts [1]: if $\alpha = (a_1, \ldots, a_h)$ and $\beta = (b_1, \ldots, b_k)$, then $\alpha \leq \beta$ whenever $h \leq k$ and $a_i \leq b_i$ for every $i = 1, 2, \ldots, h$. If $\alpha \leq \beta$ the diagram of α is contained in the diagram of β. The resulting poset is an infinite distributive lattice, called *Young lattice*. The *Young lattice* $\mathcal{Y}_\lambda$ generated by a partition λ is the principal ideal generated by λ in $\mathcal{Y}$, i.e., the set of all integer partitions α such that $\alpha \leq \lambda$. Any Young lattice $\mathcal{Y}_\lambda$ can be considered as a lattice of paths. Indeed, it is isomorphic to the lattice of all paths made of horizontal steps $X = (1, 0)$ and vertical steps $Y = (0, 1)$, confined in the region defined by the diagram Φ_λ of λ. Moreover Dyck lattices are isomorphic to the dual of the Young lattices associated with the staircase partitions $(n, n-1, \ldots, 2, 1)$ [8].

Here we will give a general formula for the number of edges in any Young lattice $\mathcal{Y}_\lambda$. To do that we need the following definitions. We say that a cell of a Young diagram is a *corner cell* whenever the diagram has no cells below and no cells on the right of such a cell. The *boundary* of the diagram Φ_λ is the set $\partial\Phi_\lambda$ of all its corner cells.

In a Ferrers diagram Φ_λ, we denote by (i, j) the cell of Φ_λ lying at the cross of the i-th row with the j-th column. For instance, $(1, 1)$ is the cell of Φ_λ in the top left corner. Given a cell (i, j) of Φ_λ, we have the partition $\lambda^{\nwarrow}_{ij}$ obtained by intersecting the first i rows and the first j columns of Φ_λ (the light colored cells in Figure 3(a)), the partition $\lambda^{\searrow}_{ij}$ obtained by deleting all cells $(i', j') \neq (i, j)$ such that $i' \geq i$ and $j' \geq j$ (i.e., by deleting the white cells in Figure 3(a)), the partition $\lambda^{\nearrow}_{ij}$ obtained by intersecting the first $i - 1$ rows and the last $\lambda_1 - j$ columns of Φ_λ (the dark cells above (i, j) in Figure 3(a)), and finally the partition $\lambda^{\swarrow}_{ij}$

obtained by intersecting the last $k-i$ rows and the first $j-1$ columns of Φ_λ (the dark cells below (i,j) in Figure 3(a)).

Theorem 23. *The number of edges in the Hasse diagram of the Young lattice $\mathcal{Y}_\lambda$ is*

$$\ell(\mathcal{Y}_\lambda) = \sum_{(i,j)\in\Phi_\lambda} |\mathcal{Y}_{\lambda_{ij}^{\swarrow}}||\mathcal{Y}_{\lambda_{ij}^{\nearrow}}|. \tag{37}$$

Proof. Let $\mu \in \mathcal{Y}_\lambda$. The partitions covered by μ are exactly the partitions obtained by removing a cell from the boundary of μ. Hence $|\nabla\mu| = |\partial\Phi_\mu|$ and

$$\ell(\mathcal{Y}_\lambda) = \sum_{\mu\in\mathcal{Y}_\lambda} |\nabla\mu| = \sum_{\mu\in\mathcal{Y}_\lambda} |\partial\Phi_\mu| = \sum_{(i,j)\in\Phi_\lambda} C_{ij}$$

where C_{ij} is the number of all partitions $\mu \in \mathcal{Y}_\lambda$ such that $(i,j) \in \partial\Phi_\mu$.

Since $\lambda_{ij}^{\nwarrow}$ and $\lambda_{ij}^{\searrow}$ are respectively the minimal and the maximal partition in $\mathcal{Y}_\lambda$ admitting (i,j) as a corner cell, a partition μ has (i,j) as a corner cell if and only if $\mu \in [\lambda_{ij}^{\nwarrow}, \lambda_{ij}^{\searrow}]$. This implies that the partition μ is equivalent to a pair (μ_1,μ_2) of partitions, where $\Phi_{\mu_1} \subseteq \Phi_{\lambda_{ij}^{\swarrow}}$ and $\Phi_{\mu_2} \subseteq \Phi_{\lambda_{ij}^{\nearrow}}$. So $C_{ij} = |\mathcal{Y}_{\lambda_{ij}^{\swarrow}}||\mathcal{Y}_{\lambda_{ij}^{\nearrow}}|$, and this yields identity (37). $\square$

To illustrate Theorem 23, we consider the particularly interesting case of the Young lattices $L(m,n) = \mathcal{Y}_\lambda$ (for $m,n \geq 1$) obtained for the partition $\lambda = (n,n,\ldots,n)$ with m parts [20, 17]. Since $L(n,n) = \mathcal{GD}_n$, the next theorem generalizes Theorem 6.

Theorem 24. *The number of edges in the Hasse diagram of the Young lattice $L(m,n)$ is*

$$\ell(L(m,n)) = \binom{m+n-1}{n} n = \frac{(m+n-1)!}{(m-1)!(n-1)!}. \tag{38}$$

Proof. Writing $\lambda = k^\ell$ for the partition $\lambda = (k,k,\ldots,k)$ with ℓ parts, for every cell $(i,j) \in \Phi_\lambda$, we have $\lambda_{ij}^{\nearrow} = (n-j)^{i-1}$ and $\lambda_{ij}^{\swarrow} = (j-1)^{m-i}$. Hence, by formula (37), we have

$$\ell(L(m,n)) = \sum_{i=1}^{m}\sum_{j=1}^{n} |L(n-j,i-1)||L(j-1,m-i)|.$$

Since $|L(m,n)| = \binom{m+n}{n}$, we obtain

$$\ell(L(m,n)) = \sum_{i=1}^{m}\sum_{j=1}^{n} \binom{m-i+j-1}{j-1}\binom{n+i-j-1}{i-1}.$$

Now, by applying the Vandermonde convolution in the form

$$\sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\beta-k}{n-k} = \binom{\alpha+\beta+1}{n},$$

we have

$$\begin{aligned}\ell(L(m,n)) &= \sum_{i=1}^{m}\left[\sum_{j=0}^{n-1}\binom{(m-i)+j}{j}\binom{(n+i-2)-j}{n-1-j}\right]\\ &= \sum_{i=1}^{m}\binom{m+n-1}{n-1} = m\binom{m+n-1}{n-1}\end{aligned}$$

which is equivalent to (38). □

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