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Kepler-Bouwkamp Radius of Combinatorial Sequences

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Abstract

The Kepler-Bouwkamp constant is defined as the limit of radii of a sequence of concentric circles that are simultaneously inscribed in a regular n-gon and circumscribed around a regular $(n+1)$-gon for $n \geq 3$. The outermost circle, circumscribed around an equilateral triangle, has radius 1. We investigate what happens when the number of sides of regular polygons from the definition is given by a sequence different from the sequence of natural numbers.

1 Introduction

Take a unit circle and inscribe in it an equilateral triangle. Then inscribe a circle in that triangle, and inscribe an equilateral triangle in that circle. Continue the procedure. It is a high-school exercise to show that the sequence of radii of inscribed circles tends to zero as $\frac{1}{2^n}$. The conclusion remains valid if we replace triangle with any other regular polygon, only the rate of convergence is changed: For a regular m-gon, the radii tend to zero as $\left(\cos \frac{\pi}{m}\right)^n$. But what happens with the limiting radius if the number of sides of inscribed polygons is not constant? What if it increases so that the number of sides of n-th polygon is given as the n-th element of a sequence (a_n) of non-negative integers that are all greater than two? For a given regular n-gon, the ratio of radii of its inscribed and its circumscribed circle is equal to $\cos\frac{\pi}{n}$. Hence the answer to our question will be given as an infinite product $\prod_{a_n\geq 3}^{\infty}\cos\frac{\pi}{a_n}$.

When $a_n = n$, the answer is well known: The limiting radius is equal to the Kepler-Bouwkamp constant $\rho = \prod_{n=3}^{\infty}\cos\frac{\pi}{n} \doteq 0.1149420448$ ([4, p. 428]; see also [7, A085365]). To the best of my knowledge, there is only one other sequence, the sequence of odd primes, for which the answer was sought. Kitson [5] computed the limiting radius as $\rho_P = 0.3128329295$. The quantities in question were also computed for a few other sequences, mostly of the form $n(n+1)$ for even and odd n, in a paper by Mathar [6], but they appear there as byproducts of some other computations. The goal of this note is to investigate for which sequences is that limit positive and to compute the limiting radii for several classes of combinatorial sequences. The results might be useful in further investigation of problems arising in computational geometry [6], and they might shed additional light on properties of integrals of the type $\int_0^\infty \prod_{k=0}^{n}\frac{\sin(a_k x)}{a_k x}dx$ [2]. They will also provide benchmarks for testing numerical methods for efficient evaluation of slowly convergent infinite products and series [3].

In the rest of this paper we consider only (weakly) increasing sequences (a_n) of non-negative integers. If a sequence has elements smaller than 3, we ignore them, along with the corresponding terms in any expressions and formulas. We call such sequences *admissible*. Whenever referring to a sequence that is not itself admissible, we mean its largest admissible subsequence.

Let (a_n) be an admissible sequence. Its *Kepler-Bouwkamp radius* is denoted by $\kappa(a_n)$ and defined by

$$\kappa(a_n) = \prod_{n}^{\infty}\cos\frac{\pi}{a_n}.$$

Hence, $\kappa(n) = \rho$, $\kappa(p_n) = \rho_P$ and $\kappa(c) = 0$ for any constant sequence $a_n = c$.

2 Growth rates and convergence

In this section we investigate under what conditions a combinatorial sequence has a positive Kepler-Bouwkamp radius. We start by some elementary observations.

Proposition 1. *Let (a_n) be an admissible sequence such that $\lim_{n\to\infty} < \infty$. Then $\kappa(a_n) = 0$.*

Proof. If an admissible sequence is bounded, it must have a convergent subsequence. The condition of integrality means that this subsequence must be constant. Then the infinite product over that subsequence diverges toward zero and the claim follows. □

Hence a sequence must grow without bounds to have a positive Kepler-Bouwkamp radius. Any sequence growing faster that the sequence of natural numbers grows fast enough:

Proposition 2. *Let an admissible sequence (a_n) be a subsequence of $\mathbb{N}$. Then $\kappa(a_n) > \rho > 0$.*

What about sequences growing slower than $\mathbb{N}$? The following example shows that linear growth, no matter how slow, still suffices for the positivity of $\kappa(a_n)$.

Proposition 3. *Let* $(a_n) = \left\lfloor \frac{n}{k} \right\rfloor$ *for a fixed* $k \geq 2$*,* $n \geq 3k$*. Then* $\kappa(a_n) = \rho^k$*.*

Proof. It is enough to look at the case $k = 2$. Since each term in the product is repeated twice, we have

$$\kappa\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = \prod_{n=3}^{\infty} \left(\cos\frac{\pi}{n}\right)^2 = \left(\prod_{n=3}^{\infty} \cos\frac{\pi}{n}\right)^2 = \rho^2.$$

□

By the same reasoning we can conclude that the infinite product will converge for all admissible sequences in which the number of repetitions of an element remains finite.

Proposition 4. *Let* (a_n) *be an admissible sequence and let there be a* $k \in \mathbb{N}$ *such that no element of* $\mathbb{N}$ *appears in* (a_n) *more than* k *times. Then* $\kappa(a_n) > \rho^k > 0$*.*

What happens when the number of repetitions of an element grows without bounds? We first consider a concrete example.

Proposition 5. $\kappa(\lfloor\sqrt{n}\rfloor) = 0$.

Proof. Let us look at the sequence $\lfloor\sqrt{n}\rfloor$, i.e., to its admissible subsequence. It start with seven 3's, continues with nine 4's, then eleven 5's and so on. In general, an integer m appears in it exactly $2m + 1$ times. Hence,

$$\kappa(\lfloor\sqrt{n}\rfloor) = \prod_{m=3}^{\infty} \left(\cos\frac{\pi}{m}\right)^{2m+1} = \rho\left(\prod_{m=3}^{\infty} \left(\cos\frac{\pi}{m}\right)^m\right)^2.$$

Now, the above infinite product converges if and only if converges the series of its logarithms $\sum_{m=3}^{\infty} m \ln\cos\frac{\pi}{m}$. By expanding $\ln\cos\frac{\pi}{m}$ we obtain $\ln\cos\frac{\pi}{m} \sim -\frac{\pi^2}{2m^2}$, and then $m\ln\cos\frac{\pi}{m} \sim -\frac{\pi^2}{2m}$. Hence the series diverges, and the infinite product goes to zero. □

From the above example we can conclude that the infinite product will converge even in cases when the number of repetitions grows without bounds, as long as the growth is sublinear. The sequence $\lfloor\sqrt{n}\rfloor$ is a limiting case – if a sequence (a_n) grows faster than $\lfloor\sqrt{n}\rfloor$, its Kepler-Bouwkamp radius will be positive. The following result summarizes our findings.

Theorem 6. $\kappa(a_n) > 0$ *if and only if* (a_n) *grows faster than* $\lfloor\sqrt{n}\rfloor$*.*

3 Explicit formulas and (semi)numerical examples

In this section we consider some admissible sequences and compute their Kepler-Bouwkamp radii. We start with a class of sequences for which it is possible to give explicit formulas. To the best of my knowledge, the class is very narrow; it contains only integer multiples of the sequence of powers of two. Hence, all sequences of this class are of the type $a_n = m \cdot 2^n$ for some integer m. The result follows from a classical infinite product formula for $\operatorname{sinc} x$.

The sinc function is defined by

$$\operatorname{sinc} x = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0; \end{cases}.$$

The following infinite product representation of $\operatorname{sinc} x$ was known already to Viète:

$$\operatorname{sinc} x = \prod_{n=1}^{\infty} \cos \frac{x}{2^n}.$$

It immediately yields formulas for the Kepler-Bouwkamp radius of an integer multiple of the sequence of powers of two.

Theorem 7. $\kappa(m \cdot 2^n) = \operatorname{sinc} \frac{\pi}{m}$.

Here are the values for some small m.

Corollary 8.

$$\kappa(2 \cdot 2^n) = \frac{2}{\pi}; \qquad \kappa(3 \cdot 2^n) = \frac{3\sqrt{3}}{2\pi}; \qquad \kappa(4 \cdot 2^n) = \frac{3\sqrt{2}}{\pi};$$
$$\kappa(5 \cdot 2^n) = \frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi}; \qquad \kappa(6 \cdot 2^n) = \frac{3}{\pi}.$$

Let us now look at a general admissible sequence (a_n) with $\kappa(a_n) > 0$. We follow, with minor modifications, the approach outlined by Kitson [5]. First we take the logarithm of both sides in the expression for $\kappa(a_n)$,

$$\ln \kappa(a_n) = \sum_n \ln \cos \frac{\pi}{a_n}.$$

The summand $\ln \cos \frac{\pi}{a_n}$ can be expanded into a series

$$\ln \cos \frac{\pi}{a_n} = -\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{2k(2k)!} |B_{2k}| \left(\frac{\pi}{a_n}\right)^{2k}$$

[1, Formula 4.3.72]. Here B_k denote the Bernoulli numbers. Their exponential generating function is given by

$$\sum_{k=0}^{\infty} \frac{B_k x^k}{k!} = \frac{x}{e^x - 1}.$$

We have the following representation of even Bernoulli numbers:

$$B_{2k} = (-1)^{k-1}\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k)$$

[1, Formula 23.1.18]. It can be obtained by expanding Bernoulli polynomials $B_n(x)$ into a cosine Fourier series

$$B_{2k}(x) = (-1)^{k-1}\frac{2(2k)!}{(2\pi)^{2k}}\sum_{l=1}^{\infty}\frac{\cos 2l\pi x}{l^{2k}}$$

and substituting $x = 0$. When this is plugged into expansion of $\ln\cos\frac{\pi}{a_n}$ a lot of cancellation occurs and we end with

$$\ln\cos\frac{\pi}{a_n} = -\sum_{k=1}^{\infty}\frac{2^{2k}-1}{k}\zeta(2k)\frac{1}{a_n^{2k}}.$$

Now we can exchange the order of summation in the double sum for $\ln\kappa(a_n)$. We obtain

$$\ln\kappa(a_n) = -\sum_{k=1}^{\infty}\frac{2^{2k}-1}{k}\zeta(2k)\sum_{n}\frac{1}{a_n^{2k}}.$$

Here the interior sum runs over all n such that $a_n \geq 3$. Hence, we have proved the following result.

Theorem 9.

$$\kappa(a_n) = \exp\left(-\sum_{k=1}^{\infty}\frac{2^{2k}-1}{k}\zeta(2k)\sum_{a_n\geq 3}a_n^{-2k}\right).$$

We see that all dependence on the sequence (a_n) is well isolated and contained in the sum $\sum_{a_n\geq 3}a_n^{-2k}$. For some sequences that sum can be expressed in closed formulas in terms of k. By plugging them into the formula from the above theorem, we obtain rapidly converging expressions for $\kappa(a_n)$; hence the (semi)numerical in the section title.

For $a_n = n$ the sum can be expressed in terms of zeta function, taking into account the corrections for terms smaller than 3: $\sum_{a_n\geq 3}n^{-2k} = \frac{4^k(\zeta(2k)-1)-1}{4^k}$. Similarly, Kitson used the prime zeta function to express the analogous sum in [5]. Let us denote by $Z(a_n)$ the sum $\sum_{a_n\geq 3}a_n^{-2k}$ for the sequence (a_n). In the following proposition we present the values of $Z(a_n)$ for several sequences that yield to this approach.

Proposition 10.

$$\begin{aligned}
Z(2n) &= 2^{-2k}(\zeta(2k)-1);\\
Z(2n+1) &= 2^{-2k}\zeta(2k,\frac{3}{2});\\
Z(m\cdot n+p) &= m^{-2k}\zeta(2k,\frac{m+p}{m}) \qquad \textit{for}\ \ m\geq 3;\\
Z(n^m) &= \zeta(2mk)-1;\\
Z(m^n) &= \frac{1}{m^{2k}-1} \qquad \textit{for}\ \ m\geq 3.
\end{aligned}$$

(Here $\zeta(2k, q)$ is the Hurwitz zeta function.)

The Kepler-Bouwkamp radii of several sequences now follow by plugging the above expressions into formula of Theorem 9 and evaluating the resulting sums.

Corollary 11.

$$\begin{aligned} \kappa(2n) &= 0.4297802164; \\ \kappa(2n+1) &= 0.2674437781; \\ \kappa(n^2) &= 0.6402929927; \\ \kappa(3^n) &= 0.4662745790. \end{aligned}$$

Here are the Kepler-Bouwkamp radii of some sequences as reported in Mathar's paper [6].

Proposition 12.

$$\begin{aligned} \kappa(n(n+1)) &= 0.8154881209; \\ \kappa((2n+1)(2n+2)) &= 0.8373758680; \\ \kappa(p_n p_{n+1}) &= 0.9729664541 \qquad \textit{for odd primes.} \end{aligned}$$

We conclude the paper by reporting the numerical values of Kepler-Bouwkamp radii for several interesting combinatorial sequences.

Proposition 13.

$$\begin{aligned} \kappa(n!) &= 0.8583138700; \\ \kappa((2n-1)!!) &= 0.4888521829; \\ \kappa((2n)!!) &= 0.9218702724; \\ \kappa(n^n) &= 0.7022723378. \end{aligned}$$

The class of sequences that allow the seminumerical approach is probably much wider than reported here, but I am not aware of any simple way to decide if a given sequence belongs to it.

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References

[1] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Math. Series **55**, 1972.

[2] D. Borwein and J. M. Borwein, Some remarkable properties of sinc and related integrals, *Ramanujan J.* **5** (2001) 73–89.

[3] M. Chamberland and A. Straub, On gamma quotients and infinite products, preprint, http://arxiv.org/abs/1309.3455, 2013.

[4] S. R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge, 2003.

[5] A. R. Kitson, The prime analog of the Kepler-Bouwkamp constant, *Math. Gazette* **92** (2008) 293–294.

[6] R. J. Mathar, Tightly circumscribed regular polygons, preprint, http://arxiv.org/abs/1301.6293, 2013.

[7] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, available electronically at http://oeis.org.

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