Journal of Integer Sequences, Vol. 17 (2014),
Article 14.5.3

Iterative Procedure for Hypersums of Powers of Integers

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Abstract

Relying on a recurrence relation for the hypersums of powers of integers put forward recently, we develop an iterative procedure which allows us to express a hypersum of arbitrary order in terms of ordinary (zeroth order) power sums. Then, we derive the coefficients of the hypersum polynomial as a function of the Bernoulli numbers and the Stirling numbers of the first kind.

1 Introduction

For every integer m, $m \geq 1$, the hypersums of powers of integers are defined recursively as follows: $P_k^{(m)}(n) = \sum_{j=1}^{n} P_k^{(m-1)}(j)$, where $P_k^{(0)}(n)$ is the sum of the first n positive integers each raised to the integer power $k \geq 0$, $P_k^{(0)}(n) = 1^k + 2^k + \cdots + n^k$ [1, 2]. The latter is given by a polynomial in n of degree $k+1$ with zero constant term. Hence, inductively $P_k^{(m)}(n)$ is given by a polynomial in n of degree $k+m+1$ with zero constant term:

$$P_k^{(m)}(n) = \sum_{r=1}^{k+m+1} c_{k,m}^r n^r. \tag{1}$$

An explicit formula for the coefficients $c_{k,m}^r$ involving the Stirling numbers of the first and second kinds has been given by the author [3]. In this paper (Section 2), by an iterative procedure, we obtain a new representation of the m-th order hypersum $P_k^{(m)}(n)$ in terms of ordinary (zeroth order) power sums. Specifically, we will show that $P_k^{(m)}(n)$ can be expressed as a linear combination of $P_k^{(0)}(n), P_{k+1}^{(0)}(n), \dots, P_{k+m}^{(0)}(n)$, as follows:

$$P_k^{(m)}(n) = \sum_{i=0}^{m} \frac{(-1)^i q_{m,i}(n)}{m!} P_{k+i}^{(0)}(n), \tag{2}$$

where $q_{m,i}(n)$ is a polynomial in n of degree $m-i$. (Note that formula (2) holds for $m=0$ if we set $q_{0,0}(n)=1$.) In Section 3, we determine the explicit form of the coefficients of $q_{m,i}(n)$. Then, using (2), we obtain the coefficients $c_{k,m}^r$ in terms of the Bernoulli numbers B_k and the (unsigned) Stirling numbers of the first kind $\left[{n \atop k}\right]$ (see A008275 in [4]). In particular, we proved that

$$c_{k,m}^1 = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i \left[{m+1 \atop i+1}\right] B_{k+i}, \tag{3}$$

in accordance with the result obtained by Inaba [2, Proposition 1]. (Please note that, throughout this paper, we use the convention that $B_1 = \frac{1}{2}$.)

For later reference, we recall that the recurrence relation defining the numbers $\left[{n \atop k}\right]$ is given by [5, p. 214]:

$$\left[{m+1 \atop i+1}\right] = m \left[{m \atop i+1}\right] + \left[{m \atop i}\right], \tag{4}$$

with $\left[{0 \atop 0}\right] = 1$, and $\left[{n \atop 0}\right] = \left[{0 \atop n}\right] = 0$ for $n \geq 1$.

Formula (2) is noteworthy since it neatly shows how the hypersum $P_k^{(m)}(n)$ is constructed out of the building blocks $P_{k+i}^{(0)}(n)$, $i = 0, 1, \dots, m$. Moreover, we point out that the polynomials $q_{m,i}(n)$ are interesting in their own right. Indeed, for fixed m, the coefficients corresponding to the set of polynomials $\{q_{m,i}(n)\}_{i=0}^{m}$ may be arranged in a Pascal-like triangular array with a specific rule of formation.

2 Iterative procedure for the hypersum

The basic tool we use to obtain $P_k^{(m)}(n)$ in terms of ordinary power sums is the following recurrence relation, a proof of which has been given by the author [3, Theorem 8]:

Theorem 1. *The hypersums $P_k^{(j)}(n)$, $P_k^{(j-1)}(n)$, and $P_{k+1}^{(j-1)}(n)$ satisfy the recurrence relation*

$$P_k^{(j)}(n) = \frac{n+j}{j} P_k^{(j-1)}(n) - \frac{1}{j} P_{k+1}^{(j-1)}(n), \quad k \geq 0,\ j \geq 1. \tag{5}$$

To obtain $P_k^{(m)}(n)$, we repeatedly apply the recurrence (5) to successive values of $j =$

$1, 2, \ldots$, up to $j = m$. Proceeding in this way, it is easy to see that, for example,

$$
\begin{aligned}
P_k^{(3)}(n) = \frac{1}{6}(n+1)(n+2)(n+3)P_k^{(0)}(n) \\
&- \frac{1}{6}\big[(n+1)(n+2) + (n+1)(n+3) + (n+2)(n+3)\big]P_{k+1}^{(0)}(n) \\
&+ \frac{1}{6}\big[(n+1) + (n+2) + (n+3)\big]P_{k+2}^{(0)}(n) - \frac{1}{6}P_{k+3}^{(0)}(n),
\end{aligned}
$$

which is of the form (2) with $q_{3,0}(n) = (n+1)(n+2)(n+3)$, $q_{3,1}(n) = (n+1)(n+2) + (n+1)(n+3) + (n+2)(n+3)$, $q_{3,2}(n) = (n+1) + (n+2) + (n+3)$, and $q_{3,3}(n) = 1$.

From this procedure, the general form of the polynomial $q_{m,i}(n)$ is argued to be

$$
q_{m,i}(n) = \sum_{1 \le s_1 < s_2 < \cdots < s_{m-i} \le m} \prod_{t=1}^{m-i} (n + s_t), \quad i = 0, 1, \ldots, m, \tag{6}
$$

where s_t, $t = 1, 2, \ldots, m-i$, is an integer. Note the special cases $q_{m,m-1}(n) = \sum_{i=1}^{m}(n+i)$ and $q_{m,0}(n) = \prod_{i=1}^{m}(n+i)$. Furthermore,

$$
q_{m,i}(0) = \sum_{1 \le s_1 < s_2 < \cdots < s_{m-i} \le m} \prod_{t=1}^{m-i} s_t = \sigma_{m-i}(1, 2, \ldots, m), \tag{7}
$$

where $\sigma_{m-i}(1, 2, \ldots, m)$ is the $(m-i)$-th elementary symmetric polynomial evaluated on the first m integers $\{1, 2, \ldots, m\}$ [6, Chapter 6].

Lemma 2. *For $m \ge 1$, the polynomials $q_{m,i}(n)$ satisfy the recurrence relation*

$$
(n+m)q_{m-1,i}(n) = q_{m,i}(n) - q_{m-1,i-1}(n), \quad i = 0, 1, \ldots, m-1, \tag{8}
$$

where it is understood that $q_{m-1,i-1}(n) = 0$ for $i = 0$.

Proof. Relation (8) follows directly from the definition of $q_{m,i}(n)$. Hence, from (6), we obtain

$$
(n+m)q_{m-1,i}(n) = \sum_{1 \le s_1 < \cdots < s_{m-i-1} \le m-1} \prod_{t=1}^{m-i-1} (n + s_t)(n+m). \tag{9}
$$

On the other hand, we have

$$
q_{m-1,i-1}(n) = \sum_{1 \le s_1 < \cdots < s_{m-i} \le m-1} \prod_{t=1}^{m-i} (n + s_t). \tag{10}
$$

Clearly, the sum of the right-hand side of (9) and (10) is identical to (6). $\square$

Now we show by induction on m that $P_k^{(m)}(n)$ have the form (2) with $q_{m,i}(n)$ given by (6). This statement is readily verified for the base cases $m = 0, 1, 2$, and 3. Assuming the inductive hypothesis holds for $P_k^{(m-1)}(n)$ (with $m \geq 1$), Equation (5) yields

$$P_k^{(m)}(n) = \frac{1}{m!}\left[(n+m)\sum_{i=0}^{m-1}(-1)^i q_{m-1,i}(n)P_{k+i}^{(0)}(n) - \sum_{i=0}^{m-1}(-1)^i q_{m-1,i}(n)P_{k+i+1}^{(0)}(n)\right].$$

Using (8), it follows that

$$\begin{aligned}
P_k^{(m)}(n) &= \frac{1}{m!}\left[\sum_{i=0}^{m-1}(-1)^i q_{m,i}(n)P_{k+i}^{(0)}(n)\right.\\
&\quad\left. - \sum_{i=1}^{m-1}(-1)^i q_{m-1,i-1}(n)P_{k+i}^{(0)}(n) + \sum_{i=1}^{m}(-1)^i q_{m-1,i-1}(n)P_{k+i}^{(0)}(n)\right]\\
&= \frac{1}{m!}\left[\sum_{i=0}^{m-1}(-1)^i q_{m,i}(n)P_{k+i}^{(0)}(n) + (-1)^m q_{m-1,m-1}(n)P_{k+m}^{(0)}(n)\right]\\
&= \frac{1}{m!}\sum_{i=0}^{m}(-1)^i q_{m,i}(n)P_{k+i}^{(0)}(n),
\end{aligned}$$

where we used that $q_{m-1,m-1}(n) = q_{m,m}(n) = 1$ to justify the last equation. This completes the inductive step and the proof of the above statement. We formally state this result in the following theorem.

Theorem 3. *The hypersum $P_k^{(m)}(n)$ admits a representation of the form (2) with $q_{m,i}(n)$ given by (6).*

3 The coefficients of the hypersum polynomial

In this section, we provide an explicit expression for the coefficients $c_{k,m}^r$ in terms of the Bernoulli numbers and the Stirling numbers of the first kind. To this end, we first put $q_{m,i}(n)$ in polynomial form as $q_{m,i}(n) = \sum_{s=0}^{m-i} q_{m,i}^s n^s$. On the other hand, according to the well-known Bernoulli formula, $P_{k+i}^{(0)}(n)$ can be written as [7, Equation 9]

$$P_{k+i}^{(0)}(n) = \frac{1}{k+i+1}\sum_{t=1}^{k+i+1}\binom{k+i+1}{t}B_{k+i+1-t}n^t.$$

(Remember that we are taking $B_1 = \frac{1}{2}$ in the above formula.) Then, substituting the aforementioned expressions for $q_{m,i}(n)$ and $P_{k+i}^{(0)}(n)$ into (2) and comparing the resulting polynomial with (1), gives

$$c_{k,m}^r = \frac{1}{m!}\sum_{i=0}^{m}(-1)^i Q_{k,m,i}^r, \quad r = 1, 2, \ldots, k+m+1, \tag{11}$$

where

$$Q^r_{k,m,i} = \frac{1}{k+i+1} \sum_{h=0}^{r-1} q^h_{m,i} \binom{k+i+1}{r-h} B_{k+i+h+1-r}. \tag{12}$$

In particular, from (11) and (12), we quickly obtain

$$c^1_{k,m} = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i q^0_{m,i} B_{k+i}, \quad k, m \geq 0. \tag{13}$$

Now let us address the question of the nature of the coefficients $q^s_{m,i}$, $s = 0, 1, \ldots, m-i$, of $q_{m,i}(n)$. Let us first look at the constant term $q^0_{m,i}$. This is the value of $q_{m,i}(n)$ at $n = 0$. Hence, from (7), we have $q^0_{m,i} = \sigma_{m-i}(1, 2, \ldots, m)$. On the other hand, the Stirling numbers of the first kind $\left[{n \atop k} \right]$ enumerate all the permutations of size n with k cycles. It turns out that $\sigma_k(1, 2, \ldots, n) = \left[{n+1 \atop n+1-k} \right]$ [5, pp. 213–214], and then

$$\sigma_{m-i}(1, 2, \ldots, m) = \begin{bmatrix} m+1 \\ i+1 \end{bmatrix}. \tag{14}$$

Thus, we have $q^0_{m,i} = \left[{m+1 \atop i+1} \right]$. Putting this in (13), we obtain formula (3).

In order to systematically derive the coefficients $q^s_{m,i}$, it is useful to note that

$$\prod_{t=1}^{m-i} (n + s_t) = \sum_{s=0}^{m-i} \sigma_{m-i-s}(s_1, s_2, \ldots, s_{m-i}) n^s,$$

where $\sigma_{m-i-s}(s_1, s_2, \ldots, s_{m-i})$ is the $(m-i-s)$-th elementary symmetric polynomial on the variables $s_1, s_2, \ldots, s_{m-i}$. Substituting this expression into (6), we deduce that

$$q^s_{m,i} = \sum_{1 \leq s_1 < s_2 < \cdots < s_{m-i} \leq m} \sigma_{m-i-s}(s_1, s_2, \ldots, s_{m-i}). \tag{15}$$

Clearly, the right-hand side of (15) is a symmetric function on $\{s_1, s_2, \ldots, s_{m-i}\}$. This function is a sum of products of $m-i-s$ distinct integers chosen from $\{1, 2, \ldots, m\}$, with a total of $\binom{m}{i}$ times $\binom{m-i}{s}$ terms. On the other hand, the elementary symmetric polynomial $\sigma_{m-i-s}(1, 2, \ldots, m)$ is a sum of $\binom{m}{i+s}$ terms, each of which is a product of $m-i-s$ distinct integers chosen from $\{1, 2, \ldots, m\}$. Therefore, since $\binom{m}{i}\binom{m-i}{s} = \binom{i+s}{s}\binom{m}{i+s}$, we conclude that the right-hand side of (15) is necessarily $\binom{i+s}{s}$ times $\sigma_{m-i-s}(1, 2, \ldots, m)$. Hence, using (14), we find that

$$q^s_{m,i} = \binom{i+s}{s} \begin{bmatrix} m+1 \\ i+s+1 \end{bmatrix}, \quad s = 0, 1, \ldots, m-i. \tag{16}$$

Note, in particular, that $q^{m-i}_{m,i} = \binom{m}{i}$. From (16), we also deduce the symmetry property $q^s_{m,i} = q^i_{m,s}$. As a concrete example, Table 1 displays the coefficients of the polynomials $q_{8,i}(n)$, $i = 0, 1, \ldots, 8$, where we use $[n^s]$ to denote the coefficient of n^s. Note that the

	$[n^0]$	$[n^1]$	$[n^2]$	$[n^3]$	$[n^4]$	$[n^5]$	$[n^6]$	$[n^7]$	$[n^8]$
$q_{8,8}(n)$	1	–	–	–	–	–	–	–	–
$q_{8,7}(n)$	36	8	–	–	–	–	–	–	–
$q_{8,6}(n)$	546	252	28	–	–	–	–	–	–
$q_{8,5}(n)$	4536	3276	756	56	–	–	–	–	–
$q_{8,4}(n)$	22449	22680	8190	1260	70	–	–	–	–
$q_{8,3}(n)$	67284	89796	45360	10920	1260	56	–	–	–
$q_{8,2}(n)$	118124	201852	134694	45360	8190	756	28	–	–
$q_{8,1}(n)$	109584	236248	201852	89796	22680	3276	252	8	–
$q_{8,0}(n)$	40320	109584	118124	67284	22449	4536	546	36	1

Table 1: The coefficients of the polynomials $q_{8,i}(n)$, $i = 0, 1, \ldots, 8$.

symmetry property implies that the table of coefficients is symmetric about a 45° diagonal. For example, we have $q^4_{8,2} = q^2_{8,4} = 8190$.

Finally, combining the Equations (11), (12), and (16), we obtain

$$c^r_{k,m} = \frac{1}{m!}\sum_{i=0}^{m}\frac{(-1)^i}{k+i+1}\sum_{h=0}^{r-1}\binom{i+h}{h}\binom{k+i+1}{r-h}\begin{bmatrix} m+1 \\ i+h+1 \end{bmatrix} B_{k+i+h+1-r},$$

which constitutes the generalization of Inaba's formula (3) to arbitrary $r = 1, 2, \ldots, k+m+1$, with $k, m \geq 0$, and $B_1 = \frac{1}{2}$.

On the other hand, from (8), we immediately derive the following recurrence relation for the coefficients $q^s_{m,i}$:

$$q^s_{m,i} = mq^s_{m-1,i} + q^s_{m-1,i-1} + q^{s-1}_{m-1,i}. \tag{17}$$

For $s = 0$, relation (17) becomes $q^0_{m,i} = mq^0_{m-1,i} + q^0_{m-1,i-1}$. Therefore, comparing this relation with (4) and noting that $q^0_{0,0} = 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we retrieve the result $q^0_{m,i} = \begin{bmatrix} m+1 \\ i+1 \end{bmatrix}$. For $s = 1$ we have $q^1_{m,i} = mq^1_{m-1,i} + q^1_{m-1,i-1} + q^0_{m-1,i}$, which is satisfied when we set $q^1_{m,i} = (i+1)q^0_{m,i+1} = (i+1)\begin{bmatrix} m+1 \\ i+2 \end{bmatrix}$. In general, the solution of the recurrence (17) is given by

$$\begin{aligned} q^s_{m,i} &= \frac{1}{s}(i+1)q^{s-1}_{m,i+1} \\ &= \frac{1}{s}\frac{1}{s-1}(i+1)(i+2)q^{s-2}_{m,i+2} \\ &\ \ \vdots \\ &= \frac{1}{s!}(i+1)(i+2)\cdots(i+s)q^0_{m,i+s}, \end{aligned}$$

so that $q^s_{m,i} = \binom{i+s}{s}q^0_{m,i+s} = \binom{i+s}{s}\begin{bmatrix} m+1 \\ i+s+1 \end{bmatrix}$, in accordance with (16).

Thus, Table 1 is generated by the rule $q^s_{m,i} = \frac{1}{s}(i+1)q^{s-1}_{m,i+1}$, $s \geq 1$, which enables one to determine the element $q^s_{m,i}$ in row $m-i$ and column s from the preceding element $q^{s-1}_{m,i+1}$ in row $m-i-1$ and column $s-1$, the elements of the starting 0-th column being given by $q^0_{m,i} = \left[{m+1 \atop i+1}\right]$.

We conclude with three brief remarks.

Remark 4. For $k=0$ the hypersum $P^{(m)}_k(n)$ is equal to $P^{(m)}_0(n) = \binom{n+m}{m+1}$. Then, letting $k=0$ in (2), we will have $\sum_{i=0}^{m} \frac{(-1)^i q_{m,i}(n)}{m!} P^{(0)}_i(n) = \binom{n+m}{m+1}$. Solving for $P^{(0)}_m(n)$, we get

$$(-1)^m P^{(0)}_m(n) = m!\binom{n+m}{m+1} + \sum_{i=0}^{m-1} (-1)^{i+1} q_{m,i}(n) P^{(0)}_i(n), \quad m \geq 1,$$

which allows us to compute recursively $P^{(0)}_m(n)$ from the power sums $P^{(0)}_0(n)$, $P^{(0)}_1(n), \ldots,$ $P^{(0)}_{m-1}(n)$, and the polynomials $q_{m,i}(n)$, $i = 0, 1, \ldots, m-1$.

Remark 5. The leading coefficient of the hypersum polynomial (1) has been given by the author [3]: $c^{k+m+1}_{k,m} = \frac{k!}{(k+m+1)!}$. On the other hand, the leading coefficients of $q_{m,i}(n)$ and $P^{(0)}_{k+i}(n)$ are given by $q^{m-i}_{m,i} = \binom{m}{i}$ and $c^{k+i+1}_{k+i,0} = \frac{1}{k+i+1}$, respectively. Therefore, equating the terms of maximum degree on the two sides of (2) yields the combinatorial identity

$$\sum_{i=0}^{m} \frac{(-1)^i}{k+i+1} \binom{m}{i} = \frac{k!\, m!}{(k+m+1)!}, \quad k, m \geq 0.$$

Remark 6. From formula (3), we deduce an identity relating the harmonic number $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ to the Bernoulli numbers and the Stirling numbers of the first kind. Indeed, from $c^1_{0,m} = 1/(m+1)$ [2, 3], recalling that $\left[{m+1 \atop 2}\right] = m! H_m$, and from (3) we obtain

$$H_m = \frac{2m}{m+1} + \frac{2}{m!} \sum_{j=1}^{\lfloor m/2 \rfloor} \left[{m+1 \atop 2j+1}\right] B_{2j}.$$

4 Acknowledgement

The author would like to thank an anonymous referee for giving an extensive list of comments and suggestions that led to an improvement of an earlier version of this paper.

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2010 *Mathematics Subject Classification*: Primary 11B57; Secondary 11C08, 11Y55.
Keywords: hypersum polynomial, iterative procedure, elementary symmetric polynomial, Bernoulli number, Stirling number of the first kind.

(Concerned with sequence A008275.)

Received October 14 2013; revised versions received January 19 2014; March 17 2014. Published in *Journal of Integer Sequences*, March 23 2014.

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