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p-adic Properties of Lengyel's Numbers

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Abstract

Lengyel introduced a sequence of numbers Z_n, defined combinatorially, that satisfy a recurrence where the coefficients are Stirling numbers of the second kind. He proved some 2-adic properties of these numbers. In this paper, we give another recurrence for the sequence Z_n, where the coefficients are Stirling numbers of the first kind. Using this formula, we give another proof of Lengyel's lower bound on the 2-adic valuation of the Z_n. We also resolve some conjectures of Lengyel about the sequence Z_n.

We also define

(a) A new sequence Y_n analogous to Z_n, exchanging the role of Stirling numbers of the first and second kind. We study its 2-adic properties.

(b) Another sequence similar to Lengyel's sequence, and we study its p-adic properties for $p \geq 3$.

1 Introduction and notation

In the text $|.|$ denotes the p-adic absolute value on $\mathbb{Q}$, and v_p denotes the corresponding p-adic valuation. Let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$ and let $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. For all the p-adic background, we refer to Amice [1]; in particular for a polynomial $P(x) \in \mathbb{C}_p[x]$ and for $r \in \mathbb{R}_+$, we write $|P|(r) := \sup_{\substack{x\in\mathbb{C}_p \\ |x|\le r}} |P(x)|$.

For an integer n, let $S_p(n)$ be the sum of the digits of the p-adic expansion of n. Throughout this article, we use the following well-known formula (cf. Amice [1, p. 102]):

$$v_p(n!) = \frac{n - S_p(n)}{p-1}\,. \tag{1}$$

Lengyel [4] studied the 2-adic properties of the sequence defined by $Z_1 = 1$ and

$$Z_n = \sum_{k=1}^{n-1} S(n,k) Z_k \tag{2}$$

for $n \geq 2$, where the $S(n,k)$ are Stirling numbers of the second kind. (See Lengyel [4] for the combinatorial properties of the Z_n). This sequence appears in Sloane's database [5] as A005121. In particular Lengyel showed the following results:

Theorem 1. *The Z_n satisfy*

(a) $v_2(Z_{2^n+L}) \geq n$ for $n \geq 1$ and $L \geq 0$;

(b) $v_2(Z_k) \geq \lfloor \log_2(k) \rfloor - 1$ for $k \geq 1$.

Lengyel also proposed some conjectures about these numbers; first [4, Conjecture 2]:

Conjecture 2. For $n \geq 3$, let Z_n be given by (2). Then $v_2(Z_{2^n}) = n$.

He also proposed [4, Conjecture 4]:

Conjecture 3. For all $n \geq 2$ we have

$$\max\{k \,|\, v_2(Z_k) = n\} = 3 \cdot 2^{n-1}\,.$$

2 The results

In this paper, we reprove the results of Lengyel by another method. This method allows us to prove the above two conjectures. More precisely we get the following theorems:

Theorem 4. *The lower bound*

$$v_2(Z_n) \geq \log_2(n) - 1$$

holds for all $n \geq 1$.

Remark 5. The inequality in Theorem 4 implies the assertion (a) of Theorem 1 when $L \geq 1$, and the assertion (b) of Theorem 1.

Theorem 6. *If $t \geq 3$, then*

$$v_2(Z_{2^t}) = t\,.$$

Remark 7. This theorem proves Conjecture 2; in addition, it completes the assertion (a) of Theorem 1, by showing that the case $L = 0$ of the assertion (a) holds.

Theorem 8. *For all $t \geq 2$, we have*

$$\max\{k \mid v_2(Z_k) = t\} = 3 \cdot 2^{t-1}$$

which proves Conjecture 3.

We also introduce a new sequence Y_n by permuting the Stirling numbers of the first and second kind. More precisely, let Y_n be the sequence of integers defined by $Y_1 = 1$ and

$$Y_n = \sum_{m=1}^{n-1} s(n,m) Y_m\,.$$

for $n \geq 2$. Let $s(n,k)$ be, as usual, $s(n,k)$ the Stirling numbers of the first kind. For properties of the Stirling numbers, we refer to Comtet [2, Chapter 5.5, pp. 212–219].

We prove the following theorem for the sequence Y_n:

Theorem 9. *Let Y_n be as above. Then*

(a) All the Y_n are odd;

(b) For $n \geq 1$, we have $v_2(Y_n + 1) = 1$ if and only if n is congruent to 0 or 1 modulo 3.

We also get some results for a sequence similar to Lengyel's sequence for primes $p \geq 3$. Define the sequence $Z_n^{\langle p\rangle}$ by $Z_0^{\langle p\rangle} = 0$, $Z_1^{\langle p\rangle} = 1$, and

$$(p-1) Z_n^{\langle p\rangle} = \sum_{k=1}^{n-1} S(n,k) Z_k^{\langle p\rangle}$$

or

$$p Z_n^{\langle p\rangle} = \sum_{k=1}^{n} S(n,k) Z_k^{\langle p\rangle}$$

for $n \geq 2$. (Note that $Z_n^{\langle p\rangle}$ is not an integer, but only a p-integer).

We prove the following result below:

Theorem 10. *The p-adic valuation of the $Z_n^{\langle p\rangle}$ satisfies*

(a) $v_p(Z_n^{\langle p\rangle}) \geq \log_p(n) - 1$ for $n \geq 1$;

(b) $v_p(Z_n^{\langle p\rangle}) = t - 1$ for $n = p^t$;

(c) $v_p(Z_n^{\langle p\rangle}) \geq t$ for $n \geq p^t + 1$.

Remark 11. Note that property (c) is an immediate consequence of property (a). Hence we have only to show that (a) and (b) hold.

3 Use of the Stirling numbers of the first kind

We now show that the Z_n satisfy another recurrence relation:

Proposition 12. *The Z_n satisfy*

$$Z_n = s(n,1) - 2\sum_{m=1}^{n-1} s(n,m) Z_m, \quad n \geq 2\,; \tag{3}$$

$$Z_n = -s(n,1) - 2\sum_{m=2}^{n-1} s(n,m) Z_m, \quad n \geq 3\,. \tag{4}$$

Proof. According to Comtet [2, Formula (6f), p. 144], the Stirling numbers of the first and second kind satisfy the formulae

$$\sum_{k=0}^{n} S(n,k)s(k,m) = \delta_{m,n}\,, \text{ and } \sum_{k=0}^{n} s(n,k)S(k,m) = \delta_{m,n}\,,$$

where $\delta_{m,n}$ is the usual Kronecker symbol. These relations are equivalent to the following Stirling inversion formula:

$$\forall\, n \in \mathbb{N} \left(f_n = \sum_{m=1}^{n} s(n,m) g_m \Longleftrightarrow g_n = \sum_{m=1}^{n} S(n,m) f_m \right). \tag{5}$$

Applying this inversion formula to the recurrence relation for the Z_n, we obtain

$$Z_n = \sum_{k=1}^{n-1} S(n,k) Z_k \Leftrightarrow 2Z_n = \sum_{m=1}^{n} S(n,m) Z_m + \sum_{m=1}^{n} S(n,m) s(m,1)\,,$$

which can also be written

$$Z_n = \sum_{m=1}^{n} S(n,m) \frac{Z_m + s(n,1)}{2}\,.$$

Next, by Stirling inversion we get

$$\frac{Z_n + s(n,1)}{2} = \sum_{m=1}^{n} s(n,m) Z_m \Leftrightarrow \frac{-Z_n + s(n,1)}{2} = \sum_{m=1}^{n-1} s(n,m) Z_m\,,$$

which is the desired result. □

Remark 13. Actually using exactly the same method, we get the following relation for the sequence $Z_n^{\langle p \rangle}$ and $n \geq 2$:

$$Z_n^{\langle p \rangle} = s(n,1) - \frac{p}{p-1} \sum_{m=1}^{n-1} s(n,m) Z_m\,. \tag{6}$$

Since $s(n,1) = (-1)^{n-1}(n-1)!$ is a p-adic unit for $n = 1, \ldots, p$, we get as an immediate consequence that $Z_n^{\langle p \rangle}$ is a p-adic unit for the same values of n.

4 Some p-adic properties of Stirling numbers of the first kind

In this section, we prove some lemmas for the Stirling numbers of the first kind.

Lemma 14. *For $n \geq m \geq 1$, we get the lower bound*

$$v_p(s(n,m)) \geq \left\lfloor \frac{n-1}{p} \right\rfloor + 1 - m\,.$$

Proof. Note that the inequality is trivial if $\left\lfloor \frac{n-1}{p} \right\rfloor + 1 - m \leq 0$. The Stirling numbers of the first kind (cf. [2, Formula (5e), p. 213]) are defined as follows:

$$P_n(x) = (x)_n := \sum_{m=1}^{n} s(n,m)x^m\,.$$

Take $x \in \mathbb{C}_p$ with $|x| = r$ with $\frac{1}{p} < r < 1$. Then $|x-k| = 1$ if k is not divisible by p and $|x-k| = r$ otherwise. This shows that $|P_n|(r) = \max\{|s(n,m)|r^m\}$ is equal to r^M, where $M = \left\lfloor \frac{n-1}{p} \right\rfloor + 1$ is the number of k, $0 \leq k \leq n-1$, that are divisible by p. Hence $|s(n,m)|r^m \leq r^M$, and $|s(n,m)| \leq r^{M-m}$.

The continuity of the map $r \mapsto |P_n|(r)$, [1, Cor. 4.2.7, p. 122], allows us to choose $r = \frac{1}{p}$ in the previous inequality, which gives the desired lower bound. □

Lemma 15. *If $t \geq 2$, and $n = p^t$, it follows that*

$$P_n(x) \equiv x^{p^{t-1}}(x^{p-1}-1)^{p^{t-1}} \pmod{p}\,.$$

Therefore $s(n,m)$ is divisible by p if $m \leq p^{t-1}-1$, and $s(p^t,p^{t-1})$ is not divisible by p.

Proof. By hypothesis, we have $n \geq p^2$. Let $r \in \{0, \cdots, p-1\}$. The number of integers k such that $0 \leq k \leq n-1$ and $k \equiv r \pmod{p}$ is $\left\lfloor \frac{n-1-r}{p} \right\rfloor + 1 = p^{t-1}$. Hence it is independent of r. It follows that

$$P_n(x) \equiv \prod_{r=0}^{p-1}(x-r)^{p^{t-1}} \mod p\,.$$

But $\prod_{r=0}^{p-1}(x-r) \equiv x^p - x = x(x^{p-1}-1) \pmod{p}$. The proof is now complete (the last two assertions are clear). □

Now, we examine the value of $s(n,k)$ modulo 4, for specific values of n.

Lemma 16. *Let $t \geq 5$, and $n = 2^t$. Then*

$$\sum_{k=0}^{n} s(n,k)x^k = P_n(x) = x^{2^{t-1}}(x^2-1)^{2^{t-2}} \pmod 4 .$$

Proof. We know that $P_n(x) = x(x-1)\cdots(x-n+1)$. Let r be such that $0 \leq r \leq 3$. There are $M_r = \left\lfloor \dfrac{n-r-1}{4} \right\rfloor + 1$ values k, $0 \leq k \leq n-1$ such that $k \equiv r \pmod 4$. Since $n = 2^t$, we immediately see that M_r is in fact independent of r. Hence M_r is equal to 2^{t-2}. Thus, modulo 4, the polynomial P_n is congruent to $\prod_{r=0}^{3}(x-r)^{2^{t-2}}$.

We first consider the factor $(x-2)^{2^{t-2}}$. Since $(x-2)^2 \equiv x^2 \pmod 4$, and $t \geq 5$, or equivalently $t-2 \geq 3$, we see that

$$(x-2)^{2^{t-2}} \equiv x^{2^{t-2}} \pmod 4 .$$

Since 3 is congruent to -1 modulo 4, we have

$$P_n(x) \equiv x^{2^{t-1}}(x^2-1)^{2^{t-2}} \pmod 4 .$$

The proof of the lemma is complete. □

Lemma 17. *Let $n = 2^t$ with $t \geq 5$. Then, for all m, $1 \leq m \leq 2^t - 1$, $m \neq 3 \cdot 2^{t-2}$ and $m \neq 2^{t-1}$, it follows that*

$$s(2^t, m) \equiv 0 \pmod 4 .$$

In addition, $s(2^t, 2^{t-1} + 2^{t-2}) \equiv 0 \pmod 2$, and $s(2^t, 2^{t-1}) \equiv 1 \pmod 4$.

Proof. From Formula (1) it follows that

$$v_2\left(\binom{2^{t-2}}{j}\right) = S_2(j) + S_2(2^{t-2} - j) - 1 .$$

If $j \neq 0$ and not equal to 2^{t-2}, then $S_2(j) + S_2(2^{t-2} - j) \geq 2$. If $S_2(j) + S_2(2^{t-2} - j) = 2$ then $j = 2^{t-3}$, because it implies that both j and $2^{t-2} - j$ are powers of 2 and hence equal (their sum is 2^{t-2}). Using the modulo 4 formula given above for the polynomial $P_n(x)$, and the binomial theorem for the factor $(x^2-1)^{2^{t-2}}$, we obtain

$$P_n(x) \equiv x^{2^t} - \binom{2^{t-2}}{2^{t-3}} x^{2^{t-1}+2^{t-2}} + x^{2^{t-1}} \pmod 4 .$$

Since the 2-adic valuation of $\binom{2^{t-2}}{2^{t-3}}$ is equal to 1, the lemma is proved. □

Lemma 18. *Let $t \geq 5$, and $n = 3 \cdot 2^t$. The following congruence holds:*

$$P_n(x) \equiv \sum_{k=0}^{n} s(n,k)x^k = x^{3\cdot 2^{t-1}}(x^2-1)^{3\cdot 2^{t-2}} \pmod 4 .$$

Proof. Let $r \in \{0, \cdots, 3\}$. The number of k such that $0 \leq k \leq n-1$ and $k \equiv r \pmod 4$ is $\left\lfloor \dfrac{n-1-r}{4} \right\rfloor + 1$ which is $3 \cdot 2^{t-2}$ for all r. Hence

$$P_n(x) \equiv \prod_{r=0}^{3}(x-r)^{3\cdot 2^{t-2}} \equiv (\prod_{r=0}^{3}(x-r)^{2^{t-2}})^3 \pmod 4 .$$

From the proof of Lemma 16 it follows that

$$\prod_{r=0}^{3}(x-r)^{2^{t-2}} \equiv x^{2^{t-1}}(x^2-1)^{2^{t-2}} \pmod 4 .$$

The proof is now complete. □

Lemma 19. *Let $t \geq 5$, $n = 3 \cdot 2^t$, and $1 \leq m \leq n$. Then*

(a) $s(n,m)$ is divisible by 4 for $m \notin \{2^{t+1}+2^{t-1}, 2^{t+1}, 3\cdot 2^{t-1}, 3\cdot 2^{t-1}+2^{t-2}, 3\cdot 2^t - 2^{t-2}, 3\cdot 2^t\}$;

(b) $s(n,m)$ is even, but not divisible by 4, for $m = 3 \cdot 2^t - 2^{t-2}$ and $m = 3 \cdot 2^{t-1} + 2^{t-2}$;

(c) $s(n,m)$ is odd for $m = 2^{t+1} + 2^{t-1}$, $m = 2^{t+1}$, $m = 3 \cdot 2^{t-1}$ and $m = 3 \cdot 2^t$.

In particular, $s(n, 3 \cdot 2^{t-1})$ is odd, and $s(n, 3 \cdot 2^t - 1)$, $s(n, 3 \cdot 2^t - 2)$, and all the $s(n,j)$ for $j \leq 3 \cdot 2^{t-1} - 1$ are divisible by 4.

Proof. We know that $P_n(x) \equiv x^{3\cdot 2^{t-1}}(x^2-1)^{3\cdot 2^{t-2}} \pmod 4$ by Lemma 18. By the binomial theorem we get

$$(x^2-1)^{3\cdot 2^{t-2}} = \sum_{j=0}^{3\cdot 2^{t-2}} (-1)^j \binom{3\cdot 2^{t-2}}{j} x^{2(3\cdot 2^{t-2}-j)} .$$

From Formula (1) we know that the 2-adic valuation of $\binom{q}{j}$ is $S_2(j) + S_2(q-j) - S_2(q)$. We have $3 \cdot 2^{t-2} = 2^{t-1} + 2^{t-2}$, and hence $S_2(3 \cdot 2^{t-2}) = 2$. To prove the lemma it suffices

(1) To find all the integers j such that

$$0 \leq j \leq 3 \cdot 2^{t-2}, \text{ and } T := S_2(j) + S_2(3\cdot 2^{t-2} - j) - 2 = \begin{cases} 0; \\ 1. \end{cases} \tag{7}$$

(2) To show that they are precisely the integers given in (b) and (c).

First suppose that $T = 0$. We have the trivial solutions of equation (7) $j = 0$ and $j = 3 \cdot 2^{t-2}$. Now we can assume $1 \le j \le 3 \cdot 2^{t-2} - 1$. Hence $S_2(j) = S_2(3 \cdot 2^{t-2} - j) = 1$ (because $S_2(n) \ge 1$), It follows that j and $3 \cdot 2^{t-2} - j$ are powers of 2, say 2^α and 2^β. Equality $3 \cdot 2^{t-2} = 2^\alpha + 2^\beta$ immediately shows that $\alpha = t-2$ and $\beta = t-1$, or vice-versa. We have found 4 values of j:

$$j = 0,\ j = 2^{t-2},\ j = 2^{t-1} \text{ and } j = 3 \cdot 2^{t-2}\,.$$

After multiplying by $x^{3 \cdot 2^{t-1}}$, we see that they are the values given in assertion (c).

Now suppose that $T = 1$. From the first case we know that $1 \le j \le 3 \cdot 2^{t-2} - 1$. From equation (7) it follows that $S(j) = 1$ or $S(3 \cdot 2^{t-2} - j) = 2$ or vice-versa. This leads to study a relation of the form $2^\alpha + (2^\beta + 2^\gamma) = 3 \cdot 2^{t-2}$, with $\beta < \gamma$. We leave to the reader to show that it implies $\alpha = \beta = t-3$ and $\gamma = t-1$. Hence the two solutions $j = 2^{t-3}$ and $j = 3 \cdot 2^{t-2} - 2^{t-3}$. After multiplying by $x^{3 \cdot 2^{t-1}}$ we see that these values are exactly the values in assertion (b).

For all other values of $1 \le m \le 3 \cdot 2^t$ $s(n,m) \equiv 0 \pmod 4$. The lemma is proved. □

5 A lemma about sums of binomial coefficients

We also need a lemma about sums of binomials.

Lemma 20. *Let j be a primitive third root of unity, $r \in \{0,1,2\}$, and $n = 3k + r$.*

For $s \in \{0,1,2\}$ define

$$\Gamma_{s,r,k} = \sum_{\substack{0 \le l \le n; \\ l \equiv s \pmod 3}} \binom{n}{l}\,.$$

It follows that

$$3\Gamma_{s,r,k} = 2^n + (-1)^n (j^{s+r} + j^{2(s+r)})\,.$$

As a consequence, we get

- *If $r + s$ is not divisible by 3, then*

$$3\Gamma_{s,r,k} = 2^n + (-1)^{n+1}\,.$$

- *If $r + s$ is divisible by 3, then*

$$3\Gamma_{s,r,k} = 2^n + 2(-1)^n\,.$$

Proof. Write

$$(1+x)^n = \sum_{l=0}^{n} \binom{n}{l} x^l\,,$$

and choose successively $x = 1$, $x = j$, and $x = j^2$. It follows that

$$\Gamma_{0,r,k} + \Gamma_{1,r,k} + \Gamma_{2,r,k} = 2^n \,, \tag{L_1}$$
$$\Gamma_{0,r,k} + j\Gamma_{1,r,k} + j^2\Gamma_{2,r,k} = (1+j)^n \,, \tag{L_2}$$
$$\Gamma_{0,r,k} + j^2\Gamma_{1,r,k} + j\Gamma_{2,r,k} = (1+j^2)^n \,. \tag{L_3}$$

Since $1+j+j^2 = 0$, we get, when computing the quantities $L_1+L_2+L_3$, $L_1+j^2L_2+jL_3$, and $L_1+jL_2+j^2L_3$,

$$\begin{aligned}
3\Gamma_{0,r,k} &= 2^n + (1+j)^n + (1+j^2)^n = 2^n + (-1)^n(j^r + j^{2r}) \,, \\
3\Gamma_{1,r,k} &= 2^n + j^2(1+j)^n + j(1+j^2)^n = 2^n + (-1)^n(j^{1+r} + j^{2r+2}) \,, \\
3\Gamma_{2,r,k} &= 2^n + j(1+j)^n + j^2(1+j^2)^n = 2^n + (-1)^n(j^{r+2} + j^{2r+1}) \,.
\end{aligned}$$

The general formula follows. $\square$

Remark 21. Analogous results exist in, e.g., [3].

6 Proof of Theorem 4

We show the result by induction on n. For $n \le 7$, we check that the lower bound holds using the explicit numerical values of the Z_n, cf. [4, p. 179]. Hence our induction hypothesis will be

$$\big(n \ge 7 \Rightarrow v_2(Z_m) \ge \log_2(m) - 1,\ 1 \le m \le n-1\big)\,. \tag{H(n)}$$

We use Formula (4), valid for $n \ge 3$:

$$Z_n = -s(n,1) - 2\sum_{m=2}^{n-1} s(n,m) Z_m \,. \tag{4}$$

We will show that all the terms in the last expression have 2-adic absolute value $\le 2^{-\log_2(n)+1}$; this will imply that this is also the case for Z_n, and the result will be proved. We consider many different cases.

6.1 Let m be such that $2 \le m \le \left\lfloor \dfrac{n-1}{2} \right\rfloor + 1$. Using Lemma 14 and the induction hypothesis, we get

$$\big|s(n,m)Z_m\big| \le 2^{-\log_2(m)+1-\left\lfloor \frac{n-1}{2}\right\rfloor -1+m} \,.$$

We want to show that this last term is bounded by $2^{-\log_2(n)+2}$ (we have taken into account the factor 2 in Formula (4)). This is equivalent to showing that

$$m - \log_2(m) \le \left\lfloor \frac{n-1}{2} \right\rfloor - \log_2(n) + 2 \,.$$

There are two cases, n even or odd.

First case: n even. Let $n = 2h$. As $n \geq 7$, we have $h \geq 4$. We also have $\left\lfloor \frac{n-1}{2} \right\rfloor = h - 1$; hence $m \leq h$. Since $\log_2(2h) = 1 + \log_2(h)$, we have to show that

$$m - \log_2(m) \leq h - \log_2(h)\,,$$

which holds because the function $x - \log_2(x)$ is increasing for $x \geq 2$.

Second case: n odd. Let $n = 2h + 1$ (hence $h \geq 3$). We get $\left\lfloor \frac{n-1}{2} \right\rfloor = h$; hence $2 \leq m \leq h + 1$. First, suppose that $m = h + 1$. We have to show

$$h + 1 - \log_2(h + 1) \leq h - \log_2(2h + 1) + 2$$

or $\log_2(2h + 1) \leq \log_2(h + 1) + 1$ which is equivalent to showing $2h + 1 \leq 2h + 2$ and hence is true.

Now we can suppose that $2 \leq m \leq h$. Since $h \geq 3$, we have $1 + \frac{1}{2h} \leq \frac{7}{6}$, and hence

$$1 - \log_2(1 + \frac{1}{2h}) \geq 1 - \log_2(\frac{7}{6}) = \frac{\log(12) - \log(7)}{\log 2} > 0\,.$$

It suffices to show that

$$m - \log_2(m) \leq h - \log_2(h)\,,$$

which holds because the function $x - \log_2(x)$ is increasing for $x \geq 2$, and $2 \leq m \leq h$.

6.2 Now suppose that m verify $\left\lfloor \frac{n-1}{2} \right\rfloor + 2 \leq m \leq n - 1$. We have only at our disposal the induction hypothesis.

We have to show, taking into account the factor 2, that $2^{-\log_2(m)+1} \leq 2^{-\log_2(n)+2}$, or $\log_2(n) \leq \log_2(m) + 1$. This is equivalent to showing $n \leq 2m$. There are two cases:

(a) If n is even, $n = 2h$, we have $m \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 2 = h + 1$; hence $2m \geq 2h + 2 = n + 2$.

(b) If n is odd, $n = 2h + 1$, we have $m \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 2 = h + 2$; hence $2m \geq 2h + 4 = n + 3$.

6.3 The only remaining term, $s(n, 1)$, is equal to $(-1)^{n-1}(n - 1)!$, cf. [2, Formula (6b), p. 214]. We want to show that, for all $n \geq 7$,

$$|(n - 1)!| = 2^{-n+1+S_2(n-1)} \leq 2^{-\log_2(n)+1}\,.$$

This is equivalent to showing that

$$S_2(n - 1) + \log_2(n) \leq n\,.$$

For $m \geq 1$, we have $S_2(m) \leq 1 + \log_2(m)$, it suffices then to show that $1 + \log_2(n - 1) + \log_2(n) \leq n$. This is equivalent to showing $n(n - 1) \leq 2^{n-1}$. We prove this inequality by noticing that it holds for $n = 6$ and using an easy induction argument.

Now the proof of Theorem 4 is complete.

7 Proof of Theorem 6

We also use induction. Looking at the numerical values of Z_n, we check the equality for $t = 3$ and $t = 4$. So suppose that equality $v_2(Z_{2^h}) = h$ holds for $3 \le h \le t-1$, and prove that $v_2(Z_{2^t}) = t$. We can suppose $t \ge 5$.

Again using Formula (4) with $n = 2^t$, we get

$$Z_{2^t} = -s(2^t, 1) - 2 \sum_{m=2}^{2^t-1} s(2^t, m) Z_m \,.$$

We will prove that all terms in Eq. (4), with the exception of the term corresponding to $m = 2^{t-1}$, (note that $t-1 \ge 4$; hence $m \ne 4$) have 2-adic valuation $\ge t+1$.

7.1 First, consider the valuation of the term $s(2^t, 1) = (-1)^{2^t-1}(2^t-1)!$. From Formula (1) it follows that

$$v_2\bigl(s(2^t, 1)\bigr) = 2^t - 1 - S_2(2^t - 1) = 2^t - 1 - t \,.$$

We want to prove that $2^t - 1 - t \ge t + 1$ or, replacing t by $t-1$, $2^{t-1} \ge 1 + t$; this last inequality holds for $t = 3$, and also for $t \ge 3$ by an easy induction.

7.2 Now take the term $2s(2^t, m)Z_m$ with $2 \le m \le 2^{t-1} - 3$. Its valuation satisfies

$$v_2\bigl(2s(2^t, m)Z_m\bigr) \ge 1 + \left\lfloor \frac{2^t - 1}{2} \right\rfloor + 1 - m + \log_2(m) - 1 = 2^{t-1} - m + \log_2(m) \,.$$

As we want to prove that $v_2(2s(2^t, m)Z_m) \ge t + 1$, it suffices to prove that

$$m - \log_2(m) \le 2^{t-1} - 1 - t \,.$$

Since the function $x - \log_2(x)$ is increasing for $x \ge 2$, we only have to prove the inequality for $m = 2^{t-1} - 3$. This is equivalent to showing that $\log_2(2^{t-1} - 3) \ge t - 2$, or $2^{t-2} \ge 3$. This last inequality holds by the hypothesis on t.

7.3 Now take a term $2s(2^t, m)Z_m$ with $m = 2^{t-1} - 1$ or $2^{t-1} - 2$. We have to prove

$$v_2\bigl(2s(2^t, m)Z_m\bigr) = 1 + v_2\bigl(s(2^t, m)\bigr) + v_2(Z_m) \ge t + 1 \,.$$

It follows from Lemma 17 that $s(2^t, 2^{t-1} - 1)$ and $s(2^t, 2^{t-1} - 2)$ are divisible by 4, and, since, for these two values, we have $m = 2^{t-2} + L$ with $L \ge 1$, we know that $v_2(Z_m) \ge t - 2$. Hence

$$v_2\bigl(2s(2^t, m)Z_m\bigr) \ge 1 + 2 + t - 2 = t + 1 \,.$$

7.4 Now examine $2s(2^t, m)Z_m$ with $2^{t-1} + 1 \le m \le 2^t - 1$ and $m \ne 2^{t-1} + 2^{t-2} = 3 \cdot 2^{t-2}$. By Lemma 17, $s(2^t, m)$ is divisible by 4 for these values of m. Hence

$$v_2\bigl(2s(2^t, m)Z_m\bigr) \ge 3 + v_2(Z_m) \ge 2 + \log_2(m)\,.$$

It suffices to prove that $\log_2(m) \ge t - 1$, or $m \ge 2^{t-1}$, which is true.

7.5 Now consider the term $2s(2^t, m)Z_m$ with $m = 2^{t-1} + 2^{t-2}$. By Lemma 17, we know that $s(2^t, m)$ is divisible by 2 and that

$$v_2(Z_m) \ge \log_2(m) - 1 = \log_2(3 \cdot 2^{t-2}) - 1 = \log_2(3) + t - 2 - 1 > t - 2\,,$$

hence $v_2(Z_m) \ge t - 1$. Therefore

$$v_2\bigl(2s(2^t, m)Z_m\bigr) \ge 1 + 1 + t - 1 = t + 1\,.$$

7.6 We are only left with the term $2s(2^t, m)Z_m$ with $m = 2^{t-1}$. By the induction hypothesis, we have $v_2(Z_m) = t - 1$, and we know that $s(2^t, 2^{t-1})$ is odd by Lemma 17. Hence the 2-adic valuation of this term is exactly t.

We have proved that, in the expression (4) of Z_{2^t}, only one term has valuation t , all the others having valuation $\ge t + 1$; hence $v_2\bigl(Z_{2^t}\bigr) = t$. The proof is complete.

8 Proof of Theorem 8

First we show that the conjecture holds for $3 \cdot 2^h$, with $2 \le h \le 4$, by numerical computations; then we proceed by induction. Suppose the result true for $2 \le h \le t - 1$; let us prove it for t. By the above, we can suppose that $t \ge 5$. We have to prove two facts:

(a) For $m = 3 \cdot 2^t$, $v_2(Z_m) = t + 1$;

(b) For $m' > 3 \cdot 2^t$, $v_2(Z_{m'}) \ge t + 2$.

First, let us prove assertion (a). Of course we use Formula (4):

$$Z_n = -s(n, 1) - 2\sum_{m=2}^{n-1} s(n, m)Z_m\,.$$

Take $n = 3 \cdot 2^t$. We will prove that all the terms in this expression are of 2-adic valuation $\ge t + 2$, with the exception of only one term.

8.1 First, look at $s(3 \cdot 2^t, 1)$. Its 2-adic valuation is $3 \cdot 2^t - 1 - S_2(3 \cdot 2^t - 1)$. Since $3 \cdot 2^t - 1 = 2^{t+1} + 2^{t-1} + \cdots + 1$ we get $S_2(3 \cdot 2^t - 1) = t + 1$. We have to show that $3 \cdot 2^t - 1 - t - 1 \ge t + 2$. This inequality holds for $t = 1$. The general case follows by an easy induction.

8.2 Now consider $2s(3\cdot 2^t, m)Z_m$, with $2 \le m \le 3\cdot 2^{t-1}-2$. Using Lemma 14 and Theorem 4, we easily see that its 2-adic valuation is $\ge 3\cdot 2^{t-1}-m+\log_2(m)$. Hence we have only to prove that $3\cdot 2^{t-1}-t-2 \ge m-\log_2(m)$. The function $x-\log_2(x)$ being increasing it suffices to prove this last inequality for $m = 3\cdot 2^{t-1}-2$. This is equivalent to showing that $\log_2\big(3\cdot 2^{t-1}-2\big) \ge t$, which is obvious since $3\cdot 2^{t-1}-2 = 2^t+(2^{t-1}-2) \ge 2^t$.

8.3 Now take $m = 3\cdot 2^{t-1}-1 = 2^t+(2^{t-1}-1)$. By Theorem 4, $v_2(Z_m) > t-1$, therefore $v_2(Z_m) \ge t$. By Lemma 19, $v_2(s(3\cdot 2^t, m)) \ge 2$, and we are done.

8.4 Now suppose that $3\cdot 2^{t-1}+1 \le m \le 3\cdot 2^t-1$, and $m \notin \{2^{t+1}, 2^{t+1}+2^{t-1}, 3\cdot 2^{t-1}, 3\cdot 2^t-2^{t-2}, 3\cdot 2^{t-1}+2^{t-2}\}$. By Lemma 19 it follows that $v_2(s(3\cdot 2^t, m)) \ge 2$. On the other hand, $m = 3\cdot 2^{t-1}+L = 2^t+L'$ with $L' > 0$. Then $v_2(Z_m) \ge t$ and this ends the case.

8.5 Now examine the two cases $m_1 = 3\cdot 2^t-2^{t-2}$ and $m_2 = 3\cdot 2^{t-1}+2^{t-2}$. By Lemma 19, we know that $v_2(s(3\cdot 2^t, m)) = 1$ for these values of m. Since $m_1 = 2^{t+1}+(2^t-2^{t-2})$, $v_2(Z_{m_1}) \ge t+1$ by Theorem 4, and $v_2(Z_{m_2}) \ge t+1$ by the induction hypothesis, because $m_2 > 3\cdot 2^{t-1}$, and this ends the case.

8.6 Now consider the two cases $m_3 = 2^{t+1}$ and $m_4 = 2^{t+1}+2^{t-1}$. By Lemma 19 we know that $s(3\cdot 2^t, m_j)$ is odd for $j = 3, 4$. We have already proved that $v_2(Z_{m_3}) = t+1$; hence $v_2(2s(n, m_3)Z_{m_3}) = t+2$. It follows that $v_2(Z_{m_4}) \ge t+1$, because $m_4 = 2^{t+1}+L$ with $L \ge 1$, and we get $v_2\big(2s(3\cdot 2^t, m_4)Z_{m_4}\big) \ge t+2$.

8.7 Only one value of m remains, namely, $m = 3\cdot 2^{t-1}$. We know that $s\big(3\cdot 2^t, 3\cdot 2^{t-1}\big)$ is odd, and by the induction hypothesis we know that that $v_2(Z_m) = t$. The 2-adic valuation of this term is $t+1$ and this is the only term with this property, all other terms having 2-adic valuation $\ge t+2$.

Hence $v_2\big(Z_{3\cdot 2^t}\big) = t+1$, and this ends the first part of the proof of Theorem 8.

Now let us prove the second part of Theorem 8.

Let n be such that $n \ge 3\cdot 2^t+1$. We want to prove that $v_2(Z_n) \ge t+2$. We use the recurrence formula (4) again:

$$Z_n = -s(n,1) - 2\sum_{m=2}^{n-1} s(n,m)Z_m\,.$$

It is enough to show that all the terms in this formula have 2-adic valuation $\ge t+2$.

8.8 First, consider the term $s(n,1)$. Its 2-adic valuation is $n-1-S_2(n-1)$. Since $S_2(n-1) \le 1+\log_2(n-1)$, we have to prove that $(n-1)-\log_2(n-1) \ge t+3$. It suffices to prove that $3\cdot 2^t-\log_2(3\cdot 2^t) \ge t+3$, and since $\log_2(3) < 2$, that $3\cdot 2^t \ge 2t+5$, or $3\cdot 2^{t-1} \ge t+3$. This last inequality holds for $t = 2$, and hence for $t \ge 2$ by an easy induction.

8.9 Next, consider the case $2 \leq m \leq 3 \cdot 2^{t-1} - 2$. Apply Lemma 14 and Theorem 4. Using the following inequality, valid for $n \geq 3 \cdot 2^t + 1$, $\left\lfloor \frac{n-1}{2} \right\rfloor \geq 3 \cdot 2^{t-1}$, we have to prove

$$3 \cdot 2^{t-1} + 1 - m + \log_2(m) \geq t + 2\,,$$

or

$$3 \cdot 2^{t-1} - t - 1 \geq m - \log_2(m)\,.$$

It suffices to prove this inequality for $m = 3 \cdot 2^{t-1} - 2$. It is equivalent to showing that $\log(3 \cdot 2^{t-1} - 2) \geq t - 1$, or $3 \cdot 2^{t-1} - 2 \geq 2^{t-1}$, which is clear.

8.10 Now consider the case $m = 3 \cdot 2^{t-1} - 1$. We have $v_2(s(n,m)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 1 - m \geq 3 \cdot 2^{t-1} + 1 - m$. Hence it suffices to prove

$$1 + 3 \cdot 2^{t-1} + 1 - 3 \cdot 2^{t-1} + 1 + v_2(Z_m) \geq t + 2\,,$$

i.e., $3 + v_2(Z_m) \geq t + 2$. Since $m = 2^t + (2^{t-1} - 1)$ it follows that $v_2(Z_m) \geq t$. The assertion is proved.

8.11 Consider the case $m = 3 \cdot 2^{t-1}$. By the induction hypothesis $v_2(Z_m) = t$ hence $v_2(s(n,m)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 1 - m \geq 3 \cdot 2^{t-1} + 1 - m = 1$. It follows that $v_2(2s(n,m)Z_m) \geq 1 + 1 + t = t + 2$.

8.12 Consider the last case, where $3 \cdot 2^{t-1} + 1 \leq m < n$. By the induction hypothesis we have $v_2(Z_m) \geq t + 1$. Hence $v_2(2s(n,m)Z_m) \geq t + 2$.

We have proved that all the terms in the formula giving Z_n have 2-adic valuation $\geq t + 2$ therefore $v_2(Z_n) \geq t + 2$ for $n \geq 3 \cdot 2^t + 1$. The proof of Theorem 8 is now complete.

9 Proof of Theorem 9

(a) With exactly the same proof as in Proposition 12, we prove another recurrence formula for the Y_n:

$$Y_n = -S(n,1) + 2\sum_{m=1}^{n} S(n,m) Y_m\,.$$

Since $S(n,1) = 1$, this gives

$$Y_n = 1 - 2\sum_{m=1}^{n-1} S(n,m) Y_m\,.$$

Hence all the Y_n are odd.

(b) We can suppose that $n \geq 2$. By the Stirling inversion formula (5), it follows that

$$2Y_n = \sum_{m=1}^{n} s(n,m) Y_m \,.$$

Taking $x = 1$ in the formula

$$x(x-1)\cdots(x-n+1) = \sum_{m=1}^{n} s(n,m) x^m \,,$$

we get that $\sum_{m=1}^{n} s(n,m) = 0$. Hence it follows that

$$2Y_n = \sum_{m=1}^{n} s(n,m)(Y_m + 1) \,.$$

By the above equality$H_j = \dfrac{Y_j + 1}{2}$ is an integer. Hence we have

$$Y_n = 2H_n - 1 = \sum_{m=1}^{n} s(n,m) H_m$$

and so

$$H_n \equiv 1 + \sum_{m=1}^{n-1} s(n,m) H_m \pmod 2 \,.$$

Now we will prove by induction that H_n is odd, for n congruent to 0 or 1 modulo 3, and even for n congruent to 2 modulo 3. Clearly this will prove the result. We easily check that the preceding assertion holds for the first few values of the sequence H_n.

By Lemma 14 we know that $v_2(s(n,m)) \geq M - m$, where $M = \left\lfloor \dfrac{n-1}{2} \right\rfloor + 1$. Therefore

$$H_n \equiv 1 + \sum_{m=M}^{n-1} s(n,m) H_m \pmod 2 \,.$$

We also have

$$x(x-1)\cdots(x-n+1) \equiv x^M (x-1)^{n-M} \pmod 2 \,.$$

This implies that $s(n, M+j) \equiv \dbinom{n-M}{j} \pmod 2$. We therefore have

$$H_n \equiv 1 + \sum_{j=0}^{n-M-1} \binom{n-M}{j} H_{j+M} \pmod 2 \,.$$

By the induction hypothesis, H_{j+M} is zero modulo 2 if $j + M$ is congruent to 2 modulo 3. Hence

$$H_n \equiv 1 + \sum_{\substack{0 \le j \le n-M-1 \\ M+j \equiv 0 \pmod 3}} \binom{n-M}{j} + \sum_{\substack{0 \le j \le n-M-1 \\ M+j \equiv 1 \pmod 3}} \binom{n-M}{j} \pmod 2 .$$

Now write $n = 6k + s$, $s \in \{0, 1, 2, 3, 4, 5\}$, and check each case.

Case s=1: We have $M = 3k$, $n - M = 3k$. Hence

$$H_n = 1 + \sum_{\substack{0 \le j \le 3k-1 \\ j \equiv 0 \pmod 3}} \binom{3k}{j} + \sum_{\substack{0 \le j \le 3k-1 \\ j \equiv 1 \pmod 3}} \binom{3k}{j} \pmod 2 ,$$

which can be written as follows:

$$H_n \equiv \Gamma_{0,0,k} + \Gamma_{1,0,k} \pmod 2 .$$

From case (b) of Lemma 20 it follows that

$$3(\Gamma_{0,0,k} + \Gamma_{1,0,k}) = 2^{3k+1} + (-1)^{3k} ,$$

hence H_n is odd.

Case s=2: It follows that $M = 3k + 1$ and $n - M = 3k$. Hence

$$H_n \equiv 1 + \sum_{\substack{0 \le j \le 3k-1 \\ j \equiv 2 \pmod 3}} \binom{3k}{j} + \sum_{\substack{0 \le j \le 3k-1 \\ j \equiv 0 \pmod 3}} \binom{3k}{j} \pmod 2 ,$$

which can be written as follows:

$$H_n \equiv \Gamma_{2,1,k} + \Gamma_{0,1,k} \pmod 2 .$$

By Lemma 20, we have

$$3(\Gamma_{2,1,k} + \Gamma_{0,1,k}) = 2^{3k+2} + (-1)^{3k+1} ,$$

and hence H_n is odd.

Case s=3: It follows that $M = 3k + 1$ and $n - M = 3k + 1$. Hence

$$H_n \equiv 1 + \sum_{\substack{0 \le j \le 3k \\ j \equiv 2 \pmod 3}} \binom{3k+1}{j} + \sum_{\substack{0 \le j \le 3k \\ j \equiv 0 \pmod 3}} \binom{3k+1}{j} \pmod 2 ,$$

which can be written as follows:

$$H_n \equiv 1 + \Gamma_{2,1,k} + \Gamma_{0,1,k} \pmod 2 .$$

By Lemma 20

$$3(\Gamma_{2,1,k} + \Gamma_{0,1,k}) = 2^{3k+2} + (-1)^{3k+1}$$

hence H_n is even.

Case s=4: It follows that $M = 3k+2$ and $n - M = 3k+1$. Hence

$$H_n \equiv 1 + \sum_{\substack{0 \le j \le 3k \\ j \equiv 1 \pmod 3}} \binom{3k+1}{j} + \sum_{\substack{0 \le j \le 3k \\ j \equiv 2 \pmod 3}} \binom{3k+1}{j} \pmod 2 ,$$

which can be written as follows:

$$H_n \equiv \Gamma_{1,1,k} + \Gamma_{2,1,k} \pmod 2 .$$

By Lemma 20 we have

$$3(\Gamma_{1,1,k} + \Gamma_{2,1,k}) = 2^{3k+2} + (-1)^{3k+1} ,$$

and hence H_n is odd.

Case s=5: It follows that $M = 3k+2$ and $n - M = 3k+2$. Hence

$$H_n \equiv 1 + \sum_{\substack{0 \le j \le 3k+1 \\ j \equiv 1 \pmod 3}} \binom{3k+2}{j} + \sum_{\substack{0 \le j \le 3k+1 \\ j \equiv 2 \pmod 3}} \binom{3k+2}{j} \pmod 2 ,$$

which can be written as follows:

$$H_n \equiv 1 + \Gamma_{1,2,k} + \Gamma_{2,2,k} \pmod 2 .$$

By Lemma 20

$$3(\Gamma_{2,1,k} + \Gamma_{0,1,k}) = 2^{3k+3} + (-1)^{3k+2} ,$$

and hence H_n is odd.

Case s=6: It follows that $M = 3k+3$ and $n - M = 3k+2$. Hence

$$H_n \equiv 1 + \sum_{\substack{0 \le j \le 3k+1; \\ j \equiv 0 \pmod 3}} \binom{3k+2}{j} + \sum_{\substack{0 \le j \le 3k+1; \\ j \equiv 1 \pmod 3}} \binom{3k+2}{j} \pmod 2 ,$$

which can be written as follows:

$$H_n \equiv 1 + \Gamma_{0,2,k} + \Gamma_{1,2,k} \pmod 2 .$$

By Lemma 20 we have

$$3(\Gamma_{0,2,k} + \Gamma_{1,2,k}) = 2^{3k+3} + (-1)^{3k+2} ,$$

and hence H_n is even.

The proof of Theorem 9 is now complete.

10 Proof of property (a) of Theorem 10

We use induction on n. For $1 \le n \le p$, the property holds, because $\log_p(n) - 1 \le 0$. Now suppose the property holds for $m \le n-1$, we have to show the property for n. By the above, one can suppose that $n \ge p+1$.

It is enough to prove that all the terms in Formula (6)

$$Z_n^{\langle p \rangle} = s(n,1) - \frac{p}{p-1} \sum_{m=1}^{n-1} s(n,m) Z_m^{\langle p \rangle}$$

have a p-adic valuation $\ge \log_p(n) - 1$.

10.1 Begin with $s(n,1) = (-1)^{n-1}(n-1)!$. Its p-adic valuation is $\dfrac{n-1-S_p(n-1)}{p-1}$. We have to prove that this quantity is $\ge \log_p(n) - 1$. Write $n-1 = a_0 + a_1 p + \cdots + a_m p^m$, with $a_k \in \{0, \cdots, p-1\}$ and $a_m \ne 0$. We have $n \le 1 + (p-1)(1+p+\cdots+p^m) = p^{m+1}$; hence $\log_p(n) \le m+1$. It suffices to prove

$$a_1 + a_2(1+p) + \cdots + a_m(1+p+\cdots+p^{m-1}) \ge m\,.$$

But in the sum $1 + p + \cdots + p^{m-1}$, all terms are ≥ 1, the second term is ≥ 3, and $a_m \ge 1$, and we are done.

10.2 Now let m be such that $1 \le m \le \left\lfloor \dfrac{n-1}{p} \right\rfloor + 1$. One can neglect the factor $\dfrac{1}{p-1}$. We have hence to prove

$$v_p(p s(n,m) Z_m^{\langle p \rangle}) = 1 + v_p(s(n,m)) + v_p(Z_m^{\langle p \rangle}) \ge \log_p(n) - 1\,.$$

Using Lemma 14 and the induction hypothesis, it suffices to prove

$$1 + \left\lfloor \frac{n-1}{p} \right\rfloor + 1 - m + \log_p(m) - 1 \ge \log_p(n) - 1\,,$$

or

$$\left\lfloor \frac{n-1}{p} \right\rfloor + 2 - \log_p(n) \ge m - \log_p(m)\,. \tag{8}$$

There are two cases $\begin{cases} \text{(a):} & n \text{ divisible by } p; \\ \text{b):} & n \text{ not divisible by } p. \end{cases}$

(a) Suppose that n is divisible by p. Let $n = kp$. It follows that $\left\lfloor \frac{n-1}{p} \right\rfloor = k - 1$; hence $1 \leq m \leq k$. We then have to prove

$$k - \log_p(k) \geq m - \log_p(m)\,,$$

which holds because the function $x - \log_p(x)$ is increasing for $x \geq 1$.

(b) Suppose that n is not divisible by p Let $n = kp + r$, with $1 \leq r \leq p - 1$. it follows that $1 \leq m \leq k + 1$. First examine the case $m = k + 1$; we have to prove that

$$k + 2 - \log_p(kp + r) \geq k + 1 - \log_p(k + 1)\,,$$

or $p(k + 1) \geq kp + r$, which is true.
Hence we can suppose $m \leq k$. Since $k + 1 - \log_p(k + 1) \geq m - \log_p(m)$, It follows that relation (8) is also satisfied in this case .

10.3 Now suppose that $\left\lfloor \frac{n-1}{p} \right\rfloor + 2 \leq m \leq n - 1$. Using the induction hypothesis again, $v_p(Z_m^{\langle p \rangle}) \geq \log_p(m) - 1$, we get

$$v_p(ps(n,m)Z_m^{\langle p \rangle}) \geq 1 + v_p(Z_m^{\langle p \rangle}) \geq \log_p(m)\,.$$

We want to prove that $\log_p(m) \geq \log_p(n) - 1$, or $p \cdot m \geq n$. Suppose on the contrary that $p \cdot m \leq n - 1$. It follows that $m \leq \left\lfloor \frac{n-1}{p} \right\rfloor$, in contradiction with hypothesis: $m \geq \left\lfloor \frac{n-1}{p} \right\rfloor + 2$.

The proof of property (a) of Theorem 10 is now complete.

11 Proof of property (b) of Theorem 10

We proceed by induction. We have seen (cf. Remark 13) that $Z_p^{\langle p \rangle}$ is a p-adic unit; hence $v_p(Z_p^{\langle p \rangle}) = 0$, which is property (b) for $t = 1$.

Now suppose property (b) satisfied for $n = p^k$, with $1 \leq k \leq t - 1$. We want to prove it for $n = p^t$. One can suppose that $t \geq 2$. We use formula (6) again:

$$Z_n^{\langle p \rangle} = s(n,1) - \frac{p}{p-1} \sum_{m=1}^{n-1} s(n,m) Z_m^{\langle p \rangle}\,.$$

We will prove that all terms in the above formula have p-adic valuation greater than t, with the only exception of the term with index $m = p^{t-1}$, which has p-adic valuation $t - 1$.

11.1 First, consider the term $s(n,1) = (-1)^{n-1}(n-1)!$. Its p-adic valuation is

$$\frac{n-1-S_p(n-1)}{p-1} = 1 + p + \cdots + p^{t-1} - t = (p-1) + \cdots + (p^{t-1} - 1)\,.$$

All the terms in this sum are ≥ 1, and for example the first one is ≥ 2 and hence the sum is $\geq t$.

11.2 Now suppose that $1 \leq m \leq p^{t-1} - 2$. It follows that

$$v_p(ps(n,m)Z_m^{\langle p \rangle}) \geq 1 + \left\lfloor \frac{n-1}{p} \right\rfloor + 1 - m + \log_p(m) - 1\,.$$

It suffices to show that

$$\left\lfloor \frac{n-1}{p} \right\rfloor + 1 - t \geq m - \log_p(m)\,.$$

Since $\left\lfloor \frac{n-1}{p} \right\rfloor + 1 = p^{t-1}$, we have to prove

$$p^{t-1} - t \geq m - \log_p(m)\,.$$

Since the function $x - \log_p(x)$ is increasing for $x \geq 1$, it suffices to show

$$p^{t-1} - t \geq p^{t-1} - 2 - \log_p(p^{t-1} - 2)\,,$$

or $p^{t-1} - 2 \geq p^{t-2}$, which is obvious.

11.3 Now consider the term of index $m = p^{t-1} - 1$. By Lemma 15, we know that $s(n,m) = s\big(p^t, p^{t-1} - 1\big)$ is divisible by p. It follows that

$$v_p(p \cdot s\big(n,m\big)Z_m^{\langle p \rangle}) \geq 1 + 1 + v_p(Z_m^{\langle p \rangle}) \geq 2 + v_p(Z_m^{\langle p \rangle})\,.$$

Since $m > p^{t-2}$, it also follows that $v_p(Z_m^{\langle p \rangle}) > t - 3$ by property (a) of Theorem 10. Hence $v_p(Z_m^{\langle p \rangle}) \geq t - 2$ and we are done.

11.4 Now take m such that $p^{t-1} + 1 \leq m \leq 2^t - 1$. By property (a) of Theorem 10, we know that $v_p(Z_m^{\langle p \rangle}) \geq t - 1$ for such m. It follows that

$$v_p(p \cdot s(2^t, m)Z_m^{\langle p \rangle}) \geq t\,,$$

and we are done for this case also.

11.5 Now consider the value $m = p^{t-1}$. By the induction hypothesis, it follows that $v_p(Z_m^{\langle p \rangle}) = t - 2$. By Lemma 15, $s(n,m) = s\big(2^t, 2^{t-1}\big))$ is not divisible by p. It follows that the p-adic valuation of $ps(n,m)Z_m^{\langle p \rangle}$ is exactly $t-1$, and this is the only term in the sum with this valuation, all the others have valuation $\geq t$.

Hence $v_p(Z_{p^t}^{\langle p \rangle}) = t - 1$. The proof of Theorem 10 is now complete.

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References

[1] Y. Amice, *Les Nombres p-adiques*, Presses universitaires de France, 1975.

[2] L. Comtet, *Advanced Combinatorics*, Reidel, 1974.

[3] F. T. Howard, Congruences for the Stirling numbers and associated Stirling numbers. *Acta Arith.* **55** (1990), 29–41.

[4] T. Lengyel, On some 2-adic properties of a recurrence involving Stirling numbers. *p-Adic Numbers Ultrametric Anal. Appl.* **4** (2012), 179–186.

[5] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, `http://oeis.org`.

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