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Constructing Exponential Riordan Arrays from Their A and Z Sequences

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

We show how to construct an exponential Riordan array from a knowledge of its A and Z sequences. The effect of pre- and post-multiplication by the binomial matrix on the A and Z sequences is examined, as well as the effect of scaling the A and Z sequences. Examples are given, including a discussion of related Sheffer orthogonal polynomials.

1 Introduction

One of the most fundamental results concerning Riordan arrays is that they have a sequence characterization [13, 18]. This normally involves two sequences, called the A-sequence and the Z-sequence. For exponential Riordan arrays [9] (see Appendix), this characterization is equivalent to the fact that the production matrix [11] of an exponential array $[g, f]$, with A-sequence $A(t)$ and Z-sequence $Z(t)$ has bivariate generating function

$$e^{zt}(Z(t) + A(t)z).$$

In this case we have

$$A(t) = f'(\bar{f}(t)), \quad Z(t) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))}.$$

Examples of exponential Riordan arrays and their production matrices may be found in the *On-Line Encyclopedia of Integer Sequences* [19, 20]. In that database, sequences are referred to by their A-numbers. For known sequences, we shall adopt this convention in this note.

A natural question to ask is the following. If we are given two suitable power series $A(t)$ and $Z(t)$, can we recover the corresponding exponential Riordan array $[g(t), f(t)]$ whose A and Z sequences correspond to the given power series $A(t)$ and $Z(t)$?

The next two simple results provide a means of doing this.

Lemma 1. *For an exponential Riordan array $[g(t), f(t)]$ with A-sequence $A(t)$, we have*

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{A(t)}.$$

Proof. By definition of the compositional inverse, we have

$$f(\bar{f}(t)) = t.$$

Differentiating this with respect to t, we obtain

$$f'(\bar{f}(t))\frac{d}{dt}\bar{f}(t) = 1$$

or

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{f'(\bar{f}(t))} = \frac{1}{A(t)}.$$

□

Lemma 2. *For an exponential Riordan array $[g(t), f(t)]$ with A-sequence $A(t)$ and Z-sequence $Z(t)$, we have*

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)}.$$

Proof. We have

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))}\frac{d}{dt}\bar{f}(t) = Z(t)\frac{1}{A(t)} = \frac{Z(t)}{A(t)}.$$

□

Thus if we can easily carry out the reversion from $\bar{f}(t)$ to $f(t)$, a knowledge of $A(t)$ and $Z(t)$, along with the equations

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{A(t)}, \qquad \frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} \tag{1}$$

will allow us to find $f(t)$ and $g(t)$. The steps to achieve this are as follows.

- Using the equation $\frac{d}{dt}\bar{f}(t) = \frac{1}{A(t)}$, solve for $\bar{f}(t)$.
- Revert $\bar{f}(t)$ to get $f(t)$.

- Sove the equation $\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)}$ and take the exponential to get $g(\bar{f}(t))$.
- Solve for $g(t)$ by substituting $f(t)$ in place of t in the last found expression.

Constants of integration may be determined using such conditions as $\bar{f}(0) = f(0) = 0$, and $g(0) = 1$.

Example 3. We seek to find $[g(t), f(t)]$ where

$$A(t) = \frac{1}{1+t}, \qquad Z(t) = -\frac{1}{1+t}.$$

We start by solving the equation

$$\frac{d}{dt}\bar{f}(t) = 1 + t.$$

Since $\bar{f}(0) = 0$, we find that

$$\bar{f}(t) = t + \frac{t^2}{2} = t\left(1 + \frac{t}{2}\right).$$

We revert this to get

$$f(t) = \sqrt{1+2t} - 1.$$

We now solve the equation

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = -1.$$

Thus we find that

$$\ln(g(\bar{f}(t))) = -t \Rightarrow g(\bar{f}(t)) = e^{-t}.$$

Thus (since $\bar{f}(f(t)) = t$) we get

$$g(t) = e^{-f(t)} = e^{1-\sqrt{1+2t}}.$$

Hence the exponential Riordan array with the given A and Z sequences is

$$[g, f] = \left[e^{1-\sqrt{1+2t}}, \sqrt{1+2t} - 1\right].$$

We note that

$$[g, f]^{-1} = \left[e^t, t + \frac{t^2}{2}\right]$$

which is the Pascal-like matrix A100862 [6].

In like manner, we can show that

$$A(t) = \frac{1}{1+2t}, \qquad Z(t) = -\frac{1}{1+2t}$$

corresponds to the exponential Riordan array

$$[g,f]=\left[e^{\frac{1-\sqrt{1+4t}}{2}},\frac{\sqrt{1+4t}-1}{2}\right],$$

whose inverse

$$[g,f]^{-1}=[e^t,t+t^2]$$

is Pascal-like [6]. In general, if $A(t)=-Z(t)=\frac{1}{1+rt}$, then

$$[g,f]=\left[e^{\frac{1}{r}(1-\sqrt{1+2rt})},\frac{1}{r}(\sqrt{1+2rt}-1)\right].$$

Then

$$[g,f]^{-1}=\left[e^t,t+r\frac{t^2}{2}\right]$$

is a Pascal-type matrix.

2 Effect of the binomial transform

The next proposition shows the effect of changing $Z(t)$ to $Z(t)+1$ and to $Z(t)+A(t)$, respectively. We recall that the binomial matrix $B=[e^t,t]$.

Proposition 4. *Let $[g,f]$ be an exponential Riordan array with A and Z sequences $A(t)$ and $Z(t)$ respectively. Then the exponential Riordan array $B\cdot[g,f]$ has A and Z sequences $A(t)$ and $Z(t)+1$ respectively, while the exponential Riordan array $[g,f]\cdot B$ has A and Z sequences $A(t)$ and $Z(t)+A(t)$ respectively.*

Proof. Firstly, we let the exponential Riordan array $[h,l]$ have A and Z sequences $A(t)$ and $Z(t)+1$ respectively. Then we have $\frac{d}{dt}\bar{l}(t)=\frac{1}{A(t)}$, which implies that $l(t)=f(t)$ (since $l(0)=f(0)=0$). Now

$$\frac{d}{dt}\ln(h(\bar{l}(t)))=\frac{d}{dt}\ln(h(\bar{f}(t)))=\frac{Z(t)+1}{A(t)}=\frac{Z(t)}{A(t)}+\frac{1}{A(t)}.$$

Thus

$$\ln(h(\bar{f}(t)))=\ln(g(\bar{f}(t)))+\bar{f}(t)\Rightarrow h(\bar{f}(t))=g(\bar{f}(t))e^{\bar{f}(t)}.$$

We obtain that

$$h(t)=g(t)l^t$$

and so

$$[h(t),l(t)]=[e^tg(t),f(t)]=[e^t,t]\cdot[g(t),f(t)]=B\cdot[g(t),f(t)].$$

Secondly, we now assume that the exponential Riordan array $[h, l]$ have A and Z sequences $A(t)$ and $Z(t) + A(t)$ respectively. As before, we see that $l(t) = f(t)$. Also,

$$\frac{d}{dt}\ln(h(\bar{l}(t))) = \frac{d}{dt}\ln(h(\bar{f}(t))) = \frac{Z(t) + A(t)}{A(t)} = \frac{Z(t)}{A(t)} + 1.$$

Thus

$$\ln(h(\bar{f}(t))) = \ln(g(\bar{f}(t))) + t \Rightarrow h(\bar{f}(t)) = g(\bar{f}(t))e^t.$$

Now substituting $f(t)$ for t gives us

$$h(t) = e^{f(t)}g(t).$$

Thus

$$[h, l] = [e^{f(t)}g(t), f(t)] = [g(t), f(t)] \cdot [e^t, t] = [g(t), f(t)] \cdot B.$$

We shall see examples of these results in the next section.

□

3 Effect of Scaling

In this section, we will assume that the exponential Riordan array with A and Z sequences $A(t)$ and $Z(t)$, respectively, is given by $[g(t), f(t)]$. We wish to characterize the exponential Riordan array $[g^*(t), f^*(t)]$ whose A and Z sequences are $A^*(t) = rA(t)$ and $Z^*(t) = sZ(t)$ respectively.

Proposition 5. *We have*

$$[g^*(t), f^*(t)] = \left[g(rt)^{\frac{s}{r}}, rf(t)\right].$$

Proof. We have

$$\frac{d}{dt}\bar{f^*}(t) = \frac{1}{rA} = \frac{1}{r}\frac{d}{dt}\bar{f}(t).$$

Thus

$$\bar{f^*}(t) = \frac{1}{r}\bar{f}(t) \Rightarrow f^*(t) = rf(t).$$

Then

$$\frac{d}{dt}\ln(g^*(\bar{f^*}(t))) = \frac{sZ}{rA} = \frac{s}{r}\frac{d}{dt}\ln(g(\bar{f}(t))),$$

and so

$$\ln(g^*(\bar{f^*}(t))) = \frac{s}{r}\ln(g(\bar{f}(t))) = \ln\left(g(\bar{f}(t))^{\frac{s}{r}}\right).$$

Thus

$$g^*(\bar{f^*}(t)) = g(\bar{f}(t))^{\frac{s}{r}} \Rightarrow g^*(\frac{1}{r}\bar{f}(t)) = g(\bar{f}(t))^{\frac{s}{r}} \Rightarrow g^*(\frac{1}{r}t) = g(t)^{\frac{s}{r}},$$

or

$$g^*(t) = g(rt)^{\frac{s}{r}}.$$

Example 6. We let

$$A(t)=1+t, \qquad Z(t)=1+2t.$$

We find that the corresponding exponential array is

$$[g,f]=\left[e^{2e^t-t-2}, e^t-1\right],$$

which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 9 & 13 & 6 & 1 & 0 & 0 & \cdots \\ 35 & 59 & 37 & 10 & 1 & 0 & \cdots \\ 153 & 301 & 230 & 85 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with production matrix which begins

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 6 & 4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 8 & 5 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 10 & 6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We now take

$$A^*(t)=3(1+t), \qquad Z^*(t)=5(1+2t).$$

The corresponding exponential Riordan array is then given by

$$[g^*(t), f^*(t)]=\left[\left(e^{2e^{3t}-3t-2}\right)^{\frac{5}{3}}, 3(e^t-1)\right].$$

This array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 55 & 33 & 9 & 0 & 0 & 0 & \cdots \\ 665 & 543 & 162 & 27 & 0 & 0 & \cdots \\ 9895 & 9033 & 3573 & 702 & 81 & 0 & \cdots \\ 165185 & 170103 & 76410 & 19575 & 2835 & 243 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with production matrix which begins

$$\begin{pmatrix} 5 & 3 & 0 & 0 & 0 & 0 & \dots \\ 10 & 8 & 3 & 0 & 0 & 0 & \dots \\ 0 & 20 & 11 & 3 & 0 & 0 & \dots \\ 0 & 0 & 30 & 14 & 3 & 0 & \dots \\ 0 & 0 & 0 & 40 & 17 & 3 & \dots \\ 0 & 0 & 0 & 0 & 50 & 20 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

4 Further examples

Example 7. We take the Stirling number related choice of

$$A(t) = 1 + t, \qquad Z(t) = 1 + t.$$

From

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{1+t},$$

we obtain

$$\bar{f}(t) = \ln(1+t) \Rightarrow f(t) = e^t - 1.$$

Then from

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = 1$$

we obtain

$$\ln(g(\bar{f}(t))) = t \Rightarrow g(\bar{f}(t)) = e^t,$$

and hence

$$g(t) = e^{e^t-1}.$$

Thus we obtain

$$[g, f] = \left[e^{e^t-1}, e^t - 1\right],$$

which is A049020. We have

$$[g, f] = S_2 \cdot B$$

where S_2 is the matrix of Stirling numbers of the second kind (A048993) and B is the binomial matrix (A007318). The production array of $[g, f]$ is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4 & 5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since this production matrix is tri-diagonal, the inverse matrix $[g, f]^{-1}$ is the coefficient array of a family of orthogonal polynomials [4, 3]. The family in question is the family of Charlier polynomials, which has the Bell numbers (with e.g.f. e^{e^t-1}) as moments. The Charlier polynomials satisfy the three-term recurrence

$$P_n(t) = (t - n)P_{n-1}(t) - (n - 1)P_{n-2}(t),$$

with $P_0(t) = 1$, $P_1(t) = t - 1$.

Example 8. We take

$$A(t) = 1 + t \qquad Z(t) = 1 + t + t^2.$$

Again, we find that

$$f(t) = e^t - 1.$$

Then

$$\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1 + t + t^2}{1 + t},$$

and hence

$$\ln(g(\bar{f}(t))) = \frac{t^2}{2} + \ln(1 + t).$$

Thus

$$g(\bar{f}(t)) = e^{\frac{t^2}{2}}(1 + t),$$

and so

$$g(t) = e^{\frac{(e^t-1)^2}{2}}(1 + e^t - 1) = e^t e^{\frac{(e^t-1)^2}{2}}.$$

In this case, the production matrix is four-diagonal and begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & 3 & 1 & 0 & 0 & \dots \\ 0 & 6 & 3 & 4 & 1 & 0 & \dots \\ 0 & 0 & 12 & 4 & 5 & 1 & \dots \\ 0 & 0 & 0 & 20 & 5 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The exponential Riordan array

$$[g, f] = \left[e^t e^{\frac{(e^t-1)^2}{2}}, e^t - 1\right]$$

begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 7 & 10 & 6 & 1 & 0 & 0 & \cdots \\ 29 & 45 & 31 & 10 & 1 & 0 & \cdots \\ 136 & 241 & 180 & 75 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The row sums of this array are the Dowling numbers A007405.

We note that the exponential Riordan array

$$B^{-1} \cdot [g, f] = [e^{-t}, t] \cdot [g, f] = \left[e^{\frac{(e^t-1)^2}{2}}, e^t - 1\right]$$

has

$$A(t) = 1 + t \qquad Z(t) = t + t^2.$$

This array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 4 & 3 & 1 & 0 & 0 & \cdots \\ 10 & 19 & 13 & 6 & 1 & 0 & \cdots \\ 45 & 91 & 75 & 35 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The first column of this array is A060311, while its row sums are given by A004211. The production matrix of this array begins

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & 2 & 1 & 0 & 0 & \dots \\ 0 & 6 & 3 & 3 & 1 & 0 & \dots \\ 0 & 0 & 12 & 4 & 4 & 1 & \dots \\ 0 & 0 & 0 & 20 & 5 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

where we see that the effect of the inverse binomial matrix is to subtract 1 from the diagonal.

In this example, we have $Z(t)=1+t+t^2=A(t)+t^2$. Thus the exponential Riordan array $[g,f]$ is equal to the product
$$[h,l]\cdot B$$
where the exponential Riordan array $[h,l]$ has A and Z sequences of $1+t$ and t^2, respectively.

Example 9. We take
$$A(t)=1+t^2, \qquad Z(t)=1+t+t^2.$$
Then Thus
$$f(t)=\tan(t).$$
Now
$$\frac{d}{dt}\ln(g(\bar{f}(t)))=\frac{Z(t)}{A(t)}=\frac{1+t+t^2}{1+t^2}=1+\frac{t}{1+t^2},$$
and so
$$\ln(g(\bar{f}(t)))=\ln\sqrt{1+t^2}+t.$$
Thus
$$g(\bar{f}(t))=e^t\sqrt{1+t^2}\Rightarrow g(t)=e^{\tan(t)}\sqrt{1+\tan^2(t)}=\frac{e^{\tan(t)}}{\cos(t)}.$$
Thus the sought-for exponential Riordan array is given by
$$[g,f]=\left[e^{\tan(t)}\sec(t),\tan(t)\right].$$
This matrix begins
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 8 & 3 & 1 & 0 & 0 & \cdots \\ 20 & 32 & 20 & 4 & 1 & 0 & \cdots \\ 92 & 156 & 100 & 40 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
with production matrix that begins
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 4 & 1 & 1 & 0 & 0 & \ldots \\ 0 & 6 & 9 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 12 & 16 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 20 & 25 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The first column is A009244. We note that we have the following factorization

$$[g, f] = \left[e^{\tan(t)}\sec(t), \tan(t)\right] = [\sec(t), \tan(t)] \cdot B.$$

Thus we can say that the exponential Riordan array $[\sec(t), \tan(t)]$, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 0 & 1 & 0 & 0 & \cdots \\ 5 & 0 & 14 & 0 & 1 & 0 & \cdots \\ 0 & 61 & 0 & 30 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

has A sequence defined by $1 + t^2$ and Z sequence defined by t. Thus its production matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can infer from this that the inverse array

$$[\sec(t), \tan(t)]^{-1} = \left[\frac{1}{\sqrt{1+t^2}}, \tan^{-1}(t)\right]$$

is the coefficient array of the family of orthogonal polynomials

$$P_n(t) = tP_{n-1}(t) - (n-1)^2 P_{n-2}(t),$$

with $P_0(t) = 1$ and $P_1(t) = t$.

Example 10. In this example, we let

$$A(t) = 1 + t, \qquad Z(t) = \frac{1}{1-t}.$$

As before, we get $f(t) = e^t - 1$. Now

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1}{1-t^2},$$

and hence
$$\ln(g(\bar{f}(t))) = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right).$$
We infer that
$$g(t) = \sqrt{\frac{e^t}{2-e^t}}.$$
The function $g(t)$ generates the sequence A014307 which begins
$$1, 1, 2, 7, 35, 226, 1787, 16717, 180560, 2211181, 30273047, \ldots.$$
It has many combinatorial interpretations [7, 15, 17].

The exponential Riordan array
$$[g, f] = \left[\sqrt{\frac{e^t}{2-e^t}}, e^t - 1\right]$$
begins
$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 7 & 10 & 6 & 1 & 0 & 0 & \cdots \\ 35 & 45 & 31 & 10 & 1 & 0 & \cdots \\ 226 & 271 & 180 & 75 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$
with production matrix that begins
$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 3 & 1 & 0 & 0 & \ldots \\ 6 & 6 & 3 & 4 & 1 & 0 & \ldots \\ 24 & 24 & 12 & 4 & 5 & 1 & \ldots \\ 120 & 120 & 60 & 20 & 5 & 6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$
In general, the exponential Riordan array with
$$A(t) = 1 + t, \qquad Z(t) = \frac{r}{1-t},$$
is given by
$$[g, f] = \left[\left(\frac{e^t}{2-e^t}\right)^{r/2}, e^t - 1\right].$$

Example 11. For this example, we take

$$A(t) = e^{-t}, \qquad Z(t) = e^t.$$

Then

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{A(t)} = \frac{1}{e^{-t}} = e^t,$$

and so we get

$$\bar{f}(t) = e^t + C = e^t - 1$$

since $\bar{f}(0) = 0$. Thus

$$f(t) = \ln(1+t).$$

Now

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{e^t}{e^{-t}} = e^{2t},$$

and so

$$\ln(g(\bar{f}(t))) = \frac{e^{2t}}{2} - \frac{1}{2} \Rightarrow g(\bar{f}(t)) = e^{\frac{1}{2}(e^{2t}-1)}.$$

Substituting $f(t)$ for t we get

$$g(t) = e^{\frac{1}{2}(e^{2\ln(1+t)}-1)} = e^{t+\frac{t^2}{2}}.$$

Thus

$$[g, f] = \left[e^{t+\frac{t^2}{2}}, \ln(1+t)\right].$$

We note that if we have

$$A(t) = Z(t) = e^{-t},$$

then we obtain

$$[g, f] = [1+t, \ln(1+t)].$$

Interestingly, this last exponential Riordan array has a production matrix that is equal the ordinary Riordan array

$$\left(\frac{1+2t}{1+t}, \frac{t}{1+t}\right)$$

with its first row removed.

5 Orthogonal polynomials

When $Z(t) = \alpha + \beta t$ and $A(t) = 1 + \gamma t + \delta t^2$, the production matrix of the corresponding exponential Riordan array $[g, f]$ is tri-diagonal, beginning as follows.

$$\begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & \cdots \\ \beta & \alpha+\gamma & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2(\beta+\delta) & \alpha+2\gamma & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3(\beta+2\delta) & \alpha+3\gamma & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4(\beta+3\delta) & \alpha+4\gamma & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5(\beta+4\delta) & \alpha+5\gamma & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As a consequence, $[g, f]^{-1}$ is the coefficient array of the family of orthogonal polynomials $P_n(t)$ defined by the three-term recurrence [8, 12, 21]

$$P_n(t) = (t - (\alpha + (n-1)\gamma))P_{n-1}(t) - (n-1)(\beta + (n-2)\delta)P_{n-2}(t),$$

with $P_0(t) = 1$ and $P_1(t) = x - \alpha$. These are precisely the Sheffer orthogonal polynomials [1, 13].

Example 12. We take the case of

$$A(t) = 1 + t + t^2, \qquad Z(t) = 1 + t.$$

We have

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{1+t+t^2}.$$

Choosing the constant of integration so that $\bar{f}(0) = 0$, we get

$$\bar{f}(t) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) - \frac{\pi}{3\sqrt{3}}.$$

Thus

$$\begin{aligned} f(t) &= \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{6}\right) - \frac{1}{2} \\ &= \frac{2\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)} \\ &= \frac{2\tan\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3} - \tan\left(\frac{\sqrt{3}t}{2}\right)}. \end{aligned}$$

We now have
$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1+t}{1+t+t^2},$$
and hence
$$\ln(g(\bar{f}(t))) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + \frac{1}{2}\ln(1+t+t^2) - \frac{\pi}{6\sqrt{3}}.$$
From this we infer that
$$g(t) = \frac{\sqrt{3}e^{\frac{x}{2}}}{\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}.$$
The function $g(t)$ generates the sequence A049774, which counts the number of permutations of n elements not containing the consecutive pattern 123.

The sought-for matrix is thus
$$[g, f] = \left[\frac{\sqrt{3}e^{\frac{x}{2}}}{\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}, \frac{2\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}\right].$$
This exponential Riordan array is A182822, which begins
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 12 & 6 & 1 & 0 & 0 & \cdots \\ 17 & 53 & 39 & 10 & 1 & 0 & \cdots \\ 70 & 279 & 260 & 95 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
with production matrix that begins
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9 & 4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 16 & 5 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 25 & 6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
Example 13. We change the previous example slightly by taking
$$A(t) = 1 + 2t + t^2 = (1+t)^2, \qquad Z(t) = 1+t.$$

Then we have

$$\frac{d}{dt}\bar{f}(t) = \frac{1}{(1+t)^2} \Rightarrow \bar{f}(t) = -\frac{1}{1+t} + 1 = \frac{t}{1+t}.$$

This means that

$$f(t) = \frac{t}{1-t}.$$

Now we have

$$\frac{d}{dt}\ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1}{1+t},$$

and hence

$$\ln(g(\bar{f}(t))) = ln(1+t) \Rightarrow g(\bar{f}(t)) = 1+t.$$

This implies that

$$g(t) = 1 + f(t) = 1 + \frac{t}{1-t} = \frac{1}{1-t}.$$

Thus

$$[g, f] = \left[\frac{1}{1-t}, \frac{t}{1-t}\right].$$

Thus $[g, f]^{-1}$ is the coefficient array of the Laguerre polynomials [5].

We finish by noting that the simple addition of t to $A(t)$ has allowed us to go from the relatively complicated exponential Riordan array

$$\left[\frac{\sqrt{3}e^{\frac{x}{2}}}{\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}, \frac{2\sin\left(\frac{\sqrt{3}t}{2}\right)}{\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}\right]$$

to the simple exponential Riordan array

$$\left[\frac{1}{1-t}, \frac{t}{1-t}\right].$$

6 Appendix: exponential Riordan arrays

The *exponential Riordan group* [6, 9, 11], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(t) = g_0 + g_1 t + g_2 t^2 + \cdots$ and $f(t) = f_1 t + f_2 t^2 + \cdots$ where $g_0 \neq 0$ and $f_1 \neq 0$. We usually assume that

$$g_0 = f_1 = 1.$$

The associated matrix is the matrix whose i-th column has exponential generating function $g(t)f(t)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. The group law is given by

$$[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, t]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n\geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}\,(t)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(t)\mathcal{U}(f(t))$. Thus the row sums of the array $[g, f]$ have exponential generating function given by $g(t)e^{f(t)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^t.

As an element of the group of exponential Riordan arrays, the binomial matrix $\mathbf{B}$ with (n, k)-th element $\binom{n}{k}$ is given by $\mathbf{B} = [e^t, t]$. By the above, the exponential generating function of its row sums is given by $e^t e^t = e^{2t}$, as expected (e^{2t} is the e.g.f. of 2^n).

To each exponential Riordan array $L = [g, f]$ is associated [10, 11] a matrix P called its *production* matrix, which has bivariate g.f. given by

$$e^{zt}(Z(t) + A(t)z)$$

where

$$A(t) = f'(\bar{f}(t)), \quad Z(t) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))}.$$

We have

$$P = L^{-1}\bar{L}$$

where $\bar{L}$ [16, 22] is the matrix L with its top row removed.

The ordinary Riordan group is described in [18].

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