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Generalized Stirling Numbers, Exponential Riordan Arrays, and Toda Chain Equations

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Abstract

We study the properties of three families of exponential Riordan arrays related to the Stirling numbers of the first and second kind. We relate these exponential Riordan arrays to the coefficients of families of orthogonal polynomials. We calculate the Hankel transforms of the moments of these orthogonal polynomials. We show that the Jacobi coefficients of two of the matrices studied satisfy generalized Toda chain equations. We finish by defining and characterizing the elements of an exponential Riordan array associated to generalized Stirling numbers studied by Lang.

1 Introduction

The Stirling numbers of the first kind $\left[{n \atop k}\right]$ are given by the elements of the exponential Riordan array

$$\left[1, \ln\left(\frac{1}{1-x}\right)\right];$$

the notation is explained in the Appendix, § 9. The numbers $\left[{n \atop k}\right]$ count the number of ways to arrange n objects into k cycles. The corresponding signed Stirling numbers of the first kind $s(n,k)=(-1)^{n-k}\left[{n \atop k}\right]$ are the elements of the exponential Riordan array

$$s=\left[1, \ln(1+x)\right].$$

The Stirling numbers of the second kind $S(n,k)=\frac{1}{k!}\sum_{i=0}^{n}(-1)^{k-i}\binom{k}{i}i^k$ A048993 are given by the elements of the exponential Riordan array

$$S=[1,e^x-1]\,.$$

They count the number of ways to partition a set of n objects into k non-empty subsets. The matrix inverse of $[1,e^x-1]$ is given by $[1,\ln(1+x)]$. The Stirling numbers of both kinds have been generalized in many ways [12, 13, 21, 25, 35]. We shall generalize their defining matrices in a number of directions. One of our main goals is to find related matrices that are associated to families of orthogonal polynomials, and to calculate the Hankel transforms of the moments of these orthogonal polynomials.

For instance, while the matrix of signed Stirling numbers of the first kind $[1,\ln(1+x)]$ A048994 does not constitute the coefficient array of a family of orthogonal polynomials, the modified matrix given by the following product

$$\left[e^{-x},x\right]\cdot[1,\ln(1+x)]=\left[e^{-x},\ln(1+x)\right]$$

is the coefficient array of a family of orthogonal polynomials. This can be seen by looking at the production matrix of the inverse matrix

$$[1,e^x-1]\cdot[e^x,x]=\left[e^{e^x-1},e^x-1\right].$$

This is the tri-diagonal matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4 & 5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The family in question is the family of Charlier polynomials, which has the Bell numbers (with e.g.f. e^{e^x-1}) as moments. They satisfy the three-term recurrence

$$P_n(x)=(x-n)P_{n-1}(x)-(n-1)P_{n-2}(x),$$

with $P_0(x)=1$, $P_1(x)=x-1$.

For our study, we will require three results from the theory of exponential Riordan arrays (see Appendix for an introduction to exponential Riordan arrays). These are [6, 5]

1. The inverse of an exponential Riordan array $[g,f]$ is the coefficient array of a family of orthogonal polynomials if and only if the production matrix of $[g,f]$ is tri-diagonal;

2. If the production matrix [15, 16, 17] of $[g, f]$ is tri-diagonal, then the elements of the first column of $[g, f]$ are the moments of the corresponding family of orthogonal polynomials;

3. The bivariate generating function of the production matrix of $[g, f]$ is given by

$$e^{xy}(Z(x) + A(x)y)$$

where

$$A(x) = f'(\bar{f}(x)),$$

and

$$Z(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))},$$

where $\bar{f}(x)$ is the compositional inverse (series reversion) of $f(x)$.

We recall that for an exponential Riordan array

$$L = [g, f]$$

the production matrix P_L of L [15, 16, 17] is the matrix

$$P_L = L^{-1}\bar{L},$$

where $\bar{L}$ is the matrix L with the first row removed.

An important classical family of orthogonal polynomials is the family of Sheffer orthogonal polynomials [1, 22, 27]. These are orthogonal polynomials $S_n(x)$ whose generating function has the following special form:

$$\sum_{n=0}^{\infty} S_n(x)t^n = A(t)e^{xu(t)}$$

where $A(t)$ and $u(t)$ are power series with $A(0) = 1$, $u(0) = 0$, $u'(0) = 1$. A necessary and sufficient condition for this to be so is that the $S_n(x)$ satisfy a three-term recurrence of the form

$$S_{n+1}(x) = (x - (an + b))S_n(x) - n(cn + d)S_{n-1}(x),$$

with $S_{-1}(x) = 0$ and $S_0(x) = 1$.

A short introduction to exponential Riordan arrays can be found in the Appendix to this note. He, Hsu and Shiue have studied links between exponential Riordan group and the Sheffer group [19, 20], including detailing the isomorphism that exists between these two groups. For general information on orthogonal polynomials and moments, see [14, 18, 32]. Continued fractions will be referred to in the sequel; [36] is a general reference, while [23, 24] discuss the connection between continued fractions and orthogonal polynomials, moments and Hankel transforms [26]. We recall that for a given sequence a_n its Hankel transform is the sequence of determinants $h_n = |a_{i+j}|_{0\le i,j\le n}$. Many interesting examples of number triangles, including exponential Riordan arrays, can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences [30, 31]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix (Pascal's triangle) $\mathbf{B}$ with (n,k)-th element $\binom{n}{k}$ is A007318. This is the exponential Riordan array $[e^x, x]$.

The following well-known results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [23], specify the links between orthogonal polynomials, the three-term recurrences that define them, the recurrence coefficients of those three-term recurrences, and the g.f. of the moment sequence of the orthogonal polynomials.

Theorem 1. *[23] (Cf. [33, Théorème 9, p. I-4] or or [36, Theorem 50.1]). Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$ Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\ge 0}$ and $(\beta_n)_{n\ge 1}$ with $\beta_n \neq 0$ for all $n \ge 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \ge 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 2. *[23] (Cf. [33, Proposition 1, (7), p. V-5] or [36, Theorem 51.1]). Let $(p_n(x))_{n\ge 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional $\mathcal{L}$. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \ge 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = \mathcal{L}(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

The *Hankel transform* [26] of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \ldots\}$ where $h_n = |a_{i+j}|_{i,j=0}^{n}$, i.e

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \rightarrow \quad h = \{h_n\}_{n\in\mathbb{N}_0} : \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \tag{1}$$

The Hankel transform of a sequence a_n and that of its binomial transform are equal. In the case that a_n has g.f. $g(x)$ expressible in the form

$$g(x) = \cfrac{a_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}$$

then we have [23]

$$h_n = a_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n = a_0^{n+1} \prod_{k=1}^{n} \beta_k^{n+1-k}. \tag{2}$$

Note that this is independent of α_n. We shall call the coefficients (α_n, β_n) above the Jacobi coefficients of the corresponding family of orthogonal polynomials. When we study related Toda chain equations, we shall use the notation (b_n, u_n) for these coefficients.

2 Stirling-related exponential Riordan arrays

We consider three families of exponential Riordan arrays, which are closely related to families of orthogonal polynomials and to generalized Stirling numbers. Thus we let

$$\begin{aligned}
A(\alpha, \beta, \gamma) &= \left[\frac{e^{-\gamma x}}{(1+\beta x)^{\alpha/\beta}}, \frac{1}{\beta} \ln(1+\beta x)\right], \\
B(\alpha, \beta, \gamma) &= \left[\frac{e^{-\gamma x}}{(1+\beta x)^{\alpha/\beta}}, \ln\left(\frac{1+(\beta+1)x}{1+\beta x}\right)\right], \\
C(\alpha, \beta, \gamma) &= \left[\frac{e^{-\gamma x}}{(1+\beta x)^{\alpha/\beta}}, \frac{x}{1+\beta x}\right].
\end{aligned}$$

We have

$$A^{-1} = \left[e^{\alpha x + \frac{\gamma}{\beta}(e^{\beta x}-1)}, \frac{e^{\beta x}-1}{\beta}\right],$$

and

$$B^{-1} = \left[e^{\frac{-\alpha p \ln(p) + \gamma(1-p)}{\beta p}}, \frac{1-p}{\beta p}\right],$$

where

$$p = 1 - \beta(e^x - 1).$$

Note that for $\gamma = 0$, we have

$$B(\alpha, \beta, 0) = \left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, \ln\left(\frac{1+(\beta+1)x}{1+\beta x}\right)\right],$$

and

$$B(\alpha,\beta,0)^{-1}=\left[\frac{1}{(1-\beta(e^x-1))^{\alpha/\beta}},\frac{e^x-1}{1-\beta(e^x-1)}\right].$$

Finally

$$C(\alpha,\beta,\gamma)^{-1}=\left[\frac{e^{\frac{\gamma x}{1-\beta x}}}{(1-\beta x)^{\alpha/\beta}},\frac{x}{1-\beta x}\right].$$

Proposition 3.

$$C(\alpha,\beta,\gamma)=A(\alpha,\beta,\gamma)\cdot\left[1,\frac{1}{\beta}(1-e^{-\beta x})\right].$$

We note that the exponential Riordan array $\left[1,\frac{1}{\beta}(1-e^{-\beta x})\right]$ has (n,k)-th term equal to

$$(-\beta)^{n-k}S(n,k).$$

Corollary 4.

$$C^{-1}=\left[1,\frac{1}{\beta}\ln\left(\frac{1}{1-\beta x}\right)\right]\cdot A^{-1}.$$

The general element of $\left[1,\frac{1}{\beta}\ln\left(\frac{1}{1-\beta x}\right)\right]$ is

$$\beta^{n-k}|s(n,k)|.$$

Clearly, we have

$$s=A(0,1,0),$$

while

$$B(0,-1,0)=\left[1,\ln\left(\frac{1}{1-x}\right)\right],$$

the matrix of (unsigned) Stirling numbers of the first kind.

We remark that

$$C(0,-1,0)=\left[1,\frac{x}{1-x}\right],$$

the so-called Lah matrix [9].

Proposition 5.

$$A^{-1}=\left[e^{\frac{\gamma}{\beta}(e^{\beta x}-1)},x\right]\cdot\left[e^{\alpha x},\frac{e^{\beta x}-1}{\beta}\right].$$

We now note that the general (n,k)-th element of the exponential Riordan array $\left[e^{\frac{\gamma}{\beta}(e^{\beta x}-1)},x\right]$ is given by

$$u_{n,k}=\binom{n}{k}\sum_{j=0}^{n-k}S(n-k,k)\gamma^j\beta^{n-k-j},$$

where $S(n,k)$ denotes the Stirling number of the second kind ($S(n,k)$ is the (n,k)-th element of the exponential Riordan array $[1, e^x - 1]$). The general (n,k)-th element of the right hand member $\left[e^{\alpha x}, \frac{e^{\beta x}-1}{\beta}\right]$ is given by

$$v_{n,k} = \sum_{j=0}^{n} \binom{n}{j} S(j,k) \alpha^{n-j} \beta^{j-k}.$$

Thus the general term of A^{-1} is given by

$$\sum_{i=0}^{n} u_{n,i} v_{i,k}.$$

Proposition 6.

$$A^{-1} = \left[e^{\alpha x}, \frac{e^{\beta x}-1}{\beta}\right] \cdot \left[e^{\gamma x}, x\right].$$

Thus we can express the general term of A^{-1} as the generalized Stirling number

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{j} S(j,i) \alpha^{n-j} \beta^{j-i} \binom{i}{k} \gamma^{i-k}.$$

Proposition 7.

$$A = \left[e^{-\gamma x}, x\right] \cdot \left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, \frac{1}{\beta}\ln(1+\beta x)\right] = \left[e^{-\gamma x}, x\right] \cdot A(\alpha, \beta, 0).$$

We now note the canonical decomposition of $A(\alpha, \beta, 0)$:

$$A(\alpha,\beta,0) = \left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, \frac{1}{\beta}\ln(1+\beta x)\right] = \left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, x\right] \cdot \left[1, \frac{1}{\beta}\ln(1+\beta x)\right].$$

Now the matrix $\left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, x\right]$ has general (n,k)-th term given by

$$\prod_{j=0}^{n-k-1} (\alpha + j\beta)(-1)^{n-k},$$

while the general term of $\left[1, \frac{1}{\beta}\ln(1+\beta x)\right]$ is given by

$$\beta^{n-k} s(n,k).$$

We thus obtain the following result.

Proposition 8. *The general (n,k)-th term of the matrix $A(\alpha,\beta,0)$ is given by*

$$\sum_{i=0}^{n} \prod_{j=0}^{n-i-1} (\alpha + j\beta)(-1)^{n-i} \beta^{i-k} s(i,k).$$

3 Orthogonal polynomials and $A(\alpha, \beta, \gamma)$

We recall that
$$A(\alpha,\beta,\gamma)=\left[\frac{e^{-\gamma x}}{(1+\beta x)^{\alpha/\beta}}, \frac{1}{\beta}\ln(1+\beta x)\right],$$
and that
$$A^{-1}=\left[e^{\alpha x+\frac{\gamma}{\beta}(e^{\beta x}-1)}, \frac{e^{\beta x}-1}{\beta}\right].$$

Proposition 9. *A is the coefficient array of a family of orthogonal polynomials, if $\beta\gamma \neq 0$.*

Proof. We calculate the production matrix of A^{-1}. For this, we have $f(x)=\frac{1}{\beta}(e^{\beta x}-1)$. Thus
$$\bar{f}(x)=\frac{1}{\beta}\ln(1+\beta x), \quad f'(x)=e^{\beta x}.$$
It follows that
$$A(x)=f'(\bar{f}(x))=1+\beta x.$$
Now
$$g(x)=e^{\alpha x+\frac{\gamma}{\beta}(e^{\beta x}-1)},$$
and hence
$$g'(x)=e^{\frac{\gamma}{\beta}e^{\beta x}+\alpha x}\left(\gamma e^{\beta x-\frac{\gamma}{\beta}}+\alpha e^{-\frac{\gamma}{\beta}}\right).$$
We deduce that
$$Z(x)=\frac{g'(\bar{f})(x)}{g(\bar{f})(x)}=\alpha+\beta\gamma x+\gamma.$$
Hence the g.f. of the production matrix of A^{-1} is given by
$$e^{xy}(\alpha+\beta\gamma x+\gamma+(1+\beta x)y).$$
The production matrix of A^{-1} therefore begins
$$\left(\begin{array}{ccccccc} \alpha+\gamma & 1 & 0 & 0 & 0 & 0 & \ldots \\ \beta\gamma & \alpha+\beta+\gamma & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2\beta\gamma & \alpha+2\beta+\gamma & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3\beta\gamma & \alpha+3\beta+\gamma & 1 & 0 & \ldots \\ 0 & 0 & 0 & 4\beta\gamma & \alpha+4\beta+\gamma & 1 & \ldots \\ 0 & 0 & 0 & 0 & 5\beta\gamma & \alpha+5\beta+\gamma & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$
Thus (when $\beta\gamma \neq 0$) A is the coefficient array of the family of orthogonal polynomials $P_n^A(x)$ with
$$P_{n+1}^A(x)=(x-(\beta n+\alpha+\gamma))P_n^A(x)-n\beta\gamma P_{n-1}^A(x),$$
with $P_0^A(x)=1$, $P_1^A(x)=x-(\alpha+\gamma)$. □

This shows that the family P_n^A is a family of Sheffer orthogonal polynomials. The moments μ_n^A of this family of orthogonal polynomials are given by the first column elements of A^{-1}, and hence have e.g.f. given by

$$e^{\alpha x}e^{\frac{\gamma}{\beta}(e^{\beta x}-1)}.$$

From this we deduce that

$$\mu_n^A = \sum_{k=0}^n \sum_{j=0}^n \binom{n}{j} S(j,k)\alpha^{n-j}\beta^{j-k}\gamma^k.$$

Corollary 10. *The sequence μ_n^A has generating function given by*

$$\cfrac{1}{1-(\alpha+\gamma)x-\cfrac{\beta\gamma x^2}{1-(\alpha+\beta+\gamma)x-\cfrac{2\beta\gamma x^2}{1-(\alpha+2\beta+\gamma)x-\cfrac{3\beta\gamma x^2}{1-\cdots}}}}.$$

Corollary 11. *The Hankel transform of μ_n^A is given by*

$$h_n = (\beta\gamma)^{\binom{n+1}{2}} \prod_{k=0}^n k!$$

4 Orthogonal polynomials and $B(\alpha, \beta, \gamma)$

We recall that

$$B(\alpha,\beta,\gamma) = \left[\frac{e^{-\gamma x}}{(1+\beta x)^{\alpha/\beta}}, \ln\left(\frac{1+(\beta+1)x}{1+\beta x}\right)\right].$$

Proposition 12. *The production array of B^{-1} is a four-diagonal matrix. When $\gamma = 0$, B is the coefficient array of a family of orthogonal polynomials.*

Note that for $\gamma = 0$, we have

$$B(\alpha,\beta,0) = \left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, \ln\left(\frac{1+(\beta+1)x}{1+\beta x}\right)\right],$$

and

$$B(\alpha,\beta,0)^{-1} = \left[\frac{1}{(1-\beta(e^x-1))^{\alpha/\beta}}, \frac{e^x-1}{1-\beta(e^x-1)}\right].$$

Proof. The production array of B^{-1} is given by the following four-diagonal matrix.

$$\left(\begin{array}{cccccc} \alpha+\gamma & 1 & 0 & 0 & 0 & \dots \\ \alpha(\beta+1)+\gamma(2\beta+1) & \alpha+(2\beta+1)+\gamma & 1 & 0 & 0 & \dots \\ 2\beta\gamma(\beta+1) & 2(\alpha+\beta)(\beta+1)+2\gamma(2\beta+1) & \alpha+2(2\beta+1)+\gamma & 1 & 0 & \dots \\ 0 & 6\beta\gamma(\beta+1) & 3(\alpha+2\beta)(\beta+1)+3\gamma(2\beta+1) & \alpha+3(2\beta+1)+\gamma & 1 & \dots \\ 0 & 0 & 12\beta\gamma(\beta+1) & 4(\alpha+3\beta)(\beta+1)+4\gamma(2\beta+1) & \dots & \dots \\ 0 & 0 & 0 & 20\beta\gamma(\beta+1) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

When $\gamma = 0$, we get the following tri-diagonal matrix.

$$\left(\begin{array}{ccccccc} \alpha & 1 & 0 & 0 & 0 & 0 & \dots \\ \alpha(\beta+1) & \alpha+(2\beta+1) & 1 & 0 & 0 & 0 & \dots \\ 0 & 2(\alpha+\beta)(\beta+1) & \alpha+2(2\beta+1) & 1 & 0 & 0 & \dots \\ 0 & 0 & 3(\alpha+2\beta)(\beta+1) & \alpha+3(2\beta+1) & 1 & 0 & \dots \\ 0 & 0 & 0 & 4(\alpha+3\beta)(\beta+1) & \alpha+4(2\beta+1) & 1 & \dots \\ 0 & 0 & 0 & 0 & 5(\alpha+4\beta)(\beta+1) & \alpha+5(2\beta+1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The family of orthogonal polynomials is then given by

$$P_{n+1}^B(x) = (x - \alpha - n(2\beta+1))P_n^B(x) - n(\alpha + (n-1)\beta)(\beta+1)P_{n-1}^B(x),$$

with $P_0^B(x) = 1$, $P_1^B(x) = x - \alpha$. □

Re-writing the above three-term recurrence as

$$P_{n+1}^B(x) = (x - ((2\beta+1)n + \alpha))P_n^B(x) - n(\beta(\beta+1)n + (\beta+1)(\alpha-\beta)),$$

we see that the family of orthogonal polynomials $P_n^B(x)$ is a family of Sheffer orthogonal polynomials. The moments μ_n^B for this family of polynomials have generating function given by

$$\frac{1}{(1-\beta(e^x-1))^{\alpha/\beta}}.$$

Proposition 13.

$$\mu_n^B = \sum_{j=0}^{n} j! \binom{\frac{\alpha}{\beta}+j-1}{j} \beta^j S(n,j).$$

Proof. We have

$$\begin{aligned}
\mu_n^B &= [x^n]\frac{1}{(1-\beta(e^x-1))^{\alpha/\beta}}\\
&= n!\,[x^n](1-\beta(e^x-1))^{-\alpha/\beta}\\
&= n![x^n]\sum_j \binom{-\alpha/\beta}{j}(-1)^j\beta^j(e^x-1)^j\\
&= n![x^n]\sum_j \binom{\alpha/\beta+j-1}{j}\beta^j(e^x-1)^j\\
&= \sum_j \binom{\alpha/\beta+j-1}{j}\beta^j n!\,[x^n](e^x-1)^j\\
&= \sum_{j=0}^n j!\binom{\frac{\alpha}{\beta}+j-1}{j}\beta^j S(n,j).
\end{aligned}$$

□

Proposition 14. *The generating function of μ_n^B may be expressed as the following continued fraction.*

$$\cfrac{1}{1-\alpha x-\cfrac{\alpha(\beta+1)x^2}{1-(\alpha+2\beta+1)x-\cfrac{2(\alpha+\beta)(\beta+1)x^2}{1-(\alpha+2(2\beta+1))x-\cfrac{3(\alpha+2\beta)(\beta+1)x^2}{1-\cdots}}}}.$$

Corollary 15. *The Hankel transform of the moment sequence μ_n^B is given by*

$$h_n^B=\prod_{k=1}^n((k(\alpha+(k-1)\beta)(\beta+1))^{n-k+1}.$$

5 Orthogonal polynomials and $C(\alpha,\beta,\gamma)$

We recall that

$$C(\alpha,\beta,\gamma)=\left[\frac{e^{-\gamma x}}{(1+\beta x)^{\alpha/\beta}},\frac{x}{1+\beta x}\right],$$

with

$$C(\alpha,\beta,\gamma)^{-1}=\left[\frac{e^{\frac{\gamma x}{1-\beta x}}}{(1-\beta x)^{\alpha/\beta}},\frac{x}{1-\beta x}\right].$$

Proposition 16. *The production array of C^{-1} is a four-diagonal matrix. When $\gamma = 0$, C is the coefficient array of a family of orthogonal polynomials.*

Proof. We find that the production matrix of C^{-1} is given by

$$\left(\begin{array}{ccccccc} \alpha+\gamma & 1 & 0 & 0 & 0 & 0 & \cdots \\ \alpha\beta+2\beta\gamma & \alpha+2\beta+\gamma & 1 & 0 & 0 & 0 & \cdots \\ 2\beta^2\gamma & 2\alpha\beta+2\beta^2+4\beta\gamma & \alpha+4\beta+\gamma & 1 & 0 & 0 & \cdots \\ 0 & 6\beta^2\gamma & 3\alpha\beta+6\beta^2+6\beta\gamma & \alpha+6\beta+\gamma & 1 & 0 & \cdots \\ 0 & 0 & 12\beta^2\gamma & 4\alpha\beta+12\beta^2+8\beta\gamma & \alpha+8\beta+\gamma & 1 & \cdots \\ 0 & 0 & 0 & 20\beta^2\gamma & 5\alpha\beta+20\beta^2+10\beta\gamma & \alpha+10\beta+\gamma & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

When $\gamma = 0$ this reduces to the tri-diagonal matrix

$$\left(\begin{array}{ccccccc} \alpha & 1 & 0 & 0 & 0 & 0 & \cdots \\ \alpha\beta & \alpha+2\beta & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2\alpha\beta+2\beta^2 & \alpha+4\beta & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3\alpha\beta+6\beta^2 & \alpha+6\beta & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4\alpha\beta+12\beta^2 & \alpha+8\beta & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5\alpha\beta+20\beta^2 & \alpha+10\beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

$\square$

Proposition 17. *$C(\alpha, \beta, 0)$ is the coefficient array of the family of orthogonal polynomials $P_n^C(x)$ that satisfy the three-term recurrence*

$$P_{n+1}(x) = (x - (2\beta n + \alpha))P_n(x) - n(\beta^2 n + \beta(\alpha - \beta))P_{n-1}(x),$$

with $P_0(x) = 1$, $P_1(x) = x - \alpha$.

Again, it is evident that these are Sheffer orthogonal polynomials. The moments μ_n^C of the orthogonal polynomials with coefficient array $C(\alpha, \beta, 0)$ are given by the first column elements of $C(\alpha, \beta, 0)^{-1}$. They have e.g.f. equal to $\frac{1}{(1-\beta x)^{\alpha/\beta}}$, and are equal to

$$\mu_n^C = \prod_{k=0}^{n-1} (\alpha + k\beta).$$

Corollary 18. *The moments μ_n^C of the orthogonal polynomials with coefficient array $C(\alpha, \beta, 0)$ have generating function given by the continued fraction*

$$\cfrac{1}{1 - \alpha x - \cfrac{\alpha\beta x^2}{1 - (\alpha + 2\beta)x - \cfrac{2\beta(\alpha+\beta)x^2}{1 - (\alpha + 4\beta)x - \cfrac{3\beta(\alpha + 2\beta)x^2}{1 - \cdots}}}}.$$

Corollary 19. *The Hankel transform of μ_n^C is given by*

$$h_n = \prod_{k=1}^{n} (k\beta(\alpha + (k-1)\beta))^{n-k+1}.$$

Note that the general (n,k)-th term of the exponential Riordan array

$$C(\alpha,\beta,0) = \left[\frac{1}{(1+\beta x)^{\alpha/\beta}}, \frac{x}{1+\beta x}\right]$$

is given by

$$C_{n,k} = \frac{n!}{k!}\binom{\frac{\alpha}{\beta}+n-1}{n-k}(-\beta)^{n-k},$$

and hence we have the explicit expression

$$P_n^C(x) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{\frac{\alpha}{\beta}+n-1}{n-k}(-\beta)^{n-k}x^k.$$

6 The rising factorials as moments

The triangular matrix of Stirling numbers of the first kind $\left[{n \atop k}\right]$ can be represented as

$$[0,1,1,2,2,3,3,\ldots] \quad \Delta \quad [1,0,1,0,1,0,\ldots]$$

in the Deleham notation [8]. This means that the triangle has bi-variate generating function given by

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{x+xy}{1-\cfrac{2x}{1-\cfrac{2x+xy}{1-\cfrac{3x}{1-\cfrac{3x+xy}{1-\cdots}}}}}}}}.$$

This is equivalent to

$$\cfrac{1}{1-xy-\cfrac{x^2y}{1-x(y+2)-\cfrac{2x^2(y+1)}{1-x(y+4)-\cfrac{3x^2(y+2)}{1-x(y+6)-\cfrac{4x^2(y+3)}{1-\cdots}}}}}.$$

Viewed as a power series in x, this generating function expands to

$$1, y, y(y+1), y(y+1)(y+2), y(y+1)(y+3)(y+3), \ldots,$$

which is the rising factorial $y^{(n)}$ in y. This form of the generating function thus exhibits $y^{(n)}$ as the moments of a family of orthogonal polynomials. We have the following.

Proposition 20. *The rising factorials $y^{(n)}$ are the moments of the family of orthogonal polynomials whose coefficient array is given by*

$$\left[\frac{1}{(1+x)^y}, \frac{x}{1+x}\right].$$

Proof. As a first step, we show that the inverse matrix

$$\left[\frac{1}{(1+x)^y}, \frac{x}{1+x}\right]^{-1} = \left[\frac{1}{(1-x)^y}, \frac{x}{1-x}\right]$$

has a tri-diagonal production matrix that corresponds to the quadratic continued fraction above. Thus let

$$[g, f] = \left[\frac{1}{(1-x)^y}, \frac{x}{1-x}\right].$$

Then

$$f'(x) = \frac{1}{(1-x)^2}, \quad \bar{f}(x) = \frac{1}{1+x}.$$

Thus

$$A(x) = f'(\bar{f})(x) = (1+x)^2.$$

We have

$$g'(x) = \frac{y}{(1-x)^{y+1}},$$

and hence

$$Z(x) = \frac{g'(\bar{f})(x)}{g(\bar{f})(x)} = y(1+x).$$

Thus the production matrix of

$$\left[\frac{1}{(1+x)^y}, \frac{x}{1+x}\right]^{-1} = \left[\frac{1}{(1-x)^y}, \frac{x}{1-x}\right]$$

is generated by

$$e^{xz}(y(1+x) + z(1+x)^2).$$

Thus the production matrix is tri-diagonal, beginning

$$\left(\begin{array}{ccccccc} y & 1 & 0 & 0 & 0 & 0 & \cdots \\ y & y+2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2(y+1) & y+4 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3(y+2) & y+6 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4(y+3) & y+8 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5(y+4) & y+10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Finally we note that

$$\frac{1}{(1-x)^y}$$

is the e.g.f. of the rising factorials $y^{(n)}$. $\square$

We now consider the generalized rising factorials $y^{(n;\beta)}$ given by the sequence

$$1, y, y(y+\beta), y(y+\beta)(y+2\beta), y(y+\beta)(y+2\beta)(y+3\beta), \ldots.$$

In a similar manner to the foregoing, we can show the following.

Proposition 21. *The rising factorials $y^{(n;\beta)}$ are the moments of the family of orthogonal polynomials whose coefficient array is given by*

$$\left[\frac{1}{(1+\beta x)^{y/\beta}}, \frac{x}{1+\beta x}\right].$$

The inverse matrix

$$\left[\frac{1}{(1-\beta x)^{y/\beta}}, \frac{x}{1-\beta x}\right]$$

of generalized (unsigned) Stirling numbers of the first kind can then be represented as

$$[0, \beta, \beta, 2\beta, 2\beta, 3\beta, 3\beta, \ldots] \quad \Delta \quad [1, 0, 1, 0, 1, 0, \ldots]$$

with the following equivalent expressions for its generating function:

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{\beta x}{1-\cfrac{\beta x+xy}{1-\cfrac{2\beta x}{1-\cfrac{2\beta x+xy}{1-\cfrac{3\beta x}{1-\cfrac{3\beta x+xy}{1-\cdots}}}}}}}},$$

and

$$\cfrac{1}{1-xy-\cfrac{\beta x^2 y}{1-x(y+2\beta)-\cfrac{2x^2\beta(y+\beta)}{1-x(y+4\beta)-\cfrac{3x^2\beta(y+2\beta)}{1-x(y+6\beta)-\cfrac{4x^2\beta(y+3\beta)}{1-\cdots}}}}}.$$

The foregoing prompts us to look at the number triangle

$$[0,\beta,1,2\beta,2,3\beta,3,\ldots] \quad \Delta \quad [1,0,1,0,1,0,\ldots].$$

We find the following.

Proposition 22. *The Deleham triangle*

$$[0,\beta,1,2\beta,2,3\beta,3,\ldots] \quad \Delta \quad [1,0,1,0,1,0,\ldots]$$

is equal to the exponential Riordan array

$$\left[1, \ln\left(\frac{\beta-1}{\beta-e^{(\beta-1)x}}\right)\right].$$

We note the following link to the Eulerian numbers [4]. We have

$$\frac{d}{dx}\ln\left(\frac{\beta-1}{\beta-e^{(\beta-1)x}}\right)=\frac{e^{\beta x}(1-\beta)}{e^{\beta x}-\beta e^{x}}.$$

This latter expression is the generating function of the triangle A123125 of Eulerian numbers $A_{n,k}$ given by

$$A_{n,k}=\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}(k-i)^n.$$

The production matrix of the exponential Riordan array $\left[1, \ln\left(\frac{\beta-1}{\beta-e^{(\beta-1)x}}\right)\right]$ is of interest. It begins

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & \beta & 1 & 0 & 0 & 0 & \ldots \\ 0 & \beta & 2\beta & 1 & 0 & 0 & \ldots \\ 0 & \beta & 3\beta & 3\beta & 1 & 0 & \ldots \\ 0 & \beta & 4\beta & 6\beta & 4\beta & 1 & \ldots \\ 0 & \beta & 5\beta & 10\beta & 10\beta & 5\beta & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The inverse of the matrix $\left[1, \ln\left(\frac{\beta-1}{\beta-e^{(\beta-1)x}}\right)\right]$ is given by

$$\left[1, \frac{1}{\beta-1}\left(\ln\left(\beta(e^x-1)+1\right)\right)-x\right)\right].$$

Then

$$\frac{d}{dx}\frac{1}{\beta-1}\left(\ln\left(\beta(e^x-1)+1\right)\right)-x\right)=\frac{1}{\beta e^x-\beta+1}$$

is the generating function of the triangle with (n,k)-th element

$$(-1)^k k! S(n,k).$$

We finish this section by noting the following. The Stirling numbers of the second kind appear in the normal ordering of powers [11] $\left(x\frac{d}{dx}\right)^n$ as

$$\left(x\frac{d}{dx}\right)^n=\sum_{m=0}^n S(n,m)x^m\left(\frac{d}{dx}\right)^m.$$

In analogy to this, we define a family of polynomials $P_n(z;k)$ by setting

$$P_n(z;k)=\frac{1}{z^k}\left(z(1+z)\frac{d}{dz}\right)^n z^k.$$

Then we can show that the coefficient array of the family $P_n(z;k)$ is given by

$$[k,0,k+1,0,k+2,0,\ldots] \quad \Delta \quad [k,1,k+1,2,k+2,3,\ldots].$$

In other words, the generating function for the coefficient array of $P_n(z;k)$ is given by

$$\cfrac{1}{1-\cfrac{kx(1+z)}{1-\cfrac{xz}{1-\cfrac{(k+1)x(1+z)}{1-\cfrac{2xz}{1-\cfrac{(k+2)x(1+z)}{1-\cfrac{3xz}{1-\cdots}}}}}}}.$$

We can equivalently write this continued fraction in the form

$$\cfrac{1}{1-k(1+z)x-\cfrac{kz(1+z)x^2}{1-(z+(k+1)(1+z))x-\cfrac{2(k+1)z(1+z)x^2}{1-(2z+(k+2)(1+z))x-\cfrac{3(k+2)z(1+z)x^2}{1-\cdots}}}}.$$

This exhibits the polynomial sequence $P_n(z;k)$ as the sequence of moments of a family of orthogonal polynomials. The Hankel transform of the sequence $P_n(z;k)$ is seen to be

$$(z(1+z))^{\binom{n+1}{2}} \prod_{i=0}^{n} i!(k+i)^{n-i}.$$

7 Generalized Toda equations

Generalized Stirling numbers have been of interest to the physics community due to their association to the boson normal ordering question [11]. In this section, we look at another physics inspired perspective on our generalized Stirling numbers. The Toda chain equations are an important example of integrable system [2]. For our purposes, the restricted Toda chain equation [7, 27, 34] is simply described by

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad n = 1, 2, \dots \quad \dot{b}_n = u_{n+1} - u_n, n = 0, 1, \dots \tag{3}$$

We wish to show that the Jacobi coefficients of $B(\beta, \beta, 0)$, that is, of

$$B(\beta, \beta, 0) = \left[\frac{1}{1+tx}, \ln\left(\frac{1+(t+1)x}{1+\beta x}\right)\right],$$

obey a generalized restricted Toda equation (we use t as the parameter in this section to conform to standard notation regarding the Toda equations).

In this special case, the production matrix of B^{-1} is given by

$$\begin{pmatrix} t & 1 & 0 & 0 & 0 & 0 & \dots \\ t(t+1) & 3t+1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4t(t+1) & 5t+2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9t(t+1) & 7t+3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16t(t+1) & 9t+4 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25t(t+1) & 11t+5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We then get

$$b_n = n + (2n+1)t,$$

and

$$u_n = n^2 t(t+1).$$

From this we calculate

$$b_n - b_{n-1} = (n + (2n+1)t) - (n - 1 + (2n-1)t) = 2t + 1.$$

Then

$$u_n(b_n - b_{n-1}) = n^2 t(t+1)(2t+1),$$

while

$$\dot{u}_n = n^2(2t+1).$$

Thus we get

$$\dot{u}_n = \frac{1}{t(t+1)} u_n(b_n - b_{n-1}). \tag{4}$$

Now

$$\dot{b}_n = 2n+1,$$

and hence we get

$$\dot{b}_n = \frac{1}{t(t+1)}(u_{n+1} - u_n). \tag{5}$$

Proposition 23. *The Jacobi coefficients corresponding to $B(t,t,0)$ satisfy the generalized restricted Toda equations (4) and (5).*

We also have in this case

$$\frac{dP^B_{n+1}(x,t)}{dt} = -(n+1)^2 P^B_n(x,t) = -\frac{u_{n+1}}{t(t+1)} P^B_n(x,t).$$

The moments of the family of orthogonal polynomials P^B_n in this case are given by

$$\mu_n(t) = \sum_{j=0}^{n} j! t^j S(n,j).$$

The Hankel transform of $\mu_n(t)$ is given by

$$h_n(t) = \prod_{k=1}^{n} (k!)^2 (t(t+1))^{n-k+1} = (t(t+1))^{\binom{n+1}{2}} \prod_{k=1}^{n} (k!)^2.$$

We now turn to the matrix $C(t,t,0)$. In this case, the Jacobi coefficients are given by

$$b_n = (2n+1)t, \quad u_n = n^2 t^2.$$

Then we get

$$\dot{u}_n = \frac{1}{t^2} u_n(b_n - b_{n-1}) \tag{6}$$

and

$$\dot{b}_n = \frac{1}{t^2}(u_{n+1} - u_n). \tag{7}$$

Proposition 24. *The Jacobi coefficients corresponding to $C(t,t,0)$ satisfy the generalized restricted Toda chain equations (6) and (7).*

We also have in this case

$$\frac{dP^C_{n+1}(x,t)}{dt} = -(n+1)^2 P^C_n(x,t) = -\frac{u_{n+1}}{t^2} P^C_n(x,t).$$

8 A Lang-inspired exponential Riordan array

We define the following array, whose reversal has already been studied [25] in the context of generalized Stirling numbers. Thus consider the exponential Riordan array defined by

$$L(h,s)=\left[1, \frac{1}{h(s-1)}\left(1-(1-hsx)^{\frac{s-1}{s}}\right)\right].$$

Proposition 25. *The general (n,k)-th term of the exponential Riordan array $L(h,s)$ is given by*

$$L_{n,k}=\frac{n!}{k!}\frac{(-hs)^n}{(h(s-1))^k}\sum_{j=0}^k \binom{k}{j}\binom{\frac{(s-1)j}{s}}{n}(-1)^j.$$

The inverse matrix $L(h,s)^{-1}$ is equal to

$$L(h,s)^{-1}=\left[1, \frac{1}{hs}\left(1-(1-h(s-1)x)^{\frac{s}{s-1}}\right)\right].$$

The general (n,k)-th element of the inverse $L(h,s)^{-1}$ is given by

$$\frac{n!}{k!}\frac{(-h(s-1))^n}{(hs)^k}\sum_{j=0}^k \binom{k}{j}\binom{\frac{sj}{s-1}}{n}(-1)^j.$$

Proof. The general (n,k)-th element of $L(h,s)$ is given by $\frac{n!}{k!}$ times

$$\begin{aligned}
[x^n]\left(\frac{1}{h(s-1)}-\frac{(1-hsx)^{\frac{s-1}{s}}}{h(s-1)}\right)^k &= \frac{1}{(h(s-1))^k}[x^n]\sum_{j=0}^k \binom{k}{j}(-1)^j(1-hsx)^{\frac{(s-1)j}{s}}\\
&= \frac{1}{(h(s-1))^k}[x^n]\sum_{j=0}^k \binom{k}{j}(-1)^j\sum_{i=0}\binom{\frac{(s-1)j}{s}}{i}(-hs)^i x^i\\
&= \frac{(-hs)^n}{(h(s-1))^k}\sum_{j=0}^k \binom{\frac{(s-1)j}{s}}{n}(-1)^j.
\end{aligned}$$

The expression for the inverse is a direct consequence of the definition of the inverse of an exponential Riordan array. The general term of the inverse is given by $\frac{n!}{k!}$ times

$$\begin{aligned}
[x^n]\left(\frac{1}{hs}\left(1-(1-h(s-1)x)^{\frac{s}{s-1}}\right)\right)^k &= \frac{1}{(hs)^k}[x^n]\sum_{j=0}^k \binom{k}{j}(-1)^j(1-h(s-1)x)^{\frac{sj}{s-1}}\\
&= \frac{1}{(hs)^k}[x^n]\sum_{j=0}^k \binom{k}{j}(-1)^j\sum_{i=0}\binom{\frac{sj}{s-1}}{i}(-h(s-1))^i x^i\\
&= \frac{(-h(s-1))^n}{(hs)^k}\sum_{j=0}^k \binom{k}{j}\binom{\frac{sj}{s-1}}{n}(-1)^j.
\end{aligned}$$

$\square$

Note that we have

$$L(1,0)=S, \quad L(1,1)=\left(\begin{bmatrix} n \\ k \end{bmatrix}\right).$$

9 Appendix: exponential Riordan arrays

The *exponential Riordan group* [10, 15, 17], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x)=g_0+g_1x+g_2x^2+\cdots$ and $f(x)=f_1x+f_2x^2+\cdots$ where $g_0 \neq 0$ and $f_1 \neq 0$. We usually assume that

$$g_0=f_1=1.$$

The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. The group law is given by

$$[g,f]\cdot[h,l]=[g(h\circ f), l\circ f].$$

The identity for this law is $I=[1,x]$ and the inverse of $[g,f]$ is $[g,f]^{-1}=[1/(g\circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix $[g,f]$, and $\mathbf{u}=(u_n)_{n\geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}(x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g,f]$ have exponential generating function given by $g(x)e^{f(x)}$ since the sequence $1,1,1,\ldots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, the binomial matrix $\mathbf{B}$ with (n,k)-th element $\binom{n}{k}$ is given by $\mathbf{B}=[e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^xe^x=e^{2x}$, as expected (e^{2x} is the e.g.f. of 2^n).

To each exponential Riordan array $L=[g,f]$ is associated [16, 17] a matrix P called its *production* matrix, which has bivariate g.f. given by

$$e^{xy}(Z(x)+A(x)y)$$

where

$$A(x)=f'(\bar{f}(x)), \quad Z(x)=\frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

We have

$$P=L^{-1}\bar{L}$$

where $\bar{L}$ [28, 36] is the matrix L with its top row removed.

The ordinary Riordan group is described in [29].

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