Journal of Integer Sequences, Vol. 16 (2013),
Article 13.7.3

On Intervals $(kn, (k+1)n)$ Containing a Prime for All $n > 1$

Vladimir Shevelev
Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva 84105
Israel
shevelev@bgu.ac.il

Charles R. Greathouse IV
3214 Whitethorn Road
Cleveland Heights, OH 44118
USA
charles.greathouse@case.edu

Peter J. C. Moses
Moparmatic Co. 1154 Evesham Road
Astwood Bank, Redditch, Worcestershire
B96 6DT England
mows@mopar.freeserve.co.uk

Abstract

We study values of k for which the interval $(kn, (k+1)n)$ contains a prime for every $n > 1$. We prove that the list of such integers k includes $1, 2, 3, 5, 9, 14$ and no others, at least for $k \le 100,000,000$. Moreover, for every known k in this list, we give a good upper bound for the smallest $N_k(m)$, such that if $n \ge N_k(m)$, then the interval $(kn, (k+1)n)$ contains at least m primes.

1 Introduction and main results

In 1850, Chebyshev proved the famous Bertrand postulate (1845) that every interval $[n, 2n]$ contains a prime (for a very elegant version of his proof, see Redmond [10, Theorem 9.2]).

Other nice proofs were given by Ramanujan in 1919 [8] and Erdős in 1932 (reproduced in Erdős and Surányi [4, p. 171–173]). In 2006, El Bachraoui [1] proved that every interval $[2n, 3n]$ contains a prime, while Loo [6] proved the same statement for every interval $[3n, 4n]$. Moreover, Loo found a lower bound for the number of primes in the interval $[3n, 4n]$. In 1952, Nagura [7] proved that there is always a prime between n and $\frac{6}{5}n$ for $n \geq 25$. From his result, it follows that the interval $[5n, 6n]$ always contains a prime. In this paper we prove the following:

Theorem 1. *The list of integers k for which every interval $(kn, (k+1)n)$, $n > 1$, contains a prime includes $k = 1, 2, 3, 5, 9, 14$ and no others, at least for $k \leq 100,000,000$.*

To prove Theorem 1, in Section 3 we introduce and study the so-called k-Chebyshev primes. We give them, and the generalized Ramanujan primes, the best estimates of the form p_{tn}, where p_n is the n-th prime. Note that the core of the proof of Theorem 1 is Proposition 9, which in turn depends on Proposition 8.

In passing, for every $k = 1, 2, 3, 5, 9, 14$, we give an algorithm for finding the smallest $N_k(m)$, such that for $n \geq N_k(m)$, the interval $(kn, (k+1)n)$ contains at least m primes.

Proof of Theorem 1 is completed in Section 7 by computer research of sequence A218831 in [13].

2 Case $k = 1$

Ramanujan [8] not only proved Bertrand's postulate, but also provided the smallest integers $\{R(m)\}$, such that if $x \geq R(m)$, then the interval $\left(\frac{x}{2}, x\right]$ contains at least m primes, or equivalently, $\pi(x) - \pi(x/2) \geq m$. It is easy to see that it is sufficient to consider *integer* x, and it is also evident that every term of $\{R(m)\}$ is prime. The numbers $\{R(m)\}$ are called *Ramanujan primes* [14]. It is sequence A104272 in [13]:

$$2, 11, 17, 29, 41, 47, 59, 67, 71, 97, \ldots \tag{1}$$

Since $\pi(x) - \pi(x/2)$ is not a monotonic function, to calculate the Ramanujan numbers one should have an effective upper bound for $R(m)$. Ramanujan [8] showed that

$$\pi(x) - \pi(x/2) > \frac{1}{\ln x}\left(\frac{x}{6} - 3\sqrt{x}\right),\ x > 300. \tag{2}$$

In particular, for $x \geq 324$, the left-hand side is positive and thus ≥ 1. Using direct descent, he found that $\pi(x) - \pi(x/2) \geq 1$ from $x \geq 2$. Thus $R(1) = 2$, which proves the Bertrand postulate. Further, e.g., for $x \geq 400$, the left-hand side of (2) is more than 1 and thus ≥ 2. Again, using direct descent, he found that $\pi(x) - \pi(x/2) \geq 2$ from $x \geq 11$. Thus $R(2) = 11$, etc. Sondow [14] found that $R(m) < 4m\ln(4m)$ and conjectured that

$$R(m) < p_{3m} \tag{3}$$

which was proved by Laishram [5]. Since, for $n \geq 2$, $p_n \leq en \ln n$ (cf. [3, Section 4]), then (3) yields $R(m) \leq 3em \ln(3m)$, $m \geq 1$. Let $x = 2n$. If $2n \geq R(m)$, then $\pi(2n) - \pi(n) \geq m$. Thus the interval $(n, 2n)$ contains at least m primes, if

$$n \geq \left\lceil \frac{R(m)+1}{2} \right\rceil = \begin{cases} 2, & \text{if } m = 1; \\ \frac{R(m)+1}{2}, & \text{if } m \geq 2. \end{cases}$$

Let $N_1(m)$ denote the smallest number such that if $n \geq N_1(m)$, then the interval $(n, 2n)$ contains at least m primes. It is clear that $N_1(1) = R(1) = 2$. If $m \geq 2$, formally the condition $x = 2n \geq 2N_1(m)$ is not stronger than the condition $x \geq R(m)$, since the latter holds for x even and odd. Therefore, for $m \geq 2$, we have $N_1(m) \leq \frac{R(m)+1}{2}$. Let us show that, in fact, we have the equality

Proposition 2. *For $m \geq 2$,*

$$N_1(m) = \frac{R(m)+1}{2}. \tag{4}$$

Proof. Note that the interval $\left(\frac{R(m)-1}{2}, R(m)-1\right)$ cannot contain more than $m-1$ primes. Indeed, it is an interval of type $\left(\frac{x}{2}, x\right)$ for integer x and the following such interval is $\left(\frac{R(m)}{2}, R(m)\right)$. By definition, $R(m)$ is the *smallest* number such that if $x \geq R(m)$, then $\{(\frac{x}{2}, x)\}$ contains $\geq m$ primes. Therefore, the supposition that the interval $\left(\frac{R(m)-1}{2}, R(m)-1\right)$ contains $\geq m$ primes contradicts the minimality of $R(m)$. Since the following interval of type $(y,\ 2y)$ with integer $y \geq \frac{R(m)-1}{2}$ is $\left(\frac{R(m)+1}{2},\ R(m)+1\right)$, Eq. (4) then follows. $\square$

So the sequence $\{N_1(m)\}$, by (1), is A084140 in [13]:

$$2, 6, 9, 15, 21, 24, 30, 34, 36, 49, \ldots \tag{5}$$

3 Generalized Ramanujan numbers

Our research is based on a generalization of Ramanujan's method. With this aim, we define generalized Ramanujan numbers (cf. [12, Section 10], and the earlier comment in A164952 in [13]).

Definition 3. Let $v > 1$ be a real number. A v-Ramanujan number, $R_v(m)$, is the smallest integer such that if $x \geq R_v(m)$, then $\pi(x) - \pi(x/v) \geq m$.

It is known [10] that all v-Ramanujan numbers are primes. In particular, $R_2(m) = R(m)$, $m = 1, 2, \ldots$ are the proper Ramanujan primes.

Definition 4. For a real number $v > 1$ the *v-Chebyshev number*, $C_v(m)$, is the smallest integer such that if $x \geq C_v(m)$, then $\vartheta(x) - \vartheta(x/v) \geq m \ln x$, where $\vartheta(x) = \sum_{p \leq x} \ln p$ is the Chebyshev function.

Since $\frac{\vartheta(x)-\vartheta(x/v)}{\ln x}$ can increase by 1 only when x is prime, then all v-Chebyshev numbers are primes.

Proposition 5. *We have*

$$R_v(m) \le C_v(m). \tag{6}$$

Proof. Let $x \ge C_v(m)$. Then we have

$$m \le \frac{\vartheta(x) - \vartheta(x/v)}{\ln x} = \sum_{\frac{x}{v}<p\le x} \frac{\ln p}{\ln x} \le \sum_{\frac{x}{v}<p\le x} 1 = \pi(x) - \pi(x/v). \tag{7}$$

Thus, if $x \ge C_v(m)$, then *always* $\pi(x) - \pi(x/v) \ge m$. By Definition 3, this means that $R_v(m) \le C_v(m)$. □

Now we give upper bounds for $C_v(m)$ and $R_v(m)$.

Proposition 6. *Let $x = x_v(m) \ge 2$ be any number for which*

$$\frac{x}{\ln x}\left(1 - \frac{1300}{\ln^4 x}\right) \ge \frac{vm}{v-1}. \tag{8}$$

Then

$$R_v(m) \le C_v(m) \le x_v(m). \tag{9}$$

Proof. We use the following inequality of Dusart [3] (see his Theorem 5.2):

$$|\vartheta(x) - x| \le \frac{1300x}{\ln^4 x},\ x \ge 2.$$

Thus we have

$$\begin{aligned}\vartheta(x) - \vartheta(x/v) &\ge x\left(1 - \frac{1}{v} - 1300\left(\frac{1}{\ln^4 x} - \frac{1}{v\ln^4 \frac{x}{v}}\right)\right)\\ &\ge x\left(1 - \frac{1}{v}\right)\left(1 - \frac{1300}{\ln^4 x}\right).\end{aligned}$$

If now

$$x\left(1 - \frac{1}{v}\right)\left(1 - \frac{1300}{\ln^4 x}\right) \ge m\ln x,\ x \ge x_v(m),$$

then

$$\vartheta(x) - \vartheta(x/v) \ge m\ln x,\ x \ge x_v(m)$$

and, by Definition 4, $C_v(m) \le x_v(m)$. So, according to (6), we conclude that $R_v(m) \le x_v(m)$. □

Proposition 6 gives the terms of sequences $\{C_v(m)\},\ \{R_v(m)\}$ for every $v > 1,\ m \geq 1$. In particular, if $k = 1$ we find $\{C_2(m)\}$:

$$\begin{aligned}&11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 223,\\ &229, 233, 239, 241, 263, 269, 281, 307, 311, 347, 349, 367, 373, 401, 409,\\ &419, 431, 433, 443, \ldots\end{aligned} \tag{10}$$

This sequence requires a separate comment. We observe that up to $C_2(100) = 1489$ only two terms of this sequence ($C_2(17) = 223$ and $C_2(36) = 443$) are not Ramanujan numbers, and the sequence is missing only the following 6 Ramanujan numbers: 181,227,439,491,1283,1301 and no others up to the 104-th Ramanujan number 1489. The latter observation shows how much the ratio $\frac{\vartheta(x)}{\ln x}$ exactly approximates $\pi(x)$. Similar observations are also valid for the following sequences for $v = \frac{k+1}{k}$ (and undoubtedly require an additional special study):

for $k = 2,\ \{C_v(m)\}$,

$$13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \ldots; \tag{11}$$

for $k = 2,\ \{R_v(m)\}$,

$$2, 13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \ldots; \tag{12}$$

for $k = 3,\ \{C_v(m)\}$,

$$29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \ldots; \tag{13}$$

for $k = 3,\ \{R_v(m)\}$,

$$11, 29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \ldots; \tag{14}$$

for $k = 5,\ \{C_v(m)\}$,

$$59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \ldots; \tag{15}$$

for $k = 5,\ \{R_v(m)\}$,

$$29, 59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \ldots; \tag{16}$$

for $k = 9,\ \{C_v(m)\}$,

$$223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \ldots; \tag{17}$$

for $k = 9,\ \{R_v(m)\}$,

$$127, 223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \ldots; \tag{18}$$

for $k = 14,\ \{C_v(m)\}$,

$$307, 347, 563, 569, 733, 821, 1427, 1429, 1433, 1439, 1447, 1481, \ldots; \tag{19}$$

for $k = 14,\ \{R_v(m)\}$,

$$127, 307, 347, 563, 569, 733, 1423, 1427, 1429, 1433, 1439, 1447, \ldots \tag{20}$$

Remark 7. In fact, Dusart [3, Theorem 5.2] gives several inequalities of the form

$$|\vartheta(x) - x| \leq \frac{ax}{\ln^b x},\ x \geq x_0(a, b)$$

From a computing point of view, the values $a = 1300,\ b = 4$ from Dusart's theorem are not always the best. The analysis for $x \geq 25$ shows that the condition

$$x\left(1 - \frac{1}{v}\right)\left(1 - \frac{ax}{\ln^b x}\right) \geq m \ln x$$

is satisfied for the smallest $x_v(m) = x_v(m; a, b)$, using the following values of a and b from Dusart's theorem:

$a = 3.965,\ b = 2$ for x in range $(25,\ 7 \cdot 10^7]$;
$a = \ 1300,\ b = 4$ for x in range $(7 \cdot 10^7,\ 10^9]$;
$a = 0.001,\ b = 1$ for x in range $(10^9,\ 8 \cdot 10^9]$;
$a = \ 0.78,\ b = 3$ for x in range $(8 \cdot 10^9,\ 7 \cdot 10^{33}]$;
$a = \ 1300,\ b = 4$ for $x > 7 \cdot 10^{33}$,

which minimizes the amount of calculations for v-Chebyshev primes.

4 Bounds of type (3)

Proposition 8. *We have*

$$C_2(m - 1) \leq p_{3m},\ m \geq 2; \tag{21}$$

$$R_{\frac{3}{2}}(m) \leq p_{4m},\ m \geq 1;\ C_{\frac{3}{2}}(m - 1) \leq p_{4m},\ m \geq 2; \tag{22}$$

$$R_{\frac{4}{3}}(m) \leq p_{6m},\ m \geq 1;\ C_{\frac{4}{3}}(m - 1) \leq p_{6m},\ m \geq 2; \tag{23}$$

$$R_{\frac{6}{5}}(m) \leq p_{11m},\ m \geq 1;\ C_{\frac{6}{5}}(m - 1) \leq p_{11m},\ m \geq 2; \tag{24}$$

$$R_{\frac{10}{9}}(m) \leq p_{31m},\ m \geq 1;\ C_{\frac{10}{9}}(m - 1) \leq p_{31m},\ m \geq 2; \tag{25}$$

$$R_{\frac{15}{14}}(m) \leq p_{32m},\ m \geq 1;\ C_{\frac{15}{14}}(m - 1) \leq p_{32m},\ m \geq 2. \tag{26}$$

Proof. Firstly, let us find some values of $m_0 = m_0(k)$, such that, at least, for $m \geq m_0$ all formulas (21)–(26) hold. According to (8) and (9), it is sufficient to show that, for $m \geq m_0$, we can take p_{tm}, where $t = 3, 4, 6, 11, 31, 32$ for formulas (21)-(26) respectively, in the capacity of $x_v(m)$. As we noted in Remark 7, in order to find possibly smaller values of m_0, we use the bound

$$\frac{x}{\ln x}\left(1 - \frac{3.965}{\ln^2 x}\right) \geq \frac{vm}{v - 1} \tag{27}$$

instead of (8). In order to get $x = p_{mt}$ satisfying this inequality, note that [11]

$$p_n \geq n \ln n.$$

Therefore, it is sufficient to consider p_{mt} satisfying the inequality

$$\ln p_{tm} \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right).$$

On the other hand, for $n \geq 2$, (see [3, (4.2)])

$$\ln p_n \leq \ln n + \ln \ln n + 1.$$

Thus, it is sufficient to choose m so large that the following inequality holds

$$\ln(tm) + \ln \ln(tm) + 1 \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right),$$

or, since $1 - \frac{1}{v} = \frac{1}{k+1}$, that

$$\frac{\ln(tm) + \ln \ln(tm) + 1}{\ln(tm)(1 - \frac{3.965}{\ln^2(tm \ln(tm))})} \leq \frac{t}{k+1}. \tag{28}$$

For example, let $k = 1,\ t = 3$. We can choose $m_0 = 350$. Then the left-hand side of (28) equals $1.4976 \cdots < 1.5$. This means that at least for $m \geq 350$, the estimate (21) is valid. Using a computer for $m \leq 350$, we obtain (21) for $m \geq 2$. Other bounds are proved in the same way. □

5 Bounds and formulas for $N_k(m)$

Proposition 9.

$$N_k(1) = 2,\ k = 2, 3, 5, 9, 14. \tag{29}$$

For $m \geq 2,\ k \geq 1$,

$$N_k(m) \leq \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil; \tag{30}$$

besides, if $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$, *then*

$$N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k}{k+1} \tag{31}$$

and, if $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$, *then*

$$N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k - 1}{k+1}. \tag{32}$$

Proof. If $m \geq 2$, formally, the condition $x = (k+1)n \geq (k+1)N_k(m)$ is not stronger than the condition $x \geq R_{\frac{k+1}{k}}(m)$, since the first one is valid only for x multiple of $k+1$. Therefore, for $m \geq 2$, (30) holds. It allows calculation of terms in the sequence $\{N_k(m)\}$ for $k > 1,\ m \geq 2$. Since $N_k(1) \leq N_k(2)$, then, having $N_k(2)$, we can also prove (29) using direct calculation. Now let $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$. Note, that for $y = (R_{\frac{k+1}{k}}(m) - 1)/(k+1)$ the interval

$$(ky,\ (k+1)y) = \left(\frac{k}{k+1}\left(R_{\frac{k+1}{k}}(m) - 1\right),\ R_{\frac{k+1}{k}}(m) - 1\right) \tag{33}$$

cannot contain more than $m-1$ primes. Indeed, it is an interval of type $\left(\frac{k}{k+1}x, x\right)$ for integer x, and the following such interval is

$$\left(\frac{k}{k+1}\left(R_{\frac{k+1}{k}}(m)\right), R_{\frac{k+1}{k}}(m)\right).$$

By definition, $R_{\frac{k+1}{k}}(m)$ is the *smallest* number such that if $x \geq R_{\frac{k+1}{k}}(m)$, then $\{(\frac{k}{k+1}x, x)\}$ contains $\geq m$ primes. Therefore, the supposition that the interval (33) contains $\geq m$ primes contradicts the minimality of $R_{\frac{k+1}{k}}(m)$. Since the following interval of type $(ky,\ (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 1)$ is

$$\left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) + k),\ R_{\frac{k+1}{k}}(m) + k\right),$$

then (31) follows.

Finally, let $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$. Again, for $y = (R_{\frac{k+1}{k}}(m) - 2)/(k+1)$ the interval

$$(ky,\ (k+1)y) = \left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2),\ R_{\frac{k+1}{k}}(m) - 2\right) \tag{34}$$

cannot contain more than $m-1$ primes. Indeed, comparing interval (34) with interval (33), we see that they contain the same integers except for $R_{\frac{k+1}{k}}(m) - 2$, which is multiple of $k+1$. Therefore, they contain the same number of primes, and this number does not exceed $m-1$. Again, since the following interval of type $(ky,\ (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2)$ is

$$\left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) + k - 1),\ R_{\frac{k+1}{k}}(m) + k - 1\right),$$

then (32) follows. □

As a corollary from (29), (31) and (32), we obtain the following formula in case $k = 2$.

Proposition 10.

$$N_2(m) = \begin{cases} 2, & \text{if } m = 1; \\ \left\lceil \frac{R_{\frac{3}{2}}(m)}{3} \right\rceil, & \text{if } m \geq 2. \end{cases} \tag{35}$$

Note that, if $k \geq 3$ and $R_{\frac{k+1}{k}}(m) \equiv j \pmod{k+1}$, $3 \leq j \leq k$, then, generally speaking, (30) is not an equality. Evidently, $N_k(m) \geq N_k(m-1)$, and it is interesting that the equality is attainable (see sequences (37)–(40) below).

Example 11. Let $k = 3$, $m = 2$. Then $v = \frac{4}{3}$ and, by (14), $R_{\frac{4}{3}}(2) = 29 \equiv 1 \pmod 4$. Therefore, by (31), $N_3(2) = \frac{29+3}{4} = 8$. Indeed, interval $(3 \cdot 7,\ 4 \cdot 7)$ already contains only prime 23.

Example 12. Let $k = 3$, $m = 3$. Then, by (14), $R_{\frac{4}{3}}(3) = 59 \equiv 3 \pmod 4$. Here $N_3(3) = 11$ which is essentially less than $\left\lceil R_{\frac{4}{3}}(3)/4 \right\rceil = 15$. Indeed, each interval

$$(3 \cdot 15,\ 4 \cdot 15),\ (3 \cdot 14,\ 4 \cdot 14),\ (3 \cdot 13,\ 4 \cdot 13),\ (3 \cdot 12,\ 4 \cdot 12),\ (3 \cdot 11,\ 4 \cdot 11)$$

contains more than 2 primes and only interval $(3 \cdot 10,\ 4 \cdot 10)$ contains only 2 primes.

In any case, Proposition 9 allows us to calculate terms of sequence $\{N_k(m)\}$ for every considered value of k. So, we obtain the following few terms of $\{N_k(m)\}$:

for $k = 2$,

$$2, 5, 13, 14, 23, 25, 33, 43, 46, 58, 60, 61, 71, 77, 80, 88, 103, 104, \ldots; \tag{36}$$

for $k = 3$,

$$2, 8, 11, 17, 26, 38, 40, 41, 48, 57, 68, 68, 70, 87, 96, 100, 108, 109, \ldots; \tag{37}$$

for $k = 5$,

$$2, 7, 17, 24, 25, 38, 41, 58, 59, 64, 65, 73, 95, 97, 103, 106, 107, 108, \ldots; \tag{38}$$

for $k = 9$,

$$2, 14, 23, 23, 34, 36, 57, 58, 60, 60, 77, 86, 100, 100, 102, 123, 149, \ldots; \tag{39}$$

for $k = 14$,

$$2, 11, 24, 37, 38, 39, 50, 96, 96, 96, 96, 97, 97, 125, 125, 132, 178, 178, \ldots \tag{40}$$

Remark 13. If, as in [1, 6], instead of intervals $(kn,\ (k+1)n)$, we consider intervals $[kn,\ (k+1)n]$, then sequences (5) and (36)–(38) would begin with 1.

6 Method of small intervals

If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $(x,\ (1+\frac{1}{\Delta})x]$ contains a prime, then we can calculate a bounded number of the first terms of sequences (5) and (36)–(40). Indeed, put $x_1 = kn$, such that $n \geq \frac{x_0}{k}$. Then $(k+1)n = \frac{k+1}{k}x_1$ and, if $1+\frac{1}{\Delta} < \frac{k+1}{k}$, i.e., $\Delta > k$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})x_1\right] \subset (kn, (k+1)n).$$

Thus, if $n \geq \frac{x_0}{k}$, then the interval $(kn, (k+1)n)$ contains a prime, and using method of finite descent, we can find $N_k(1)$. Further, put $x_2 = (1+\frac{1}{\Delta})x_1$. Then interval $(x_2,\ (1+\frac{1}{\Delta})x_2]$ also contains a prime. Thus the union

$$\left(x_1,\ (1+\frac{1}{\Delta})x_1\right] \cup \left(x_2,\ (1+\frac{1}{\Delta})x_2\right] = \left(x_1, (1+\frac{1}{\Delta})^2 x_1\right]$$

contains at least two primes. This means that if $(1+\frac{1}{\Delta})^2 x_1 < (k+1)n$ or $(1+\frac{1}{\Delta})^2 < 1+\frac{1}{k}$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})^2 x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least two primes; again, using method of finite descent, we can find $N_k(2)$ etc. If $(1+\frac{1}{\Delta})^m < 1+\frac{1}{k}$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})^m x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least m primes and we can find $N_k(m)$. In this way, we can find $N_k(m)$ for $m < \frac{\ln(1+\frac{1}{k})}{\ln(1+\frac{1}{\Delta})}$. In 2002, Ramaré and Saouter [9] proved that the interval $(x(1-28314000^{-1}),\ x)$ always contains a prime if $x > 10726905041$, or, equivalently, the interval $(x,\ (1+28313999^{-1})x)$ contains a prime if $x > 10726905419$. This means that, e.g., we can find $N_{14}(m)$ for $m \leq 1954471$. Unfortunately, this method cannot give the exact bounds and formulas for $N_k(m)$ as (30)–(32).

We can also consider a more general application of this method. Consider a fixed infinite set P of primes which we call P-primes. Furthermore, consider the following generalization of v-Ramanujan numbers.

Definition 14. For $v > 1$, a (v, P)-Ramanujan number, $R_v^{(P)}(m)$, is the smallest integer such that if $x \geq R_v^{(P)}(m)$, then $\pi_P(x) - \pi_P(x/v) \geq m$, where $\pi_P(x)$ is the number of P-primes not exceeding x.

Note that every (v, P)-Ramanujan number is P-prime. If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $\left(x,\ (1+\frac{1}{\Delta})x\right]$ contains a P-prime, then using the above described algorithm, we can calculate a bounded number of the first (v, P) -Ramanujan numbers. For example, let P be the set of primes $p \equiv 1 \pmod 3$. From the result of Cullinan and Hajir [2] it follows, in particular, that for $x \geq 106706$, the interval $(x,\ 1.048x)$ contains a P-prime. Using the same algorithm, we can calculate the first 14 $(2, P)$-Ramanujan numbers. They are

$$7, 31, 43, 67, 97, 103, 151, 163, 181, 223, 229, 271, 331, 337. \tag{41}$$

Analogously, if P is the set of primes $p \equiv 2 \pmod 3$, then the sequence of $(2, P)$-Ramanujan numbers begins

$$11, 23, 47, 59, 83, 107, 131, 167, 227, 233, 239, 251, 263, 281, \ldots; \tag{42}$$

if P is the set of primes $p \equiv 1 \pmod 4$, then the sequence of $(2, P)$-Ramanujan numbers begins

$$13, 37, 41, 89, 97, 109, 149, 229, 233, 241, 257, 277, 281, 317, \ldots; \tag{43}$$

and, if P is the set of primes $p \equiv 3 \pmod 4$, then the sequence of $(2, P)$-Ramanujan numbers begins

$$7, 23, 47, 67, 71, 103, 127, 167, 179, 191, 223, 227, 263, 307, \ldots; \tag{44}$$

Let $N_k^{(P)}(m)$ denote the smallest number such that for $n \geq N_k^{(P)}(m)$, the interval $(kn, (k+1)n)$ contains at least m P-primes. It is easy to see that formulas (30)–(32) hold for $N_k^{(P)}(m)$ and $R_{\frac{k+1}{k}}^{(P)}(m)$. In particular, in cases $k = 1, 2$ we have the formulas

$$N_1^{(P)}(m) = \frac{R_2^{(P)}(m) + 1}{2}, \quad N_2^{(P)}(m) = \left\lceil \frac{R_{\frac{3}{2}}^{(P)}(m)}{3} \right\rceil. \tag{45}$$

Therefore, the following sequences for $N_1^{(P)}(m)$, correspond to sequences (41)–(44) respectively:

$$4, 16, 22, 34, 49, 52, 76, 82, 91, 112, 115, 136, 166, 169, \ldots; \tag{46}$$

$$6, 12, 24, 30, 42, 54, 66, 84, 114, 117, 120, 126, 132, 141, \ldots; \tag{47}$$

$$7, 19, 21, 45, 49, 55, 75, 115, 117, 121, 129, 139, 141, 159, \ldots; \tag{48}$$

$$4, 12, 24, 34, 36, 52, 64, 84, 90, 96, 112, 114, 132, 154, \ldots \tag{49}$$

7 The proof of Theorem 1

For $k \geq 1$, let $a(k)$ denote the least integer $n > 1$ for which the interval $(kn,\ (k+1)n)$ contains no prime; if no such n exists, we put $a(k) = 0$. Taking into account (29), note that

$a(k) = 0$ for $k = 1, 2, 3, 5, 9, 14$. Consider sequence $\{a(k)\}$. Its first few terms are (A218831 in [13])

$$0, 0, 0, 2, 0, 4, 2, 3, 0, 2, 3, 2, 2, 0, 6, 2, 2, 3, 2, 6, 3, 2, 4, 2, 2, 7, 2, 2, 4, 3, \ldots \tag{50}$$

Calculations of $a(k)$ for $k \in [1, 15]$, except for $k = 1, 2, 3, 5, 9, 14$, give positive values of $a(k)$. Computer calculations of $a(k)$ in the range $\{16, \ldots, 10^8\}$ show that all values of $a(k)$ in this range are positive and belong to the interval $[2, 16]$. This completes the proof. $\square$

In conclusion, we present a distribution of numbers of values $a(k) = 2, 3, \ldots, 16$ within intervals $\{[1, 10^7(i)]\}, i = 1, \ldots, 10$. All these numerical results are obtained using the following *Mathematica* program:

```
start=2;cutOff=100;
a218831=Table[
        NestWhile[#+1&,start ,
                Union[PrimeQ[Range[# k+1,# (k+1)-1]]]!={False}&,
                1,cutOff],
        {k,1,100000000}]/.{cutOff+start->0};
```

We have for $a(k) = 2$ the numbers

$$8729394, 17566347, 26437886, 35330619, 44238546,$$
$$53158353, 62087802, 71025543, 79969616, 88921064.$$

In general, here we have a simple explicit formula: the number of $a(k) = 2, k \le K$ is $K + 1 - \pi(2K + 1)$. Further, let

$$A_t(K) = |\{k \le K : a(k) = t\}|.$$

In cases $t \ge 3$ we have no explicit formulas. But, taking into account the distribution of primes into residue classes, a rough argument suggests that $A_t(K) \asymp c_t K(\ln K)^{2-t}$. For example, for $a(k) = 3$ within the considered intervals we have the numbers

$$1061880, 2050703, 3014798, 3963752, 4901317,$$
$$5830488, 6752801, 7668802, 8580597, 9486975,$$

and one can hope that $c_3 \approx 1.7\cdots$. In other cases we have

$$t = 4 : \ 173835, 321315, 461745, 597249, 729660, 859605, 987238, 1113288, 1237558, 1360344;$$

$$t = 5 : \ 25108, 45086, 63177, 80407, 97199, 113213, 128850, 144474, 159648, 174577;$$

$$t = 6 : \ 7312, 12542, 17150, 21536, 25714, 29734, 33616, 37243, 40952, 44503;$$

$$t = 7 : \ 1753, 2918, 3841, 4749, 5590, 6373, 7201, 7950, 8691, 9378;$$

$$t = 8 : \ 449, 703, 918, 1109, 1309, 1507, 1670, 1810, 1977, 2141;$$

$$t = 9: \ 149, 216, 278, 342, 400, 440, 508, 558, 606, 647;$$

$$t = 10: \ 73, 109, 138, 164, 186, 203, 222, 232, 249, 262;$$

$$t = 11: \ 18, 25, 29, 31, 35, 36, 42, 46, 48, 49;$$

$$t = 12: \ 13, 15, 17, 19, 21, 25, 26, 29, 30, 31;$$

$$t = 13: \ 2, 3, 3, 3, 3, 3, 3, 4, 7, 7;$$

$$t = 14: \ 4, 5, 6, 6, 6, 6, 7, 7, 7, 8;$$

$$t = 15: \ 0, 2, 3, 3, 3, 3, 3, 3, 3, 3;$$

$$t = 16: \ 4, 5, 5, 5, 5, 5, 5, 5, 5, 5.$$

For those t when the difference

$$\frac{A_t(10^8)}{10^8}(8 \ln 10)^{t-2} - \frac{A_t(10^7)}{10^7}(7 \ln 10)^{t-2}$$

remains less than 0.5, we can get an impression about the change of c_t depending on t. So, $c_2 = 1$ and approximately $c_3 = 1.7, \ c_4 = 4.6, \ c_5 = 11, \ c_6 = 49$.

8 Acknowledgment

The authors are grateful to N. J. A. Sloane for several important remarks. They thank also an anonymous referee for useful and constructive comments that promoted an essential improvement of the paper.

References

[1] M. El Bachraoui, Primes in the interval $[2n, 3n]$, *Int. J. Contemp. Math. Sciences* **1** (2006), no. 13, 617–621.

[2] J. Cullinan, and F. Hajir, Primes of prescribed congruence class in short intervals, *INTEGERS* **12** (2012), Article A56.

[3] P. Dusart, Estimates of some functions over primes without R.H., preprint. Available at http://arxiv.org/abs/1002.0442.

[4] P. Erdős, and J. Surányi, *Topics in the Theory of Numbers*, Undergraduate Texts in Mathematics, Springer-Verlag, 2003.

[5] S. Laishram, On a conjecture on Ramanujan primes, *Int. J. Number Theory* **6** (2010), 1869–1873.

[6] A. Loo, On the primes in the interval $[3n, 4n]$, *Int. J. Contemp. Math. Sciences* **6** (2011), no. 38, 1871–1882.

[7] J. Nagura, On the interval containing at least one prime number, *Proc. Japan Academy, Ser. A* **28** (1952), 177–181.

[8] S. Ramanujan, A proof of Bertrand's postulate, *J. Indian Math. Soc.* **11** (1919), 181–182.

[9] O. Ramaré, and Y. Saouter, Short effective intervals containing primes, *J. Number Theory* **98** (2003), 10–33.

[10] D. Redmond, *Number Theory, An Introduction*, Marcel Dekker, 1996.

[11] B. Rosser, The n-th prime is greater than $n \ln n$, *Proc. London Math. Soc.* **45** (1939), 21–44.

[12] V. Shevelev, Ramanujan and Labos primes, their generalizations, and classifications of primes, *J. Integer Seq.* **15** (2012), Article 12.5.4.

[13] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at `http://oeis.org`.

[14] J. Sondow, Ramanujan primes and Bertrand's postulate, *Amer. Math. Monthly*, **116** (2009), 630–635.

2010 *Mathematics Subject Classification*: Primary 11A41.
Keywords: prime number, generalized Ramanujan prime.

(Concerned with sequences A084140, A104272, A164952, A185004, A185005, A185006, A185007, A218769, A218831, A218850, A220268, A220269, A220273, A220274, A220281, A220293, A220462, A220463, A220474, and A220475.)

Received January 1 2013; revised versions received May 7 2013; June 10 2013; July 25 2013. Published in *Journal of Integer Sequences*, July 31 2013.

Return to Journal of Integer Sequences home page.