Journal of Integer Sequences, Vol. 16 (2013),
Article 13.2.16

Exponential Sums Involving the k-th Largest Prime Factor Function

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Dedicated to Jean-Paul Allouche on the occasion of his 60^{th} birthday

Abstract

Letting $P_k(n)$ stand for the k-th largest prime factor of $n \geq 2$ and given an irrational number α and a multiplicative function f such that $|f(n)| = 1$ for all positive integers n, we prove that $\sum_{n \leq x} f(n) \exp\{2\pi i \alpha P_k(n)\} = o(x)$ as $x \to \infty$.

1 Introduction

In 1954, Vinogradov [7] showed that, given any irrational number α, if $p_1 < p_2 < \cdots$ stands for the sequence of primes, then

$$\sum_{n \leq x} e(\alpha p_n) = o(x) \qquad \text{as } x \to \infty, \tag{1}$$

where we used the standard notation $e(z) = \exp\{2\pi i z\}$. In light of the well known Weyl criteria (see the book of Kuipers and Niederreiter [5]), statement (1) is equivalent to asserting that the sequence αp_n, $n = 1, 2, \ldots$, is uniformly distributed mod 1.

In 2005, Banks, Harman and Shparlinski [1] proved that for every irrational number α,

$$\sum_{n \le x} e(\alpha P(n)) = o(x) \qquad \text{as } x \to \infty, \tag{2}$$

where $P(n)$ stands for the largest prime factor of the integer $n \ge 2$ with $P(1) = 1$.

Let $\mathcal{M}$ denote the set of all complex valued multiplicative arithmetical functions and let $\mathcal{M}_1$ be those $f \in \mathcal{M}$ for which $|f(n)| = 1$ for all positive integers n. In [2], we generalized (2) by showing that for any irrational number α and any function $f \in \mathcal{M}_1$, we have $\sum_{n \le x} f(n) e(\alpha P(n)) = o(x)$ as $x \to \infty$.

Let $\omega(n)$ stand for the number of distinct prime divisors of $n \ge 2$ with $\omega(1) = 0$. Given an integer $k \ge 1$, for each integer $n \ge 2$, we let $P_k(n)$ stand for the k-th largest prime factor of n if $\omega(n) \ge k$, while we set $P_k(n) = 1$ if $\omega(n) \le k - 1$. Thus, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ stands for the prime factorization of n, where $p_1 < p_2 < \cdots < p_s$, then

$$P_1(n) = P(n) = p_s, \qquad P_2(n) = p_{s-1}, \qquad P_3(n) = p_{s-2}, \ldots$$

In this paper, we prove that, given any integer $k \ge 2$ and any irrational number α, then $\displaystyle\sum_{n \le x} f(n) e(\alpha P_k(n)) = o(x)$ as $x \to \infty$.

2 Main result

Theorem 1. *Given an integer $k \ge 2$ and an irrational number α, let $f \in \mathcal{M}_1$ and consider the sum*

$$S_f(x) = \sum_{n \le x} f(n) e(\alpha P_k(n)).$$

Then

$$S_f(x) = o(x) \qquad \textit{as } x \to \infty. \tag{3}$$

3 Notation and preliminary results

We say that a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ is slowly oscillating if $\lim_{y \to \infty} L(cy)/L(y)$ for each real number $c > 0$.

In 1968, Halász [4] established the following result.

Lemma 2 (Halász's theorem)**.** *Let f be a complex-valued multiplicative arithmetical function such that $|f(n)| \le 1$ for all positive integers n. The following two statements hold:*

(a) If there exists a real number τ_0 for which the series

$$\sum_p \frac{1-\Re(f(p)/p^{i\tau_0})}{p}$$

is convergent, then, as $x\to\infty$,

$$\sum_{n\le x} f(n) = x\cdot\frac{x^{i\tau_0}}{1+\imath\tau_0}\prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{r=1}^{\infty}\frac{f(p^r)}{p^{r(1+i\tau_0)}}\right)+o(x).$$

(b) If the series

$$\sum_p \frac{1-\Re(f(p)/p^{i\tau})}{p}$$

is divergent for every real number τ, then

$$\sum_{n\le x} f(n) = o(x) \qquad \textit{as } x\to\infty.$$

Proof. For a proof, see the book of Schwarz and Spilker ([6, Thm. 3.1]). □

Fix an integer $k\ge 2$ and for each real number τ, let

$$R_\tau(x) := \sum_{n\le x} f(n)n^{i\tau}e(\alpha P_k(n)).$$

We then have the following result.

Lemma 3. *Let $\tau_1,\tau_2\in\mathbb{R}$. Then, as $x\to\infty$,*

$$\text{(a)}\quad R_{\tau_1}(x)=o(x) \qquad \Longleftrightarrow \qquad \text{(b)}\quad R_{\tau_2}(x)=o(x). \tag{4}$$

Proof. It is clear that (a) holds if and only if, given any $\varepsilon>0$,

$$\frac{1}{\varepsilon x}\sum_{x\le n\le(1+\varepsilon)x} f(n)n^{i\tau_1}e(\alpha P_k(n))\to 0 \quad \text{as } x\to\infty,$$

while (b) holds if and only if, given any $\varepsilon>0$,

$$\frac{1}{\varepsilon x}\sum_{x\le n\le(1+\varepsilon)x} f(n)n^{i\tau_2}e(\alpha P_k(n))\to 0 \quad \text{as } x\to\infty.$$

But since each $n\in[x,(1+\varepsilon)x]$ can be written as $n=x+\delta x$ for some $0\le\delta\le\varepsilon$, we have

$$n^{i\tau_2}=(x+\delta x)^{i\tau_2}=x^{i\tau_2}\,(1+\delta)^{i\tau_2}=x^{i\tau_2}(1+O(\varepsilon)),$$

and similarly

$$n^{i\tau_1}=x^{i\tau_1}(1+O(\varepsilon)).$$

It follows that (a) and (b) are equivalent, thus proving (4). □

Lemma 4. *For all $2 \le y \le x$, let $\Psi(x,y) := \#\{n \le x : P(n) \le y\}$. Then,*

(a) As $x \to \infty$,

$$\Psi(x,y) = (1+o(1))\rho(u)\,x,$$

where $u = \log x/\log y$ and $\rho(u)$ is the Dickman function defined by the initial condition $\rho(u) = 1$ for $0 \le u \le 1$ and thereafter as the continuous solution of the differential equation with shift differences

$$u\rho'(u) + \rho(u-1) = 0 \qquad (u > 1).$$

(b) For all $2 \le y \le x$, $\Psi(x,y) \ll x \exp\left\{-\frac{1}{2}\frac{\log x}{\log y}\right\}$.

Proof. Proofs of these results can be found in the book of De Koninck and Luca ([3], pages 134 and 138). □

Lemma 5. *Given an arbitrary irrational number α, set*

$$S_1(x) = \sum_{n \le x} e(\alpha P_k(n)).$$

Then

$$S_1(x) = o(x) \qquad \text{as } x \to \infty.$$

Proof. Let $\varepsilon > 0$ be a small number. It is easy to see that in the sum representing $S_1(x)$, we may drop three types of integers $n \le x$, namely (i) those for which $\omega(n) \le k+1$, (ii) those for which $P_{k+1}(n) \le x^\varepsilon$ and finally (iii) those for which $p^2|n$ for some prime $p \ge P_k(n)$, the reason being that the number of these exceptional n's is $O(\varepsilon x)$. So, let us write the remaining integers $n \le x$ as

$$n = \nu p_k p_{k-1} \cdots p_1, \quad \text{where } x^\varepsilon < P(\nu) < p_k < p_{k-1} < \cdots < p_1$$

and set

$$Q_k = p_k p_{k-1} \cdots p_1 \ (< x^{1-\varepsilon}).$$

Using this set up, we may write

$$S_1(x) = \sum_{\substack{x^\varepsilon < p_k < \cdots < p_1 \\ Q_k < x^{1-\varepsilon}}} e(\alpha p_k) \Psi\left(\frac{x}{Q_k}, p_k\right) + O(\varepsilon x).$$

Let

$$T_1(x) = \sum_{\substack{x^\varepsilon < p_k < \cdots < p_1 \\ Q_k < x^{1-\varepsilon}}} e(\alpha p_k) \Psi\left(\frac{x}{Q_k}, p_k\right),$$

so that

$$S_1(x) = T_1(x) + O(\varepsilon x), \tag{5}$$

Now, observe that, using Lemma 4 and the fact that $Q_k = p_k Q_{k-1}$, we have

$$\begin{aligned}\Psi\left(\frac{x}{Q_k}, p_k\right) &= \frac{x}{Q_k}\rho\left(\frac{\log x - \log Q_k}{\log p_k}\right) + o\left(\frac{x}{Q_k}\right)\\ &= \frac{x}{Q_k}\rho\left(\frac{\log x - \log Q_{k-1}}{\log p_k} - 1\right) + o\left(\frac{x}{Q_k}\right).\end{aligned}$$

Substituting this last identity in (5), we get

$$\begin{aligned}T_1(x) &= \sum_{\substack{x^\varepsilon < p_k < \cdots < p_1\\ p_k Q_{k-1} < x^{1-\varepsilon}}} e(\alpha p_k)\frac{x}{Q_k}\rho\left(\frac{\log x - \log Q_{k-1}}{\log p_k} - 1\right) + o(x)\\ &= \sum_{\substack{x^\varepsilon < p_k < \cdots < p_1\\ p_k Q_{k-1} < x^{1-\varepsilon}}} \frac{x}{Q_{k-1}} \sum_{x^\varepsilon < p_k < \min\left(p_{k-1}, \frac{x^{1-\varepsilon}}{Q_{k-1}}\right)} \frac{e(\alpha p_k)}{p_k}\rho\left(\frac{\log x - \log Q_{k-1}}{\log p_k} - 1\right)\\ &\quad + \quad o(x). \qquad (6)\end{aligned}$$

Setting $t(p_{k-1}, \ldots, p_1) := \min\left(p_{k-1}, \dfrac{x^{1-\varepsilon}}{Q_{k-1}}\right)$, we now subdivide the above inner sum into two separate sums, depending if

$$t(p_{k-1}, \ldots, p_1) \le 2x^\varepsilon \quad \text{or} \quad t(p_{k-1}, \ldots, p_1) > 2x^\varepsilon,$$

and thus we write $T_1(x) = T_1'(x) + T_1''(x)$.

On the one hand, using the fact that $\displaystyle\sum_{x^\varepsilon < p_k \le 2x^\varepsilon} \frac{1}{p_k} \ll \frac{1}{\varepsilon \log x}$, we obtain

$$|T_1'(x)| \ll \frac{x}{\varepsilon \log x}\left(\sum_{x^\varepsilon < p_k \le 2x^\varepsilon} \frac{1}{p_k}\right)^k \ll_\varepsilon \frac{x}{\log x}. \qquad (7)$$

On the other hand, using the Vinogradov theorem (see (1)) and the continuity of the ρ function, we obtain that, as $x \to \infty$,

$$\begin{aligned}|T_1''(x)| &\le \sum_{x^\varepsilon < p_k < t(p_{k-1}, \ldots, p_1)} \frac{e(\alpha p_k)}{p_k}\rho\left(\frac{\log x - \log Q_{k-1}}{\log p_k} - 1\right)\\ &= o\left(\sum_{x^\varepsilon < p_k < t(p_{k-1}, \ldots, p_1)} \frac{1}{p_k}\rho\left(\frac{\log x - \log Q_{k-1}}{\log p_k} - 1\right)\right)\\ &= o(x). \qquad (8)\end{aligned}$$

Substituting (7) and (8) in (6), and thus in light of (5) completes the proof of Lemma 5.

□

4 Proof of Theorem 1

Let us first assume (case (b) of Halász's theorem) that

$$\sum_p \frac{1-\Re(f(p)p^{i\tau})}{p} = \infty \quad \text{for all } \tau \in \mathbb{R}.$$

Let us set

$$E(x) = \sum_{n\le x} f(n) \qquad \text{and} \qquad E(x|y) = \sum_{\substack{n\le x\\ P(n)\le y}} f(n).$$

It follows from Halász's theorem (Lemma 2) that $E(x) = o(x)$ as $x\to\infty$, in which case there exists a positive decreasing function $\delta(x)$ which tends to 0 as $x\to\infty$ and for which we have

$$|E(x)| \le x\delta(x). \tag{9}$$

Let $\varepsilon > 0$ be a fixed small number and choose y satisfying $x^\varepsilon \le y \le x$. Further set $\Pi_y := \prod_{y\le p\le x} p$. We then have

$$\begin{aligned} E(x|y) &= \sum_{n\le x} f(n) \sum_{d|(n,\Pi_y)} \mu(d) = \sum_{d|\Pi_y} \mu(d) \sum_{md\le x} f(md) \\ &= \sum_{d|\Pi_y} \mu(d) f(d) \sum_{m\le x/d} f(m) + O\left(\sum_{\substack{dm\le x\\ d|\Pi_y\\ (d,m)>1}} 1 \right) \\ &= \sum_{d|\Pi_y} \mu(d) f(d) E(x/d) + O\left(x\sum_{p>y} \frac{1}{p^2} \right). \end{aligned}$$

Consequently, uniformly for $x^\varepsilon \le y\le x$, and in light of (9), we have

$$|E(x|y)| \le x \sum_{d|\Pi_y} \frac{\delta(x/d)}{d} + O\left(\frac{x}{y}\right). \tag{10}$$

In light of (10), in order to show that

$$E(x|y) = o(x) \qquad \text{as } x\to\infty, \tag{11}$$

we only need to show that

$$T_0 := \sum_{d|\Pi_y} \frac{\delta(x/d)}{d} = o(1) \quad \text{ as } x\to\infty. \tag{12}$$

We split the above sum in two parts as follows:

$$\begin{aligned} T_0 &= \sum_{\substack{d\mid \Pi_y\\ d\le x/\log x}} \frac{\delta(x/d)}{d} + \sum_{\substack{d\mid \Pi_y\\ x/\log x<d\le x}} \frac{\delta(x/d)}{d} \\ &\le \delta(\log x) \sum_{\substack{d\mid \Pi_y\\ d\le x/\log x}} \frac{1}{d} + c \sum_{\substack{d\mid \Pi_y\\ x/\log x<d\le x}} \frac{1}{d} \\ &= \delta(\log x) T_1 + cT_2, \end{aligned} \tag{13}$$

where c is some positive constant. On the one hand, we have

$$T_1 \le \prod_{x^\varepsilon\le p\le x}\left(1+\frac{1}{p}\right) \ll \exp\left\{\sum_{x^\varepsilon\le p\le x}\frac{1}{p}\right\} \ll \frac{1}{\varepsilon}. \tag{14}$$

On the other hand, setting $U_0 = x/\log x$ and letting j_0 be the smallest positive integer satisfying $2^{j_0+1}U_0 > x$, we have

$$\begin{aligned} T_2 &\le \sum_{j=0}^{j_0} \frac{1}{2^j U_0} \sum_{\substack{2^j U_0\le d<2^{j+1}U_0\\ p(d)>x^\varepsilon}} 1 \\ &\le \sum_{j=1}^{j_0} \prod_{p<x^\varepsilon}\left(1-\frac{1}{p}\right) \ll \frac{j_0}{\log x} \ll \frac{\log\log x}{\log x}. \end{aligned} \tag{15}$$

Combining (14) and (15), we immediately obtain (12), from which (11) follows.

On the other hand,

$$\Psi(x,y) \ge_\varepsilon x \qquad \text{for } x^\varepsilon \le y \le x. \tag{16}$$

Combining (11) and (16), we get that

$$\lim_{x\to\infty} \max_{x^\varepsilon\le y\le x} \frac{|E(x|y)|}{\Psi(x,y)} = 0. \tag{17}$$

Given a positive integer k and a positive integer n, it will be convenient to write

$$Q_k(n) = Q_k = P_k(n)P_{k-1}(n)\cdots P_1(n).$$

Then, write

$$S_f(x) = \sum_{\substack{n\le x\\ P_k(n)\le x^\varepsilon}} f(n)e(\alpha P_k(n)) + \sum_{\substack{n\le x\\ P_k(n)>x^\varepsilon}} f(n)e(\alpha P_k(n)) = S'_f(x) + S''_f(x), \tag{18}$$

say.

First, observe that it is an easy consequence of the Turán-Kubilius inequality that

$$\sum_{n\le x}\left(\sum_{\substack{p\mid n\\ x^\varepsilon<p\le x}} 1 - \sum_{x^\varepsilon<p\le x}\frac{1}{p}\right)^2 \ll x\sum_{x^\varepsilon<p\le x}\frac{1}{p} \ll x\log(1/\varepsilon),$$

from which it follows that

$$\sum_{\substack{n\le x\\ P_k(n)\le x^{\varepsilon}}} (k-\log(1/\varepsilon))^2 \ll x\log(1/\varepsilon).$$

Using this, we conclude that

$$\left|S_f'(x)\right| \le \#\{n\le x : P_k(n)\le x^{\varepsilon}\} \ll \frac{x}{\log(1/\varepsilon)}. \tag{19}$$

Similarly, we can say that

$$\#\{n\le x : P_{k+1}(n)\le x^{\varepsilon}\} \ll \frac{x}{\log(1/\varepsilon)}.$$

This implies that $Q_k(n)\le x^{1-\varepsilon}$ for all but $O\left(\dfrac{x}{\log(1/\varepsilon)}\right)$ integers $n\le x$.

This means that

$$\left|S_f''(x)\right| \le |e(\alpha P_k)f(Q_k)E(x/Q_k|P_k)| + O\left(\frac{x}{\log(1/\varepsilon)}\right). \tag{20}$$

Using (17), we obtain that the summation on the right-hand side of (20) is

$$o\left(x\sum_{\substack{p_k\cdots p_1\le x\\ x^{\varepsilon}<p_k<\cdots<p_1}}\frac{1}{p_k\cdots p_1}\right) = o\left(x\frac{1}{k!}\left(\sum_{x^{\varepsilon}<p\le x}\frac{1}{p}\right)^k\right) = o\left(x\left(\log\frac{1}{\varepsilon}\right)^k\right),$$

implying that

$$S_f''(x) = o(x) \qquad \text{as } x\to\infty. \tag{21}$$

Substituting (19) and (21) in (18), we obtain (3).

It remains to consider case (a) of Halász's theorem (Lemma 2), that is when there exists a real number τ_0 for which the series

$$\sum_p \frac{1-\Re(f(p)/p^{i\tau_0})}{p}$$

is convergent. In light of Lemma 3 we can assume that $\tau_0=0$, that is that

$$\sum_p \frac{1-\Re(f(p))}{p} < \infty. \tag{22}$$

For each prime power p^a, let us write $f(p^a)=\exp\{iu(p^a)\}$ where $u(p^a)\in[-\frac{\pi}{2},\pi)$. It follows that

$$\sum_p \frac{u^2(p)}{p} < \infty.$$

Now let D be a large number and define the multiplicative functions f_D and g_D on prime powers p^a by

$$f_D(p^a) = \begin{cases} f(p^a), & \text{if } p \le D; \\ 1, & \text{if } p > D; \end{cases} \qquad \text{and} \qquad g_D(p^a) = \begin{cases} 1, & \text{if } p \le D; \\ f(p^a), & \text{if } p > D. \end{cases}$$

Then define the arithmetical function $t(n)$ implicitly by the relation $f_D(n) = \sum_{\delta | n} t(\delta)$. Since one easily sees that $t(p) = 0$ if $p > D$, it follows that the above summation runs over only those divisors δ for which $P(\delta) \le D$.

Further define

$$a_D(x) := \sum_{D < p \le x} \frac{u(p)}{p}.$$

Using the Turán-Kubilius inequality, we obtain that

$$\sum_{n \le x} \left(\sum_{\substack{p^a \| n \\ p > D}} u(p^a) - a_D(x) \right)^2 \ll x \sum_{p > D} \frac{u^2(p^a)}{p^a} \ll x\eta_D^2,$$

say, where $\eta_D \to 0$ as $D \to \infty$.

It follows from this that

$$\sum_{n \le x} \left| f(n) - f_D(n) e^{i a_D(x)} \right|^2 \ll \eta_D^2 x,$$

and therefore that

$$\sum_{n \le x} \left| f(n) - f_D(n) e^{i a_D(x)} \right| \ll \eta_D x.$$

We may conclude from this that

$$S_f(x) = e^{-i a_D(x)} A_D(x) + O(\eta_D x),$$

where

$$A_D(x) := \sum_{n \le x} f_D(n) e(\alpha P_k(n)).$$

For each integer $\delta \ge 1$, let

$$B_\delta(y) = \sum_{m \le y} e(\alpha P_k(\delta m)).$$

With this definition, we may write

$$A_D(x) = \sum_{\substack{\delta \le x \\ P(\delta) \le D}} t(\delta) B_\delta\left(\frac{x}{\delta}\right). \tag{23}$$

Now if $P_k(\delta m) \ne P_k(m)$, then either $\omega(m) \le k-1$ or $P_k(m) \le D$. Thus

$$\left| B_\delta\left(\frac{x}{\delta}\right) - B_1\left(\frac{x}{\delta}\right) \right| \le \sum_{\substack{m \le x/\delta \\ \omega(m) \le k-1}} 1 + \sum_{\substack{Q\nu \le x/\delta \\ \omega(Q) \le k-1,\ P(\nu) \le D}} 1 = U_1(x) + U_2(x), \tag{24}$$

say. Write

$$U_1(x) = \sum_{\delta \le \sqrt{x}} * + \sum_{\sqrt{x} < \delta \le x} * = U_1'(x) + U_1''(x), \tag{25}$$

say. Then, it is clear that

$$U_1''(x) \le \sum_{m \le \sqrt{x}} 1 \le \sqrt{x}. \tag{26}$$

On the other hand, using the Hardy-Ramanujan inequality (see, for instance, [3, Theorem 10.1]), it follows that there exist two absolute positive constants c_1 and c_2 such that

$$U_1'(x) \le \frac{c_1 x}{\delta \log x} \frac{(\log\log x + c_2)^{k-2}}{(k-2)!}. \tag{27}$$

On the other hand,

$$U_2(x) \le U_2'(x) + U_2''(x), \tag{28}$$

where in $U_2'(x)$, we sum over those $Q \le \sqrt{x/\delta}$, while in $U_2''(x)$, we sum over those $\nu \le \sqrt{x/\delta}$. To estimate $U_2'(x)$, we proceed as follows. First, using Lemma 4 (b), we get

$$U_2'(x) \le \sum_{\substack{Q \le \sqrt{x/\delta} \\ \omega(Q) \le k-1}} \sum_{\substack{\nu \le x/\delta Q \\ P(\nu) \le D}} 1 \ll \sum_{\substack{Q \le \sqrt{x/\delta} \\ \omega(Q) \le k-1}} \frac{x}{\delta Q} \exp\left\{ -\frac{1}{2} \frac{\log(x/\delta Q)}{\log D} \right\}. \tag{29}$$

Since $\dfrac{x}{\delta Q} \ge \left(\dfrac{x}{\delta}\right)^{1/4} \ge x^{1/8}$, it follows from (29) that

$$U_2'(x) \ll \sum_{\substack{Q \le \sqrt{x/\delta} \\ \omega(Q) \le k-1}} \frac{x}{\delta Q} \exp\left\{ -\frac{1}{16} \frac{\log x}{\log D} \right\}. \tag{30}$$

Since

$$\sum_{\substack{Q \le x \\ \omega(Q) \le k-1}} \frac{1}{Q} \ll (\log\log x)^{k-1},$$

it follows from (30) that, given any positive number K,

$$U_2'(x) \ll_D \frac{x}{\delta} (\log x)^{-K}. \tag{31}$$

On the other hand, setting $\pi_k(x) := \#\{n \le x : \omega(n) = k\}$ and again using the Hardy-Ramanujan inequality, it follows that

$$\begin{aligned}
U_2''(x) &\le \sum_{\substack{\nu \le \sqrt{x/\delta} \\ P(\nu) \le D}} \sum_{\substack{Q \le x/\delta\nu \\ \omega(Q) \le k-1}} 1 \\
&\le (k-1) \sum_{\substack{\nu \le \sqrt{x/\delta} \\ P(\nu) \le D}} \pi_{k-1}\left(\frac{x}{\delta\nu}\right) \\
&\le c_1 \sum_{P(\nu) \le D} \frac{kx}{\delta\nu} \cdot \frac{1}{\log x} \frac{(\log\log x + c_2)^{k-2}}{(k-2)!} \\
&\le \frac{c_1 x}{\delta \log x} (\log\log x + c_2)^{k-2} \prod_{p \le D} \left(1 - \frac{1}{p}\right)^{-1} \\
&\ll \frac{x}{\delta} \frac{\log D}{\log x} (\log\log x)^{k-2}. \qquad (32)
\end{aligned}$$

Substituting (26) and (27) in (25), and then using (31) and (32) in (28), we obtain from (24) that

$$\max_{\delta \le \sqrt{x}} \frac{1}{x/\delta} \left| B_\delta\left(\frac{x}{\delta}\right) - B_1\left(\frac{x}{\delta}\right) \right| \ll \frac{1}{\sqrt{\log x}},$$

say. It follows from this last estimate and (23) that for some positive constant c_3

$$\begin{aligned}
|A_D(x)| &\le x \sum_{\substack{\sqrt{x} < \delta < x \\ P(\delta) \le D}} \frac{|t(\delta)|}{\delta} + \sum_{\substack{\delta \le \sqrt{x} \\ P(\delta) \le D}} |t(\delta)| \left| B_1\left(\frac{x}{\delta}\right) \right| + \frac{c_3}{\sqrt{\log x}} \sum_{\delta \le \sqrt{x}} |t(\delta)| \frac{x}{\delta} \\
&= xW_1(x) + W_2(x) + \frac{c_3 x}{\sqrt{\log x}} W_3(x), \qquad (33)
\end{aligned}$$

say.

Since

$$W_3(x) \le \prod_{p \le D} \left(1 + \frac{|t(p)|}{p} + \frac{|t(p^2)|}{p^2} + \cdots\right)$$

and since $|t(p^a)| = |f(p^a) - f(p^{a-1})| \le 2$, it follows that

$$W_3(x) \le c(\log D)^2. \qquad (34)$$

Using Lemma 5, we obtain that, as $x \to \infty$,

$$W_2(x) = o\left(xW_3(x)\right) = o\left(x(\log D)^2\right). \qquad (35)$$

In order to estimate $W_1(x)$, let us first find an upper bound for

$$\kappa(v) := \sum_{\substack{v \le \delta \le 2v \\ P(\delta) \le D}} t(\delta) \qquad \text{for } \sqrt{x} \le v \le x.$$

We have

$$\kappa(v) \le 2 \sum_{\substack{k \le \sqrt{2v} \\ P(k) \le D}} \sum_{\substack{\ell \in [v/k, 2v/k] \\ P(\ell) \le D}} 1 \le 2 \sum_{\substack{k \le \sqrt{2v} \\ P(k) \le D}} \Psi\left(\frac{2v}{k}, D\right). \qquad (36)$$

Since $\frac{2v}{k} \geq \sqrt{2v} \geq \sqrt{x}$, it follows that, given any arbitrary large number $R > 0$,

$$\Psi\left(\frac{2v}{k}, D\right) \leq \frac{2vc}{k}(\log x)^{-R}. \tag{37}$$

Let $v_0 = \sqrt{x}$ and, for each integer $j \geq 1$, let $v_j = 2^j\sqrt{x}$. Letting j_0 be the smallest positive integer such that $v_{j_0} \geq x$, so that $j_0 = O(\log x)$, we obtain, using (37) in (36), that

$$W_1(x) \leq \sum_{j=0}^{j_0} \frac{\kappa(v_j)}{v_j} \ll \frac{j_0 + 1}{(\log x)^R}. \tag{38}$$

Substituting (34), (35) and (38) in (33), we obtain that

$$A_D(x) = o(x) \qquad \text{as } x \to \infty,$$

thus completing the proof of Theorem 1.

5 Acknowledgments

Research of the first author was supported by a grant from NSERC and that of the second author by ELTE IK.

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2010 *Mathematics Subject Classification*: Primary 11L07; Secondary 11N37.
Keywords: exponential sum, largest prime factor.

Received October 7 2012; revised version received October 27 2012. Published in *Journal of Integer Sequences*, March 2 2013.