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On the Entropy of Curves

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To Jean-Paul Allouche for his 60th birthday

Abstract

Using geometric probability, we apply the formal definitions of Shannon entropy and Rényi's generalization to study the complexity of planar curves of finite length within a convex set. The bounds for the Shannon and Rényi entropies depend on the arc length of the curve and on that of the boundary of the convex set; they involve a Gibbs distribution and a power law distribution, respectively. We also obtain explicit formulae for the two entropies and determine convex sets that maximize the entropy of curves.

1 Introduction

Planar curves range from a simple straight line segment or a circle to a complicated tangle. Their complexity has been analysed using both the thermodynamic concept of Shannon entropy [9, 10] and the geometrical concept of dimension [7, 8]. Examples can be found in

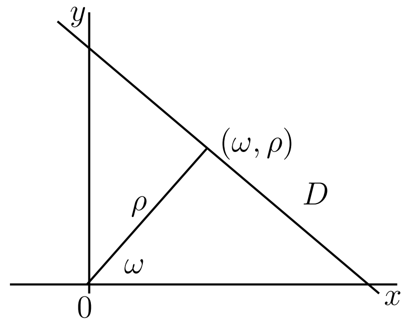

Figure 1: A straight line D determined by polar coordinates (ω, ρ)

various other domains [1, 2, 3, 4, 5, 6, 11, 12]. A brief overview of the fundamental ideas is given and the Shannon and the more general Shannon-Rényi entropies are defined. The question of bounds for these entropies is considered; the analysis of the general Rényi entropy turns out to be quite different from that of the Shannon entropy and involves a modified Hurwitz zeta function. A more precise viewpoint which gives exact formulae for the entropies is then developed.

2 Geometric probability

Given a plane curve Γ of finite length $|\Gamma|$, we recall the classical definition of the probability that a straight line D intersects Γ in exactly n points [9, 15]. In the plane, a straight line D that does not pass through the origin O is determined by the polar coordinates (ω, ρ), where ω, $0 \leqslant \omega < 2\pi$, is the angle the normal to D makes with the x-axis and $\rho > 0$ is the distance from O. Straight lines that pass through the origin are not counted since, as we will see, the family of such lines has measure 0.

In the (ω, ρ) plane, a point in the strip $\mathcal{S} := [0, 2\pi) \times (0, \infty)$ represents a straight line not passing through the origin in the usual (x, y)-plane. The measure μ in the strip is defined by

$$d\mu := d\rho\, d\omega,$$

i.e., the usual Lebesgue measure. Now let K be a bounded convex set in the (x, y)-plane and let $F(K)$ be the set of straight lines that intersect K. The μ-measure of $F(K)$ is known to be equal to the length $|\partial K|$ of the boundary ∂K of K, i.e.,

$$\mu(F(K)) = |\partial K|.$$

It follows that the measure given by

$$dp = \frac{d\mu}{|\partial K|} = \frac{d\rho\, d\omega}{|\partial K|}$$

is a probability measure defined on $F(K)$.

Suppose $K_0 \subseteq K$ is a convex subset of K. Then $|\partial K_0| \leqslant |\partial K|$ and the probability that a straight line meeting K also meets K_0 is

$$\frac{|\partial K_0|}{|\partial K|}.$$

Consequently the probability that a straight line meeting K misses K_0 is

$$1 - \frac{|\partial K_0|}{|\partial K|}. \tag{1}$$

Consider a curve Γ_0 of finite length lying in the convex set K. Denote by $p_n(\Gamma_0, K)$ the probability that a straight line D meeting K intersects Γ_0 in exactly n points. Observe that if $K = K_0 = \mathrm{con}(\Gamma_0)$, the convex hull of Γ_0, then D must hit Γ_0, so that $p_0(\Gamma_0, K_0) = 0$.

By the definition of probability,

$$\sum_{n=0}^{\infty} p_n(\Gamma_0, K) = 1. \tag{2}$$

A classical result of H. Steinhaus [17] states that the expected number λ say of intersection points of Γ_0 with random straight lines is

$$\lambda := \sum_{n=1}^{\infty} n\, p_n(\Gamma_0, K) = \frac{2|\Gamma_0|}{|\partial K|}. \tag{3}$$

The next moment $\sum_{n=1}^{\infty} n^2\, p_n(\Gamma_0, K)$ corresponds to the "energy" of the curve but is much less well-behaved and will not be considered.

3 Entropy

The celebrated Shannon entropy of thermodynamics and information theory has a generalization due to Rényi [13, 14]. Their definitions are used in conjunction with geometric probability to define formally notions of entropy for curves of finite length within a convex set K. Note that unlike the thermodynamic setting, there is no underlying mechanism or dynamic here that leads to the entropies increasing.

3.1 The Shannon entropy

The sequence $(p_n(\Gamma_0, K)\colon n = 0, 1, 2, \ldots)$ of probabilities of the number of intersection points can be substituted into the formula for Shannon entropy. To be precise, the *Shannon entropy* $h(\Gamma_0, K)$ of the curve Γ_0 relative to a convex set K containing Γ_0 is defined by the formula

$$h(\Gamma_0, K) = \sum_{n=0}^{\infty} p_n(\Gamma_0, K) \log \frac{1}{p_n(\Gamma_0, K)} = \sum_{n=0}^{\infty} p_n(K) \log \frac{1}{p_n(K)}, \tag{4}$$

where as usual we put $p\log(1/p) = 0$ if $p = 0$ and where for simplicity here and elsewhere whenever the argument is clear, we suppress Γ_0 or K and write the probability

$$p_n(\Gamma_0, K) = p_n(K) \text{ or } p_n.$$

Similarly we will write the Shannon entropy

$$h(\Gamma_0, K) = h(K) = h.$$

3.2 The Rényi entropy

Rényi [13, 14] has defined a form of entropy $h^{(s)}$, where $s \geqslant 0$, by

$$h^{(s)} := \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s, \tag{5}$$

which tends to the Shannon entropy h as $s \to 1$:

$$h^{(1)} := \lim_{s\to 1} h^{(s)} = \lim_{s\to 1} \left(\frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s \right) = \sum_{n=0}^{\infty} p_n \log 1/p_n = h\,. \tag{6}$$

In addition it is readily verified that $d\,h^{(s)}/ds = 0$ at $s = 1$ and that $h^{(s)}$ is minimal for $s = 1$. The Rényi entropy can also be applied formally to curves. We will write

$$h^{(s)}(\Gamma_0, K) = \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s(\Gamma_0, K) \tag{7}$$

for the Rényi entropy for a curve Γ_0 relative to a convex set $K \supseteq \Gamma_0$ and, as with the Shannon entropy, we will write

$$h^{(s)}(\Gamma_0, K) = h^{(s)}(K) = h^{(s)}$$

when the meaning is clear.

The two series have finitely many terms for an algebraic curve, since $p_n = 0$ for all n larger than its degree, but they converge (absolutely) in general by the following results. A separate argument is needed for the two entropies. The Shannon entropy ($s = 1$) has been treated in this context [9, 10] and is now discussed for completeness.

3.3 A bound for the Shannon entropy of curves

Theorem 1. *The Shannon entropy $h(\Gamma_0, K)$ of a curve Γ_0 of finite length lying in the convex set K satisfies*

$$h(\Gamma_0, K) \leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K|} + 1\right) + \frac{2|\Gamma_0|}{|\partial K|} \log\left(1 + \frac{|\partial K|}{2|\Gamma_0|}\right) \leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K|} + 1\right) + 1 \tag{8}$$

and more precisely in the case $K = K_0 = \mathrm{con}(\Gamma_0)$ *(so that* $p_0 = 0$*),*

$$\begin{aligned} h(\Gamma_0, K_0) &\leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K_0|}\right) + \left(\frac{2|\Gamma_0|}{|\partial K_0|} - 1\right)\log\left(\frac{2|\Gamma_0|}{2|\Gamma_0| - |\partial K_0|}\right) \\ &\leqslant \log\left(\frac{2|\Gamma_0|}{|\partial K_0|}\right) + 1. \end{aligned} \tag{9}$$

Proof. By (4), it suffices to bound the sum $\sum_{n=k}^{\infty} p_n \log 1/p_n$, where $p_n \in [0,1]$ and $k = 0$ in (8) and $k = 1$ in (9), subject to the two constraints (2) and (3), i.e.,

$$\sum_{n=k}^{\infty} p_n = 1$$

and

$$\sum_{n=k}^{\infty} n\, p_n = \frac{2|\Gamma_0|}{|\partial K|} = \lambda,$$

where $k = 0$ or 1. This is done using Lagrange multipliers. Let $\alpha, \beta \in \mathbb{R}$ and consider

$$U = \sum_{n=k}^{\infty} \widehat{p}_n \log 1/\widehat{p}_n - \alpha \sum_{n=k}^{\infty} \widehat{p}_n - \beta \sum_{n=k}^{\infty} n\widehat{p}_n.$$

Then for each $n = k, k+1, \ldots$, $\partial U/\partial \widehat{p}_n = 0$ implies

$$-\log \widehat{p}_n - 1 - \alpha - n\beta = 0,$$

whence the 'maximal' probability distribution $\widehat{p}_n$ is a negative exponential with constant factor. More precisely,

$$\widehat{p}_n = e^{-1-\alpha-n\beta} = Ce^{-\beta n}, \tag{10}$$

where $C = e^{-1-\alpha}$ and $\widehat{p}_n$ is a Gibbs distribution [16]. Recall that Gibbs measure is a probability measure of importance in thermodynamics: it is the unique measure that maximizes the entropy for a given expected energy. It underlies maximum entropy methods and related algorithms and its appearance here is accordingly natural.

The constant $C = e^{-1-\alpha}$ (or α) is determined from the power series

$$\sum_{n=k}^{\infty} \widehat{p}_n = C \sum_{n=k}^{\infty} e^{-\beta n} = 1$$

which gives $C = C(\beta) = e^{\beta k}(1 - e^{-\beta})$. In statistical mechanics, $C(\beta)$ is the reciprocal of the partition function and in physics β corresponds to the inverse temperature [16]. Moreover $C \sum_{n=k}^{\infty} n\, e^{-\beta n} = \lambda$ whence

$$\lambda = \frac{1}{1 - e^{-\beta}} \left(k - (k-1)\, e^{-\beta}\right) = \frac{1}{e^{\beta} - 1} \left(k\, e^{\beta} - k + 1\right).$$

Thus

$$e^{\beta} = \frac{\lambda - k + 1}{\lambda - k} \text{ and } e^{\beta} - 1 = \frac{1}{\lambda - k}.$$

Hence the entropy h satisfies

$$h \leqslant C \sum_{n=k}^{\infty} e^{-\beta n} \log \frac{e^{\beta n}}{C} = H$$

say, so that

$$\begin{aligned}
H &= \frac{\beta}{1 - e^{-\beta}} \left(k - (k-1)e^{-\beta}\right) + \log \frac{e^{-k\beta}}{1 - e^{-\beta}} \\
&= \frac{\beta}{e^{\beta} - 1} \left(ke^{\beta} - k + 1\right) - \beta\, k + \log \frac{e^{\beta}}{e^{\beta} - 1} \\
&= \log(\lambda - k + 1) + (\lambda - k) \log \frac{\lambda - k + 1}{\lambda - k} \\
&\leqslant \log(\lambda - k + 1) + 1.
\end{aligned}$$

Case 1 $k = 0$, corresponds to the inequality (8).

Case 2 $k = 1$ corresponds to the inequality (9).

□

Remark 2. The first inequality (8) in the theorem shows that the entropy vanishes as $|\partial K|$ increases to infinity, a fact we shall rediscover in §4.

Remark 3. In the above calculation,

$$\lambda = \sum_{n=k}^{\infty} n\, \widehat{p}_n \geqslant k \sum_{n=k}^{\infty} \widehat{p}_n = k.$$

Thus obviously if $\lambda \to k$ from above, then $H = 0$. Hence

Remark 4. In the case $k = 1$, if $2|\Gamma|/|\partial K| \to 1$ from above, then $h = 0$; which is otherwise obvious since $2|\Gamma| = |\partial K|$ implies Γ is a segment of straight line.

3.4 Bounds for the Rényi entropy of curves

Recall from equation (7) that the Rényi entropy $h^{(s)}(\Gamma_0, K)$ of Γ_0 relative to the convex set $K \supset \Gamma_0$ is given by

$$h^{(s)}(\Gamma_0, K) := \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s(\Gamma_0, K).$$

For simplicity, we let $K = K_0 = \mathrm{con}(\Gamma_0)$, so that $p_0(\Gamma_0, K_0) = 0$, and denote the Rényi entropy $h^{(s)}(\Gamma_0, K_0)$ of Γ_0 relative to its convex hull K_0 by $h_0^{(s)}$. Two bounds for the Rényi entropy $h_0^{(s)}$ are obtained, one for the range $0 < s < 1$ and the other for $1/2 < s < 1$. Although the Rényi entropy coincides with Shannon entropy at $s = 1$, the bounds for the former may well be infinite at $s = 1$.

Two auxiliary functions are introduced. For each $\gamma \in [0, \infty)$ and $s \in [1/2, 1]$, we write

$$F_s(\gamma) = \frac{\zeta_1\left(\frac{s}{1-s}; \gamma\right)}{\zeta_1\left(\frac{1}{1-s}; \gamma\right)},$$

where $\zeta_1(u, c)$, $c \geqslant 0$, is the modified Hurwitz zeta function given by

$$\zeta_1(u, c) := \sum_{k=1}^{\infty} \frac{1}{(k+c)^u} = \sum_{k=0}^{\infty} \frac{1}{(k+c)^u} - \frac{1}{c^u} = \zeta(u, c) - \frac{1}{c^u}.$$

Note that $\zeta_1(u, 0) = \zeta(u)$ and that $F_s(0) > 1$.

For convenience, the function F_s is simplified when $1/2 \leqslant s < 1$ by putting $u = 1/(1-s) \in [2, \infty)$, so that $s/(1-s) = u - 1$ and

$$F_s(\gamma) = \frac{\zeta_1(u-1; \gamma)}{\zeta_1(u; \gamma)} := G_u(\gamma) = G_{1/(1-s)}(\gamma).$$

Then for $u > 2$ and $\gamma \geqslant 0$,

$$\begin{aligned} (\gamma+1)\zeta_1(u; \gamma) &= \sum_{k=1}^{\infty} \frac{\gamma+1}{(k+\gamma)^u} = \sum_{k=1}^{\infty} \left(\frac{k+\gamma}{(k+\gamma)^u} - \frac{k-1}{(k+\gamma)^u} \right) \\ &< \sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{u-1}} = \zeta_1(u-1; \gamma), \end{aligned}$$

and $G_u(\gamma) := \zeta_1(u-1; \gamma)/\zeta_1(u; \gamma) > 1 + \gamma$.

A more precise estimate is possible using the familiar inequality

$$\int_1^{\infty} f(x)dx \leqslant \sum_{k=1}^{\infty} f(k) \leqslant f(1) + \int_1^{\infty} f(x)dx \tag{11}$$

for a positive increasing integrable function $f\colon \mathbb{R} \to [0, \infty)$. It follows from (11) that

$$\frac{(1+\gamma)^{-u+1}}{u-1} < \zeta_1(u, \gamma) := \sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^u} < \frac{1}{(1+\gamma)^u} + \frac{1}{(u-1)(1+\gamma)^{u-1}}. \tag{12}$$

Hence for any fixed $u > 2$,

$$\begin{aligned} G_u(\gamma) &= \frac{\zeta_1(u-1, \gamma)}{\zeta_1(u, \gamma)} > \frac{1}{(u-2)(1+\gamma)^{u-2}} \left(\frac{1}{(1+\gamma)^u} + \frac{1}{(u-1)(1+\gamma)^{u-1}} \right)^{-1} \\ &> \frac{(u-1)}{(u-2)} \frac{(1+\gamma)^2}{(u+\gamma)}. \end{aligned}$$

Thus given any $u > 2$, for all sufficiently large γ, there exists an $\varepsilon_u > 0$ such that

$$G_u(\gamma) > (1 + \varepsilon_u)(1 + \gamma).$$

It follows that given any real κ, $\varepsilon_u(1 + \gamma) > \kappa$ for all sufficiently large γ. Hence for fixed u the inequality

$$G_u(\gamma) > \gamma + \kappa \tag{13}$$

holds for any κ and all sufficiently large γ. This lower bound for the function $F_s = G_u$ is used to show that the equation $F_s(\gamma) = \gamma + \lambda$, where $\lambda = 2|\Gamma_0|/|\partial K_0| \in (0, \infty)$ (see (3)), is soluble when $\lambda \geqslant F_s(0) = G_u(0)$, i.e., when $\lambda \geqslant \zeta(u - 1)/\zeta(u) > 1$.

Lemma 5. *Suppose $\lambda \geqslant G_u(0)$. Then there exists a unique $\gamma_0 \geqslant 0$ such that*

$$G_u(\gamma_0) = \gamma_0 + \lambda.$$

Proof. Suppose without loss of generality that $\lambda > G_u(0)$. By definition

$$G_u(0) - \lambda < 0.$$

But by equation (13), for fixed u and sufficiently large γ,

$$G_u(\gamma) > \gamma + \lambda$$

and by continuity, there exists a $\gamma_0 > 0$ such that $G_u(\gamma_0) = \gamma_0 + \lambda$.

The uniqueness of the root γ_0 follows from considering the derivative $\partial G_u(\gamma)/\partial\gamma$ for $\gamma > 0$. The partial derivative $\partial\zeta_1(u; \gamma)/\partial\gamma = -u\,\zeta_1(u + 1; \gamma)$. Thus

$$\begin{aligned} \frac{\partial G_u(\gamma)}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\frac{\zeta_1(u-1,\gamma)}{\zeta_1(u;\gamma)} = \frac{-(u-1)\,\zeta_1(u;\gamma)\zeta_1(u;\gamma) + u\,\zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)}{\zeta_1(u;\gamma)^2} \\ &= u\left(\frac{\zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)}{\zeta_1(u;\gamma)^2} - 1\right) + 1 > 1 \end{aligned}$$

if $\zeta_1(u; \gamma)^2 < \zeta_1(u - 1; \gamma)\zeta_1(u + 1; \gamma)$. Now by the Cauchy-Schwarz inequality,

$$\begin{aligned} \zeta_1(u;\gamma) &= \sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{(u-1)/2}} \frac{1}{(k+\gamma)^{(u+1)/2}} \\ &\leqslant \left(\sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{u-1}}\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{(k+\gamma)^{u+1}}\right)^{1/2} = \left(\zeta_1(u-1;\gamma)\zeta_1(u+1;\gamma)\right)^{1/2} \end{aligned}$$

and $\zeta_1(u; \gamma)^2 < \zeta_1(u-1; \gamma)\zeta_1(u+1; \gamma)$, since the inequality is readily seen to be strict. Hence $\partial G_u(\gamma)/\partial\gamma > 1$, so that the graph of G_u crosses that of $\gamma \mapsto \gamma + \lambda$ no more than once. $\square$

Theorem 6. *Let Γ_0 be a curve of finite length with convex hull K_0. Then for $0 < s < 1$,*

$$h_0^{(s)} \leqslant \frac{1}{1-s} \log\left(\frac{s}{\beta(1-s)}\right) + \log \zeta_1\left(\frac{1}{1-s};\gamma\right), \tag{14}$$

where $\beta = \beta(s, 2|\Gamma_0|/|\partial K_0|)$, $\gamma = \gamma(s, 2|\Gamma_0|/|\partial K_0|)$. Moreover, provided $1/2 < s < 1$ and

$$\frac{2|\Gamma_0|}{|\partial K_0|} > \frac{\zeta(s/(1-s))}{\zeta(1/(1-s))} > 1, \tag{15}$$

there exists a unique γ_0 such that

$$h_0^{(s)} \leqslant \frac{1}{1-s} \log\left(\frac{2|\Gamma_0|}{|\partial K_0|} + \gamma_0\right) + \log \zeta_1\left(\frac{1}{1-s};\gamma_0\right). \tag{16}$$

Proof. As with Shannon entropy, we use Lagrange multipliers under the same two constraints (2) and (3) to find a bound for $h_0^{(s)}$. Consider

$$V = \frac{1}{1-s}\log\sum_{n=1}^{\infty}\widehat{p}_n^s - \alpha\sum_{n=1}^{\infty}\widehat{p}_n - \beta\sum_{n=1}^{\infty}n\widehat{p}_n.$$

Then for each $n = 1, 2, \ldots,$

$$\frac{\partial V}{\partial \widehat{p}_n} = \frac{s\widehat{p}_n^{s-1}}{(1-s)\sum_{n=1}^{\infty}\widehat{p}_n^s} - \alpha - \beta n.$$

Hence $\partial V/\partial\widehat{p}_n = 0$ implies

$$\widehat{p}_n^{s-1} = \beta\left(\frac{1-s}{s}\right)\left(\sum_{n=1}^{\infty}\widehat{p}_n^s\right)(\gamma+n),$$

where $\gamma = \alpha/\beta$. Hence the 'maximal' distribution is given by

$$\widehat{p}_n = \left(\frac{s}{\beta(1-s)}\right)^{1/(1-s)}\left(\sum_{n=1}^{\infty}\widehat{p}_n^s\right)^{-1/(1-s)}\left(\frac{1}{\gamma+n}\right)^{1/(1-s)}. \tag{17}$$

We compute $\sum_{n=1}^{\infty}\widehat{p}_n^s$ in two ways. First using (2),

$$1 \;=\; \sum_{n=1}^{\infty}\widehat{p}_n = \left(\frac{s}{\beta(1-s)}\right)^{1/(1-s)}\left(\sum_{n=1}^{\infty}\widehat{p}_n^s\right)^{-1/(1-s)}\sum_{n=1}^{\infty}\left(\frac{1}{\gamma+n}\right)^{1/(1-s)},$$

whence

$$\left(\sum_{n=1}^{\infty}\widehat{p}_n^s\right)^{1/(1-s)} = \left(\frac{s}{\beta(1-s)}\right)^{1/(1-s)}\zeta_1\left(\frac{1}{1-s};\gamma\right).$$

i.e.,

$$\sum_{n=1}^{\infty} \widehat{p}_n^s = \left(\frac{s}{\beta(1-s)}\right) \zeta_1 \left(\frac{1}{1-s}; \gamma\right)^{1-s}, \tag{18}$$

where the right hand side converges for $s \in (0,1)$ and (14) follows.

Secondly, substitute (18) in (17) to get

$$\widehat{p}_n = \zeta_1 \left(\frac{1}{1-s}; \gamma\right)^{-1} \left(\frac{1}{\gamma+n}\right)^{1/(1-s)},$$

so that, by contrast with the Shannon entropy case (10), the distribution $\widehat{p}_n$ obeys an inverse power law, with factor the reciprocal of a modified Hurwitz zeta function. Hence

$$\sum_{n=1}^{\infty} \widehat{p}_n^s = \zeta_1 \left(\frac{1}{1-s}; \gamma\right)^{-s} \sum_{n=1}^{\infty} \left(\frac{1}{\gamma+n}\right)^{s/(1-s)} = \frac{\zeta_1\left(\frac{s}{1-s}; \gamma\right)}{\zeta_1\left(\frac{1}{1-s}; \gamma\right)^s}, \tag{19}$$

which converges when $1/2 < s < 1$ but diverges for $s \leqslant 1/2$.

To determine γ, use the constraint $2|\Gamma_0|/|\partial K_0| = \lambda$ given by (3):

$$\begin{aligned}
\frac{2|\Gamma_0|}{|\partial K_0|} = \lambda \quad &= \quad \sum_{n=1}^{\infty} n\widehat{p}_n = \zeta_1 \left(\frac{1}{1-s}; \gamma\right)^{-1} \sum_{n=1}^{\infty} n \left(\frac{1}{\gamma+n}\right)^{1/(1-s)} \\
&= \quad \zeta_1 \left(\frac{1}{1-s}; \gamma\right)^{-1} \sum_{n=1}^{\infty} \left[\frac{\gamma+n}{(\gamma+n)^{1/(1-s)}} - \frac{\gamma}{(\gamma+n)^{1/(1-s)}}\right] \\
&= \quad \zeta_1 \left(\frac{1}{1-s}; \gamma\right)^{-1} \left(\zeta_1\left(\frac{s}{1-s}; \gamma\right) - \gamma \zeta_1 \left(\frac{1}{1-s}; \gamma\right)\right) \\
&= \quad \frac{\zeta_1\left(\frac{s}{1-s}; \gamma\right)}{\zeta_1\left(\frac{1}{1-s}; \gamma\right)} - \gamma.
\end{aligned} \tag{20}$$

By (15) and Lemma 5 with $u = 1/(1-s)$, $1/2 < s < 1$, there is a $\gamma_0 = \gamma_0(s, \lambda)$ satisfying (20), i.e., such that

$$G_u(\gamma_0) = F_s(\gamma_0) = \frac{\zeta_1\left(\frac{s}{1-s}; \gamma_0\right)}{\zeta_1\left(\frac{1}{1-s}; \gamma_0\right)} = \lambda + \gamma_0.$$

Now by (19),

$$h^{(s)} \quad = \quad \frac{1}{1-s} \log \left(\sum_{n=1}^{\infty} p_n^s\right) \leqslant \frac{1}{1-s} \log \left(\sum_{n=1}^{\infty} \widehat{p}_n^s\right) = \widehat{h}^{(s)}$$

and for $u = 1/(1-s)$,

$$\begin{aligned}
\widehat{h}^{(s)} \quad &= \quad \frac{1}{1-s} \log \left[\frac{\zeta_1\left(\frac{s}{1-s}; \gamma_0\right)}{\zeta_1\left(\frac{1}{1-s}; \gamma_0\right)}\right] + \log \zeta_1 \left(\frac{1}{1-s}; \gamma_0\right) \\
&= \quad \frac{1}{1-s} \log(\lambda + \gamma_0) + \log \zeta_1 \left(\frac{1}{1-s}; \gamma_0\right),
\end{aligned}$$

giving (16).

The corresponding parameters β_0 and $\alpha_0 = \beta_0\,\gamma_0$, are fixed by the equation

$$\frac{s}{\beta_0(1-s)} = \frac{\zeta_1\left(\frac{s}{1-s};\gamma_0\right)}{\zeta_1\left(\frac{1}{1-s};\gamma_0\right)} = F_s(\gamma_0) = \lambda + \gamma_0\,,$$

obtained by dividing (18) by (19) and using (20). This equation implies that as $s \to 1$ from below, $\beta_0 \to \infty$ and therefore $\alpha_0 \to \infty$, while $F_s(\gamma_0) \to \infty$ as $s \to 1/2$ from above, which implies that β_0, $\alpha_0 \to 0$. ☐

Remark 7. The bound for the Rényi entropy $h^{(s)}$ is finite for $1/2 < s < 1$. The equation $F_s(\gamma_0) = \gamma_0 + \lambda$ only holds if $\lambda = 2|\Gamma_0|/|\partial K_0| \geqslant F_s(0) = \zeta(s/(s-1))/\zeta(1/(s-1)) > 1$. This and the values α, β do not have an obvious interpretation. Nor does the bound for the Shannon entropy appear to follow from the limit of the Rényi entropy case as $s \to 1$.

4 Changing the viewpoint: the parameter t

In this section, we discuss the Shannon-Rényi entropy $h^{(s)}$ of the curve Γ_0 relative to a compact subset K containing K_0; this involves a parameter t. Consider the curve Γ_0, its convex hull K_0 and a bounded convex set $K \supseteq K_0$. The set K can be thought of as a variable set that "increases" and within which the curve can increase its entropy.

4.1 The parameter t

Let $t \geqslant 1$ be the ratio of the lengths of the boundaries of K and K_0, i.e., let

$$t = \frac{|\partial K|}{|\partial K_0|} \in [1, \infty).$$

In this construction, the length $|\partial K|$ of the boundary of the convex set K increases.

Lemma 8. *The probability $p_n(\Gamma_0, K)$ that a straight line D meeting K intersects Γ_0 in exactly n points is given by*

$$p_n(\Gamma_0, K) = p_n(K) = \begin{cases} \frac{1}{t}\,p_n(K_0), & \text{if } n \geqslant 1; \\ 1 - \frac{1}{t}, & \text{if } n = 0. \end{cases} \tag{21}$$

Proof. If $n \geqslant 1$,

$$\begin{aligned} p_n(\Gamma_0, K) &= \frac{\mu\{D\colon \operatorname{card}(D\cap\Gamma_0) = n\}}{\mu\{D\colon D\cap K \neq \emptyset\}} \\ &= \frac{\mu\{D\colon D\cap K_0 \neq \emptyset\}}{\mu\{D\colon D\cap K \neq \emptyset\}}\,\frac{\mu\{D\colon \operatorname{card}(D\cap\Gamma_0) = n\}}{\mu\{D\colon D\cap K_0 \neq \emptyset\}} \\ &= \frac{|\partial K_0|}{|\partial K|}\,\frac{\mu\{D\colon \operatorname{card}(D\cap\Gamma_0) = n\}}{\mu\{D\colon D\cap K_0 \neq \emptyset\}} = \frac{1}{t}\,p_n(K_0, \Gamma_0). \end{aligned}$$

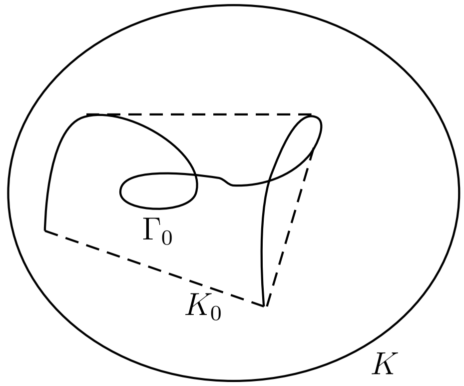

Figure 2: The curve Γ_0 inside a bounded convex set K.

If $n = 0$,

$$p_0(\Gamma_0, K) = 1 - \frac{|\partial K_0|}{|\partial K|} = 1 - \frac{1}{t},$$

as in (1) in §2.

$\square$

Let $h_0^{(s)} := h^{(s)}(\Gamma_0, K_0)$, $0 < s \leqslant 1$, be the Shannon-Rényi entropy of Γ_0 with respect to its convex hull K_0. In the preceding section §3, $h_0^{(s)}$ is written by h or $h^{(1)}$ when $s = 1$ (Shannon entropy) and $h^{(s)}$ when $0 < s < 1$. We begin with the case $s = 1$.

4.2 The Shannon entropy case

The Shannon entropy $h(\Gamma_0, K) = h$ of Γ_0 relative to K is now determined in terms of the Shannon entropy $h(\Gamma_0, K_0) = h_0$ of Γ_0 relative to the convex hull K_0 of Γ_0 (h_0 is finite by Theorem 1).

Theorem 9. *Let h_0 be the Shannon entropy of Γ_0 with respect to K_0. The entropy $h(\Gamma_0, K)$ of Γ_0 with respect to K is given by*

$$h(\Gamma_0, K) = \frac{h_0}{t} + \log t - \frac{t-1}{t}\log(t-1) = \frac{h_0}{t} + (-1 + \frac{1}{t})\log(1 - \frac{1}{t}) + \frac{\log t}{t} \to 0$$

as $t \to \infty$. Moreover $h(\Gamma_0, K)$ is maximal when $t = t_1 = 1 + e^{-h_0}$, with value $h_1 = \log(1 + e^{h_0}) > h_0$.

Proof. Let $h = h(K)$ be the entropy of Γ_0 relative to K and let $h_0 = h(K_0)$ be the entropy

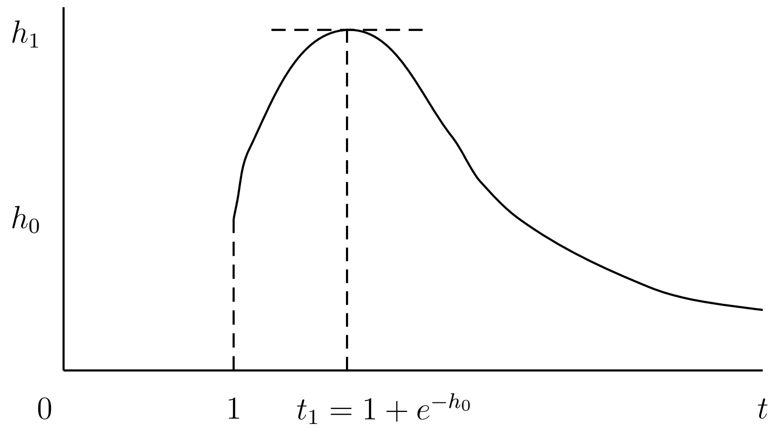

Figure 3: The graph of h with maximum at t_1

of Γ_0 relative to K_0. Then by (4) and Lemma 8 (and writing $p_n(K) = p_n(\Gamma_0, K)$),

$$\begin{aligned}
h &= \sum_{n=0}^{\infty} p_n(K) \log \frac{1}{p_n(K)} \\
&= \left(1 - \frac{1}{t}\right) \log \left(1 - \frac{1}{t}\right)^{-1} + \sum_{n=1}^{\infty} \frac{1}{t}\, p_n(K_0) \log \frac{t}{p_n(K_0)} \\
&= \left(1 - \frac{1}{t}\right) \log \frac{t}{t-1} + \frac{1}{t} \sum_{n=1}^{\infty} p_n(K_0) \log t + \frac{1}{t} \sum_{n=1}^{\infty} p_n(K_0) \log \frac{1}{p_n(K_0)} \\
&= \frac{h_0}{t} + \log t - \frac{t-1}{t} \log(t-1).
\end{aligned}$$

We wish to find the maximum value of h when $t \in (1, \infty)$. The derivative

$$\frac{dh}{dt} = -\frac{1}{t^2}(h_0 + \log(t-1)) = 0$$

for $t_1 = |\partial K_1|/|\partial K_0| = 1 + e^{-h_0}$, with entropy h a maximum at $t = t_1$ and corresponding value

$$h_1 := \log(1 + e^{h_0}) > h_0.$$

Here K_1 is any convex set whose boundary has length $t_1 |\partial K_0|$ and which contains K_0. $\square$

4.3 The Rényi entropy case

For convenience $h^{(s)}(\Gamma_0, K)$, the Rényi entropy with respect to the convex set $K \supset \Gamma_0$ will be written $h^{(s)}$, $p_n(\Gamma_0, K)$ will be written $p_n(K)$ and $h^{(s)}(\Gamma_0, K_0)$, the Rényi entropy with respect to the convex hull K_0 of Γ_0 will be written $h_0^{(s)}$. The dependence of $h^{(s)}$ on the parameter t will be indicated where helpful; thus when $t = 1$, $h^{(s)}(t) = h^{(s)}(1) = h_0^{(s)}$. Note that by Theorem 6, the quantity $h_0^{(s)}$ is finite for $s \in (0, 1)$.

Theorem 10. *The Rényi entropy $h^{(s)}(\Gamma_0, K)$, $0 < s < 1$, of Γ_0 with respect to K is given by*

$$\begin{aligned} h^{(s)}(\Gamma_0, K) &= \frac{1}{1-s} \log\left(\left(1-\frac{1}{t}\right)^s + \sum_{n=1}^{\infty} \frac{1}{t^s} p_n^s(K_0)\right) \\ &= \frac{1}{1-s} \log\left(1+(t-1)^{-s} e^{(1-s)h_0^{(s)}}\right) + \frac{s}{1-s} \log\left(1-\frac{1}{t}\right), \end{aligned}$$

so that $h^{(s)} \to 0$ as $t \to \infty$. Moreover $h^{(s)}(\Gamma_0, K)$ is maximal for

$$t_1 = 1 + \left(\sum_{n=1}^{\infty} p_n^s(K_0)\right)^{-1/(1-s)} = 1 + e^{-h_0^{(s)}},$$

with value $h^{(s)}(t_1) = \log(1 + e^{h_0^{(s)}})$.

Proof. Write $h^{(s)}(\Gamma_0, K) = h^{(s)}$. It follows from (21) that

$$\begin{aligned} h^{(s)} &= h^{(s)}(t) = \frac{1}{1-s} \log \sum_{n=0}^{\infty} p_n^s(K) = \frac{1}{1-s} \log\left(\left(1-\frac{1}{t}\right)^s + \sum_{n=1}^{\infty} \frac{1}{t^s} p_n^s(K_0)\right) \\ &= \frac{s}{1-s} \log\left(1-\frac{1}{t}\right) + \frac{1}{1-s} \log\left(1+\frac{e^{(1-s)h_0^{(s)}}}{(t-1)^s}\right), \end{aligned}$$

as claimed.

We now ask which $t \geqslant 1$ maximizes $h^{(s)} = h^{(s)}(t)$. The derivative

$$\frac{dh^{(s)}}{dt} = \frac{s}{(1-s)\left(\sum_{n=0}^{\infty} p_n^s(K_0)\right)} \left((1-\frac{1}{t})^{s-1}\frac{1}{t^2} - \frac{1}{t^{s+1}} \sum_{n=1}^{\infty} p_n^s(K_0)\right) = 0$$

iff

$$(t-1)^{s-1} = \sum_{n=1}^{\infty} p_n^s(K_0) = e^{(1-s)\, h_0^{(s)}},$$

i.e., iff

$$t = 1 + \left(\sum_{n=1}^{\infty} p_n^s(K_0)\right)^{-1/(1-s)} = 1 + e^{-h_0^{(s)}}. \tag{22}$$

The entropy $h^{(s)}(t)$ is maximal at the value t_1 given by (22), i.e.,

$$\begin{aligned} h^{(s)}(t_1) &= \frac{1}{1-s} \log \sum_{n=1}^{\infty} p_n^s(K_0) + \log\left(1 + \left(\sum_{n=1}^{\infty} p_n^s(K_0)\right)^{-\frac{1}{(1-s)}}\right) \\ &= h_0^{(s)} + \log\left(1 + e^{-h_0^{(s)}}\right), \end{aligned} \tag{23}$$

whence $h^{(s)}(t_1) = \log\left(1 + e^{h_0^{(s)}}\right)$. The first term on the right hand side of (23) is the s-entropy $h_0^{(s)} = h^{(s)}(1)$ of Γ_0 relative to its convex hull K_0, so that $h^{(s)}(t_1) > h^{(s)}(1)$. $\square$

Remark 11. For a certain 'dilution' corresponding to $|\partial K|/|\partial K_0| = t_1$, the Rényi s-entropy attains a maximal value and then decreases to 0. Suppose now $s \to 1$. Then it is easily seen that $t_1 \to 1 + e^{-h_0}$ and $h^{(s)}(t_1) \to \log(1 + e^{h_0})$, which is consistent with the results above in §4.2.

If we agree to identify entropy and complexity, we see that the complexity of a curve depends on the point of observation. Seen from a certain distance, the curve increases its complexity, while seen from infinity the curves reduces to a point with entropy 0.

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