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Generalized Akiyama-Tanigawa Algorithm for Hypersums of Powers of Integers

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Abstract

In this paper we consider the hypersum polynomials, $P_k^{(m)}(n) = \sum_{r=0}^{k+m+1} c_{k,m}^r n^r$, and give an explicit formula for the coefficients $c_{k,m}^r$. We show that the $c_{k,m}^r$'s satisfy a generalized Akiyama-Tanigawa recurrence relation, thus extending some previous results due to Inaba. We also give a number of identities involving the Stirling numbers of the first and second kinds, as well as Bernoulli and harmonic numbers.

1 Introduction

Akiyama and Tanigawa, in the course of their investigation of multiple zeta values at non-positive integers [1], found an algorithm to calculate the Bernoulli numbers in a manner similar to Pascal's triangle for binomial coefficients. The Akiyama-Tanigawa algorithm, as reformulated by Kaneko [2] and Chen [3], is described by the sequence $a_{k,m}$ defined recursively by

$$a_{k,m} = (m+1)(a_{k-1,m} - a_{k-1,m+1}), \quad k \geq 1, m \geq 0,$$

for a given initial sequence $a_{0,m}$, $m = 0, 1, 2, \ldots$. If we start with $a_{0,m} = 1/(m+1)$, then it can be shown [2] that the leading element $a_{k,0}$ is the k-th Bernoulli number B_k (with $B_1 = \frac{1}{2}$).

Later, Inaba [4] considered hypersums of powers of integers and found that the coefficient of the first-degree term in the hypersum polynomial coincides with the element $a_{k,m}$ of the Akiyama-Tanigawa matrix. In this paper (Section 2) we give an explicit expression for the coefficients of the hypersum polynomials in terms of the Stirling numbers of the first and second kinds. Moreover, in Section 3, we derive a recursive relationship for the hypersums. Based on this relationship, in Section 4 we show that the coefficients of the hypersum polynomials satisfy a generalized Akiyama-Tanigawa recurrence relation. Further, in Section 5, as an illustration of the general theory, we give a detailed treatment of the coefficient of the second-degree term in the hypersum polynomial, and provide the general result for the third-degree term. We conclude in Section 6 with a brief historical account of the work of Johann Faulhaber on power sums.

2 Hypersums of powers of integers

Using Inaba's notation [4], the hypersums of powers of integers are defined recursively as

$$P_k^{(m)}(n) = \sum_{j=1}^{n} P_k^{(m-1)}(j), \qquad m \geq 1,$$

where $P_k^{(0)}(n)$ is the sum of the first n positive integers each raised to the integer power $k \geq 0$, $P_k^{(0)}(n) = 1^k + 2^k + 3^k + \cdots + n^k$. There exist several formulations for $P_k^{(0)}(n)$ (see, for instance, [5]). A convenient formula for our purposes is given in terms of the Stirling numbers of the second kind $S(k,i)$ (Sloane's A008277 [6])

$$P_k^{(0)}(n) = \sum_{i=1}^{k} i! \binom{n+1}{i+1} S(k,i), \qquad k \geq 1,$$

with $\binom{n+1}{i+1} = 0$ for $n < i$. A detailed derivation of this formula appears, for example, in the article [7]. For hypersums of arbitrary order m this formula generalizes to [4]

$$P_k^{(m)}(n) = \sum_{i=1}^{k} i! \binom{n+m+1}{i+m+1} S(k,i), \qquad k \geq 1. \tag{1}$$

In addition, for $k = 0$, $P_0^{(m)}(n)$ turns out to be

$$P_0^{(m)}(n) = \binom{n+m}{m+1}. \tag{2}$$

From (1), it is readily seen that $P_k^{(m)}(n)$ is a polynomial in n of degree $k+m+1$ with constant term zero. This follows from the fact that each $\binom{n+m+1}{i+m+1}$ can be expanded as a polynomial in n of degree $i+m+1$, and that the maximum value taken by i is k. Further, from (1) we get $P_k^{(m)}(0) = 0$ since $\binom{m+1}{i+m+1}$ is zero for each $1 \leq i \leq k$. Thus $P_k^{(m)}(n)$ admits a polynomial representation of the form

$$P_k^{(m)}(n) = \sum_{r=0}^{k+m+1} c_{k,m}^r n^r, \tag{3}$$

with $c_{k,m}^0 = 0$ for all k and m.

The following proposition gives us an explicit formula for the coefficients $c_{k,m}^r$. From now on, $|s(r,t)|$ will denote the (unsigned) Stirling numbers of the first kind, also known as Stirling cycle numbers. (The Stirling numbers of the first kind (signed) are given by $s(r,t) = (-1)^{r-t}|s(r,t)|$, see A008275 in [6].)

Proposition 1. *For $k = 0$ and $1 \leq r \leq m + 1$, we have*

$$c^r_{0,m} = \frac{1}{(m+1)!} C(m, r), \tag{4}$$

where

$$C(m, r) = \sum_{t=r}^{m+1} (-1)^{m+1-t} |s(m + 1, t)| \binom{t}{r} m^{t-r}. \tag{5}$$

On the other hand, for $k \geq 1$ and $1 \leq r \leq m + k + 1$, $c^r_{k,m}$ is given by

$$c^r_{k,m} = \sum_{i=max\{1,r-m-1\}}^{k} \frac{i!}{(i+m+1)!} C(i, m, r) S(k, i), \tag{6}$$

where

$$C(i, m, r) = \sum_{t=r}^{i+m+1} (-1)^{i+m+1-t} |s(i + m + 1, t)| \binom{t}{r} (m + 1)^{t-r}. \tag{7}$$

Proof. Let us first prove relation (6). For that, write the hypersum $P^{(m)}_k(n)$ in (1) as

$$P^{(m)}_k(n) = \sum_{i=1}^{k} \frac{i!}{(i+m+1)!} [n + m + 1]_{i+m+1} S(k, i),$$

where $[n]_k$ denotes the falling factorial $n(n-1)(n-2)\cdots(n-k+1)$. Considering n as a variable, $[n+m+1]_{i+m+1}$ can be regarded as a polynomial in n of degree $i+m+1$. Therefore, to prove (6) it suffices to show that the coefficient of the r-degree term in $[n + m + 1]_{i+m+1}$ is equal to $C(i, m, r)$. But, by definition of the Stirling numbers of the first kind in terms of the falling factorial, $[n + m + 1]_{i+m+1}$ can be expressed as

$$[n + m + 1]_{i+m+1} = \sum_{t=1}^{i+m+1} (-1)^{i+m+1-t} |s(i + m + 1, t)| (n + m + 1)^t.$$

Furthermore, by the binomial theorem we have

$$(n + m + 1)^t = \sum_{r=0}^{t} \binom{t}{r} n^r (m + 1)^{t-r},$$

and then the r-degree coefficient in $(n + m + 1)^t$ is $\binom{t}{r}(m + 1)^{t-r}$, from which we in turn deduce that the r-degree coefficient in $[n + m + 1]_{i+m+1}$ is just $C(i, m, r)$. Moreover, since $C(i, m, r) = 0$ for $i+m+1 < r$, we can restrict the summation in (6) to the values $i = 1, \ldots, k$ if $r - m \leq 1$, or else to the values $i = r - m - 1, \ldots, k$ if $r - m \geq 2$.

Similarly, to prove (4), we first note that (2) can be written as

$$\binom{n+m}{m+1} = \frac{[n + m]_{m+1}}{(m + 1)!}.$$

As before, $[n+m]_{m+1}$ can be expanded as

$$[n+m]_{m+1} = \sum_{t=1}^{m+1} (-1)^{m+1-t} |s(m+1,t)| (n+m)^t.$$

Since the r-degree coefficient of $(n+m)^t$ is $\binom{t}{r} m^{t-r}$ we conclude that the r-degree coefficient in $[n+m]_{m+1}$ is equal to $C(m,r)$, and hence the proof of (4) is done. $\square$

Next we give an alternative, closed formula for $c_{0,m}^r$ that will be used later when we discuss the generalized Akiyama-Tanigawa algorithm in Section 4.

Proposition 2. *For* $1 \leq r \leq m+1$

$$c_{0,m}^r = \frac{|s(m+1,r)|}{(m+1)!}. \tag{8}$$

Proof. This follows immediately when we write $\binom{n+m}{m+1}$ as

$$\binom{n+m}{m+1} = \frac{[n]^{m+1}}{(m+1)!},$$

where $[n]^k$ denotes the rising factorial $n(n+1)(n+2)\cdots(n+k-1)$. To prove the proposition we have to show that $|s(m+1,r)|$ constitutes the r-degree term of $[n]^{m+1}$. But, from the algebraic definition of the (unsigned) Stirling numbers of the first kind in terms of the rising factorial, we have that

$$[n]^{m+1} = \sum_{r=1}^{m+1} |s(m+1,r)| n^r,$$

and thus relation (8) is proved. $\square$

A direct consequence of (8) is that, for any given r, all the coefficients $c_{0,m}^r$ $(m = 0,1,2,\ldots)$ are non-negative. On the other hand, from (6) and (7) we quickly deduce the leading coefficient of $P_k^{(m)}(n)$,

$$c_{k,m}^{k+m+1} = \frac{k!}{(k+m+1)!}, \tag{9}$$

which applies to all $k \geq 0$ and $m \geq 0$. Note that, as expected, for $m = 0$ we retrieve the well-known result $c_{k,0}^{k+1} = \frac{1}{k+1}$. In addition to $c_{k,m}^{k+m+1}$, from (6) and (7) we can also obtain closed-form expressions for the successive high-degree coefficients of $P_k^{(m)}(n)$. Here we give the resultant formulas for the first five trailing coefficients next to $c_{k,m}^{k+m+1}$:

$$\begin{aligned}
c_{k,m}^{k+m} &= \frac{k!}{(k+m)!}\frac{m+1}{2}, \quad k \geq 1, m \geq 0, \\
c_{k,m}^{k+m-1} &= \frac{k!}{(k+m-1)!}\frac{(m+1)(3m+2)}{24}, \quad k \geq 2, m \geq 0, \\
c_{k,m}^{k+m-2} &= \frac{k!}{(k+m-2)!}\frac{m(m+1)^2}{48}, \quad k \geq 3, m \geq 0, \\
c_{k,m}^{k+m-3} &= \frac{k!}{(k+m-3)!}(m+1)\frac{15m^3+15m^2-10m-8}{5760}, \quad k \geq 4, m \geq 0, \\
c_{k,m}^{k+m-4} &= \frac{k!}{(k+m-4)!}\frac{m(m+1)^2}{11520}(3m^2-m-6), \quad k \geq 5, m \geq 0.
\end{aligned} \tag{10}$$

Likewise, for $m = 0$, from these formulas we readily get $c_{k,0}^{k} = \frac{1}{2}$, $c_{k,0}^{k-1} = \frac{k}{12}$, $c_{k,0}^{k-2} = 0$, $c_{k,0}^{k-3} = -\frac{k(k-1)(k-2)}{720}$, and $c_{k,0}^{k-4} = 0$.

In the following proposition we give an alternative formula for $C(i,m,r)$, which is most suitable for determining the low-degree coefficients of $P_k^{(m)}(n)$.

Proposition 3. *For $1 \leq p \leq m+2$ and $1 \leq q \leq i$,*

$$C(i,m,r) = \sum_{p+q=r+1} (-1)^{i-q}|s(m+2,p)||s(i,q)|. \tag{11}$$

Proof. We start from the equality $[n+m+1]_{i+m+1} = \frac{1}{n}[n]^{m+2}[n]_i$. Now, putting $[n]^{m+2} = \sum_{p=1}^{m+2}|s(m+2,p)|n^p$ and $[n]_i = \sum_{q=1}^{i}(-1)^{i-q}|s(i,q)|n^q$, we have

$$\frac{1}{n}[n]^{m+2}[n]_i = \sum_{p=1}^{m+2}\sum_{q=1}^{i}(-1)^{i-q}|s(m+2,p)||s(i,q)|n^{p+q-1}.$$

From this expression it is clear that the r-degree coefficient in $\frac{1}{n}[n]^{m+2}[n]_i$ is attained whenever $p+q = r+1$. Thus, recalling that $C(i,m,r)$ denotes the coefficient of the r-degree term in $[n+m+1]_{i+m+1}$, the proposition follows. □

For example, for the case $r = 1$ we must have $p = q = 1$ and then

$$C(i,m,1) = (-1)^{i-1}|s(m+2,1)||s(i,1)| = (-1)^{i-1}(m+1)!(i-1)!, \tag{12}$$

where we have used the fact that $|s(n,1)| = (n-1)!$. Then, taking into account (8) and (6), we find that

$$\begin{aligned}
c_{0,m}^{1} &= \frac{1}{m+1}, \\
c_{k,m}^{1} &= \sum_{i=1}^{k}\frac{(-1)^{i-1}(m+1)!(i-1)!i!}{(i+m+1)!}S(k,i), \quad k \geq 1.
\end{aligned}$$

These relations were previously derived in [4, Prop. 2]. In Section 5 we shall obtain, using relations (8), (6), and (11), the second and third-degree coefficients $c_{k,m}^2$ and $c_{k,m}^3$ of $P_k^{(m)}(n)$.

Next we give a general expression for the leading elements $c_{k,0}^r$ in terms of the Bernoulli numbers.

Proposition 4. *For* $1 \leq r \leq k+1$,

$$c_{k,0}^r = \frac{1}{k+1}\binom{k+1}{r} B_{k+1-r}. \tag{13}$$

with $B_1 = \frac{1}{2}$.

Proof. This expression readily follows from the well-known Bernoulli formula for the ordinary power sums (see, for example, the articles [5] and [8]):

$$P_k^{(0)}(n) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j} B_j n^{k+1-j},$$

where B_1 is taken to be $\frac{1}{2}$. This can also be written as

$$P_k^{(0)}(n) = \frac{1}{k+1}\sum_{r=1}^{k+1}\binom{k+1}{r} B_{k+1-r} n^r.$$

On the other hand, recalling that $c_{k,0}^0 = 0$, the polynomial (3) for $m = 0$ reads as

$$P_k^{(0)}(n) = \sum_{r=1}^{k+1} c_{k,0}^r n^r.$$

Thus, comparing like terms in the previous two expressions, we get (13). $\square$

From (13) we see at once that $c_{k,0}^1 = B_k$ (cf. [2]). Now Proposition 4, in conjunction with equation (6), yields the following identity.

Corollary 5. *For* $1 \leq r \leq k+1$ *and* $k \geq 1$,

$$\sum_{i=max\{1,r-1\}}^{k} \frac{1}{i+1}\left(\sum_{t=r}^{i+1}(-1)^{i+1-t}|s(i+1,t)|\binom{t}{r}\right) S(k,i) = \frac{1}{k+1}\binom{k+1}{r} B_{k+1-r}. \tag{14}$$

Proof. Set $m = 0$ in (6) and identify the resulting expression with (13). $\square$

Putting $r = 1$ in (14) yields the following explicit formula for the Bernoulli numbers (with $B_1 = \frac{1}{2}$):

$$B_k = \sum_{i=1}^{k} \frac{1}{i+1}\left(\sum_{t=1}^{i+1}(-1)^{i+1-t}|s(i+1,t)|t\right) S(k,i), \quad k \geq 1.$$

Furthermore, putting $m=0$ and $r=1$ in (7) and equating the resulting expression to that obtained in (12) for $m=0$ gives

$$\sum_{t=1}^{i+1}(-1)^{i+1-t}|s(i+1,t)|t=(-1)^{i-1}(i-1)!,$$

and then the previous expression for B_k reduces to the classical identity for the Bernoulli numbers in terms of the Stirling numbers of the second kind.

Corollary 6.

$$B_k=\sum_{i=1}^{k}\frac{(-1)^{i-1}(i-1)!}{i+1}S(k,i),\quad k\geq 1.$$

with $B_1=\frac{1}{2}$.

3 A recursive relationship for the hypersums

In this section we derive a recursive relationship for the hypersums $P_k^{(m)}(n)$ that shall constitute the core of the generalized Akiyama-Tanigawa algorithm developed in Section 4. This recurrence relation is presented in Theorem 8 below. To prove this theorem, we shall need the following preliminary result.

Lemma 7.

$$\sum_{j=1}^{n} jP_k^{(m)}(j)=(n+1)P_k^{(m+1)}(n)-P_k^{(m+2)}(n). \tag{15}$$

Proof. Set $l=m+1$. Then, in view of (1), it is clear that proving (15) is tantamount to proving the combinatorial identity

$$\sum_{j=1}^{n} j\binom{j+l}{i+l}=(n+1)\binom{n+l+1}{i+l+1}-\binom{n+l+2}{i+l+2}. \tag{16}$$

To prove (16), we shall repeatedly use the well-known identity

$$\sum_{j=0}^{n}\binom{j}{i}=\binom{n+1}{i+1},$$

with $\binom{j}{i}=0$ for $j<i$. Another crucial ingredient is the absorption property

$$(i+1)\binom{j+1}{i+1}=(j+1)\binom{j}{i},\quad 0\leq i\leq j,$$

which we rewrite in the form

$$j\binom{j}{i}=(i+1)\binom{j+1}{i+1}-\binom{j}{i}. \tag{17}$$

With this in mind we have

$$\begin{aligned}\sum_{j=1}^{n} j\binom{j+l}{i+l} &= \sum_{j=1}^{n}(j+l)\binom{j+l}{i+l} - l\sum_{j=1}^{n}\binom{j+l}{i+l} \\ &= \sum_{s=i+l}^{n+l} s\binom{s}{i+l} - l\sum_{s=i+l}^{n+l}\binom{s}{i+l} \\ &= \sum_{s=i+l}^{n+l} s\binom{s}{i+l} - l\binom{n+l+1}{i+l+1}.\end{aligned}$$

Now from relation (17) it follows that

$$\begin{aligned}\sum_{s=i+l}^{n+l} s\binom{s}{i+l} &= (i+l+1)\sum_{s=i+l+1}^{n+l+1}\binom{s}{i+l+1} - \sum_{s=i+l}^{n+l}\binom{s}{i+l} \\ &= (i+l+1)\binom{n+l+2}{i+l+2} - \binom{n+l+1}{i+l+1},\end{aligned}$$

and then

$$\sum_{j=1}^{n} j\binom{j+l}{i+l} = (i+l+1)\binom{n+l+2}{i+l+2} - (l+1)\binom{n+l+1}{i+l+1}. \tag{18}$$

Applying the absorption property once more, we get

$$(i+l+2)\binom{n+l+2}{i+l+2} = (n+l+2)\binom{n+l+1}{i+l+1}.$$

From this we obtain

$$(i+l+1)\binom{n+l+2}{i+l+2} = (n+l+2)\binom{n+l+1}{i+l+1} - \binom{n+l+2}{i+l+2},$$

and then, substituting this into (18), we retrieve the identity (16) and thus the lemma is proved. □

The extended Akiyama-Tanigawa algorithm for hypersums of powers of integers is based on the following recursive relationship for the hypersums, which is stated in the next theorem.

Theorem 8. *The hypersums $P_{k+1}^{(m)}(n)$, $P_k^{(m)}(n)$, and $P_k^{(m+1)}(n)$ are constrained to obey the relation*

$$P_{k+1}^{(m)}(n) = (m+1)\big(P_k^{(m)}(n) - P_k^{(m+1)}(n)\big) + nP_k^{(m)}(n), \quad k,m \geq 0. \tag{19}$$

Proof. Assume that k and n take fixed (but otherwise arbitrary) integer values $k \geq 0$ and $n \geq 1$, and proceed by induction on m. First we prove the base case where $m = 0$. For this case we must show that

$$(n+1)P_k^{(0)}(n) = P_{k+1}^{(0)}(n) + P_k^{(1)}(n).$$

This is most easily seen by displaying $(n+1)P_k^{(0)}(n)$ as the sum of the $n+1$ rows

$$\left.\begin{array}{l} 1^k + 2^k + 3^k + \cdots + n^k \\ 1^k + 2^k + 3^k + \cdots + n^k \\ \vdots \\ 1^k + 2^k + 3^k + \cdots + n^k \end{array}\right\} n+1,$$

and noting that this sum can be decomposed as the sum of the following two pieces

$$\left.\begin{array}{l} 1^k \\ 1^k + 2^k \\ 1^k + 2^k + 3^k \\ \vdots \\ 1^k + 2^k + 3^k + \cdots + n^k \end{array}\right\} n \quad + \quad \left.\begin{array}{r} 1^k + 2^k + 3^k + \cdots + n^k \\ 2^k + 3^k + \cdots + n^k \\ 3^k + \cdots + n^k \\ \vdots \\ n^k \end{array}\right\} n.$$

Clearly, summing the rows of the piece on the left gives $P_k^{(1)}(n)$. On the other hand, the entries in the i-th column of the piece on the right sum to i^{k+1}, and so the sum of all these columns amounts to $P_{k+1}^{(0)}(n)$.

Next we take as the inductive hypothesis the assumption that, for any given $m \geq 1$, it happens that

$$P_{k+1}^{(m-1)}(j) = m\big(P_k^{(m-1)}(j) - P_k^{(m)}(j)\big) + jP_k^{(m-1)}(j), \tag{20}$$

for arbitrary integers $k \geq 0$ and $j \geq 1$. The task is to derive (19) starting from (20). This is rather immediate once we have established Lemma 7. Indeed, recalling that, by definition,

$$\sum_{j=1}^{n} P_{k+1}^{(m-1)}(j) = P_{k+1}^{(m)}(n), \quad \sum_{j=1}^{n} P_k^{(m-1)}(j) = P_k^{(m)}(n), \text{ and } \sum_{j=1}^{n} P_k^{(m)}(j) = P_k^{(m+1)}(n),$$

from (20) we obtain that

$$P_{k+1}^{(m)}(n) = m\big(P_k^{(m)}(n) - P_k^{(m+1)}(n)\big) + \sum_{j=1}^{n} jP_k^{(m-1)}(j).$$

From Lemma 7 we have

$$\sum_{j=1}^{n} jP_k^{(m-1)}(j) = (n+1)P_k^{(m)}(n) - P_k^{(m+1)}(n),$$

and then substituting this expression into the previous equation we get the recursion formula (19). Having completed the base case and the inductive case, the overall proof of Theorem 8 is done. $\square$

4 Generalized Akiyama-Tanigawa algorithm for hypersums of powers of integers

The Akiyama-Tanigawa algorithm for computing Bernoulli numbers starts with the initial row (the 0-th row) given by the sequence $a_{0,m} = 1/(m+1)$ (for $m \geq 0$), and then generates the row k (for $k \geq 1$) by the rule [2, 3, 4]

$$a_{k,m} = (m+1)(a_{k-1,m} - a_{k-1,m+1}). \tag{21}$$

Then the leading element $a_{k,0}$ of each row is seen to be the k-th Bernoulli number B_k with $B_1 = \frac{1}{2}$ [2].

As noted in the Introduction, Inaba [4] showed that the coefficient of the first-degree term in $P_k^{(m)}(n)$, $c_{k,m}^1$, satisfies a recurrence relation given by (21), namely,

$$c_{k,m}^1 = (m+1)(c_{k-1,m}^1 - c_{k-1,m+1}^1), \tag{22}$$

with the same initial condition $c_{0,m}^1 = 1/(m+1)$, and so we have in fact that $c_{k,m}^1 = a_{k,m}$. Based on Theorem 8, we are going to show that, actually, relation (22) is just a particular case of a more general relationship for the coefficients $c_{k,m}^r$ of the hypersum polynomials (3). This is established in the following proposition, which constitutes one of the main results of this paper.

Proposition 9 (*Generalized Akiyama-Tanigawa algorithm*)**.** *For $1 \leq r \leq k+m+1$, $k \geq 1$, and $m \geq 0$, we have the recurrence relation*

$$c_{k,m}^r = (m+1)\big(c_{k-1,m}^r - c_{k-1,m+1}^r\big) + c_{k-1,m}^{r-1}, \tag{23}$$

with the starting sequence given by

$$c_{0,m}^r = \frac{|s(m+1,r)|}{(m+1)!}. \tag{24}$$

Proof. Write the recursive relationship (19) for the hypersums as

$$P_k^{(m)}(n) = (m+1)\big(P_{k-1}^{(m)}(n) - P_{k-1}^{(m+1)}(n)\big) + nP_{k-1}^{(m)}(n).$$

Replacing $P_k^{(m)}(n)$ for its polynomial expression (3) (with $c_{k,m}^0 = 0$) we obtain

$$\sum_{r=1}^{k+m+1} c_{k,m}^r n^r = (m+1)\left(\sum_{r=1}^{k+m} c_{k-1,m}^r n^r - \sum_{r=1}^{k+m+1} c_{k-1,m+1}^r n^r\right) + \sum_{r=1}^{k+m} c_{k-1,m}^r n^{r+1},$$

or, equivalently,

$$\sum_{r=1}^{k+m+1} c_{k,m}^r n^r = \sum_{r=1}^{k+m+1} (m+1)(c_{k-1,m}^r - c_{k-1,m+1}^r)n^r + \sum_{r=1}^{k+m+1} c_{k-1,m}^{r-1} n^r,$$

on the understanding that $c_{k-1,m}^{k+m+1} = 0$. Thus, by equating like powers of n on both sides of this equation we are finally left with relation (23). On the other hand, the proof of (24) is given in Proposition 2. □

Clearly, for the case $r = 1$ relation (23) reduces to (22), as $c^0_{k-1,m} = 0$ for all $k \geq 1$ and $m \geq 0$. Finally, one can check that the coefficients given in (10) in fact satisfy the recurrence (23). So, for example, for $k \geq 5$ and $m \geq 0$, we have that

$$\begin{aligned}
c^{k+m-3}_{k,m} &= \frac{k!}{(k+m-3)!}(m+1)\frac{15m^3+15m^2-10m-8}{5760}, \\
c^{k+m-3}_{k-1,m} &= \frac{(k-1)!}{(k+m-3)!}\,\frac{m(m+1)^2}{48}, \\
c^{k+m-3}_{k-1,m+1} &= \frac{(k-1)!}{(k+m-3)!}(m+2)\frac{15m^3+60m^2+65m+12}{5760}, \\
c^{k+m-4}_{k-1,m} &= \frac{(k-1)!}{(k+m-4)!}(m+1)\frac{15m^3+15m^2-10m-8}{5760},
\end{aligned}$$

which satisfy the relation

$$c^{k+m-3}_{k,m} = (m+1)\big(c^{k+m-3}_{k-1,m} - c^{k+m-3}_{k-1,m+1}\big) + c^{k+m-4}_{k-1,m}.$$

5 Second-degree coefficient of the hypersum polynomial

In this section we examine explicitly the coefficient of the second-degree term in $P^{(m)}_k(n)$, $c^2_{k,m}$, by means of the generalized Akiyama-Tanigawa algorithm. For $r = 2$, recurrence (23) becomes

$$c^2_{k,m} = (m+1)(c^2_{k-1,m} - c^2_{k-1,m+1}) + c^1_{k-1,m}, \tag{25}$$

with initial sequence (the row 0)

$$c^2_{0,m} = \frac{|s(m+1,2)|}{(m+1)!} = \frac{H_m}{m+1}, \tag{26}$$

where we have used that $|s(n,2)| = (n-1)!H_{n-1}$, H_n denoting the harmonic number $H_n = \sum_{i=1}^n 1/i$, with $H_0 = 0$. To obtain the sequence corresponding to row 1, $c^2_{1,m}$, we feed (26) into (25) and, recalling that $c^1_{0,m} = 1/(m+1)$, we get

$$c^2_{1,m} = \frac{H_{m+1}}{m+2}.$$

Similarly, inserting this expression into (25), and noting that $c^1_{1,m} = 1/(m+2)$, we find the elements of row 2

$$c^2_{2,m} = \frac{2+(m+1)H_{m+1}}{(m+2)(m+3)}.$$

Likewise, proceeding this way, we obtain the sequences for rows 3 and 4 as

$$\begin{aligned}
c^2_{3,m} &= \frac{(m+1)(6+mH_{m+1})}{(m+2)(m+3)(m+4)}, \\
c^2_{4,m} &= \frac{(m+1)\big(4+14m+(m-4)(m+1)H_{m+1}\big)}{(m+2)(m+3)(m+4)(m+5)},
\end{aligned}$$

$k\backslash m$	0	1	2	3	4	5	$\cdots$
0	0	1/2	1/2	11/24	5/12	137/360	$\cdots$
1	1/2	1/2	11/24	5/12	137/360	7/20	$\cdots$
2	1/2	5/12	3/8	31/90	23/72	167/560	$\cdots$
3	1/4	1/4	29/120	7/30	227/1008	73/336	$\cdots$
4	0	1/20	3/40	113/1260	25/252	887/8400	$\cdots$
5	−1/12	−1/12	−11/168	−1/21	−23/720	−31/1680	$\cdots$
6	0	−5/84	−5/56	−127/1260	−13/126	−621/6160	$\cdots$
7	1/12	1/12	1/24	0	−53/1584	−31/528	$\cdots$
8	0	7/60	7/40	361/1980	65/396	162641/1201200	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

Table 1: The Akiyama-Tanigawa matrix $\{c^2_{k,m}\}$ for $k = 0/8$ and $m = 0/5$.

and so on.

In Table 1 we display the matrix for $c^2_{k,m}$ that results for the first few values of k and m. From (13), it follows that the sequence corresponding to the 0-th column, $c^2_{k,0}$, $k = 1, 2, 3, \ldots$, is given by $c^2_{k,0} = \frac{1}{2}kB_{k-1}$, with $c^2_{0,0} = 0$.

We can deduce a general expression for $c^2_{k,m}$ by first using Proposition 3 to get $C(i, m, 2)$, and then using (6). For $r = 2$ the allowed values of the ordered pair (p, q) are $(1, 2)$ and $(2, 1)$, and then from (11) we get

$$C(i, m, 2) = (-1)^{i-2}|s(m + 2, 1)||s(i, 2)| + (-1)^{i-1}|s(m + 2, 2)||s(i, 1)|.$$

Recalling that $|s(n, 1)| = (n - 1)!$ and $|s(n, 2)| = (n - 1)!H_{n-1}$, this gives

$$\begin{aligned} C(i, m, 2) &= (-1)^{i-2}(m + 1)!(i - 1)!H_{i-1} + (-1)^{i-1}(m + 1)!H_{m+1}(i - 1)! \\ &= (-1)^{i-1}(m + 1)!(i - 1)!\big(H_{m+1} - H_{i-1}\big). \end{aligned}$$

Provided with $C(i, m, 2)$, we can now use (26) and (6) to obtain $c^2_{k,m}$:

Proposition 10.

$$c^2_{k,m} = \begin{cases} \dfrac{H_m}{m+1}, & \textit{if } k = 0, \\ \displaystyle\sum_{i=1}^{k} \frac{(-1)^{i-1}(m + 1)!(i - 1)!i!\big(H_{m+1} - H_{i-1}\big)}{(i + m + 1)!} S(k, i), & \textit{if } k \geq 1. \end{cases}$$

The following corollary gives us an alternative explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind and the harmonic numbers.

Corollary 11.

$$B_k = \frac{2}{k+1} \sum_{i=1}^{k+1} \frac{(-1)^{i-1}(i - 1)!\big(1 - H_{i-1}\big)}{i + 1} S(k + 1, i), \quad k \geq 0.$$

with $B_1 = \frac{1}{2}$.

Proof. We already know that $c^2_{k,0} = \frac{1}{2}kB_{k-1}$ for $k \geq 1$ or, equivalently, $c^2_{k+1,0} = \frac{1}{2}(k+1)B_k$ for $k \geq 0$. Then put $m = 0$ in the second relation of Proposition 10 and shift k to $k+1$. Equal the resulting expression to $\frac{1}{2}(k+1)B_k$, and hence the corollary follows. □

In the same way, by applying relations (8), (6), and (11), one can determine the successive low-degree coefficients $c^3_{k,m}, c^4_{k,m}, \ldots$. In the following proposition we give the expression for $c^3_{k,m}$ (note that, in this case, we have $c^3_{0,0} = c^3_{0,1} = c^3_{1,0} = 0$.)

Proposition 12.

$$c^3_{k,m} = \begin{cases} \dfrac{1}{2(m+1)}\Big((H_m)^2 - H_m^{(2)}\Big), & \textit{if } k = 0, \\ \displaystyle\sum_{i=1}^{k} \frac{(-1)^{i-1}(m+1)!(i-1)!i!}{2(i+m+1)!}\Big(\big(H_{m+1} - H_{i-1}\big)^2 - H_{m+1}^{(2)} - H_{i-1}^{(2)}\Big)S(k,i), & \textit{if } k \geq 1, \end{cases}$$

where $H_m^{(2)} = \sum_{i=1}^{m} 1/i^2$.

6 Concluding remarks

In his 1631 *Academia Algebrae*, the German mathematician and engineer Johann Faulhaber (1580–1635) presented novel formulas for (in our notation) $P_k^{(0)}(n)$ for values of k ranging from $k = 13$ up to $k = 23$. (He had previously published formulas for $P_k^{(0)}(n)$ up to $k = 12$). These formulas were expressed for the first time in terms of the variable $N = n(n+1)/2$. Moreover, Faulhaber also produced remarkable formulas for sums of power sums. Indeed, as Knuth explains in his comprehensive study of Faulhaber's work on sums of powers [9], he exhibited a totally correct 17-degree polynomial in n for the hypersum $P_6^{(10)}(n)$ (which in Knuth's notation is $\Sigma^{11} n^6$) that, in Knuth's words, would have been quite difficult to obtain by repeated summation. Here we argue that, in fact, $P_6^{(10)}(n)$ can be obtained by repeated summation using the generalized Akiyama-Tanigawa algorithm described in equations (23) and (24). Of course this entails a lot of work since one has to construct 17 tables of coefficients $c^r_{k,m}$, $r = 1, 2, \ldots, 17$, and, for each of these tables, one has to determine, starting from (24) and by repeated application of (23), the pair of elements $c^r_{5,10}$ and $c^r_{5,11}$. Once the values of $c^r_{5,10}$ and $c^r_{5,11}$ have been ascertained for each r, a last application of (23) would allow us to obtain the desired coefficients $c^r_{6,10}$.

Fortunately, 382 years after the time of publication of *Academia Algebrae*, we can routinely run a modern computer to perform the calculations in a fraction of a second. Furthermore, besides relations (23) and (24), we can alternatively use the formulas (6) and (7) to directly evaluate the coefficients we want to. So, putting $k = 6$ and $m = 10$ in (6) and (7) yields

$$c^r_{6,10} = \sum_{i=\max\{1,r-11\}}^{6} \frac{i!}{(i+11)!}\left(\sum_{t=1}^{i+11}(-1)^{i+11-t}|s(i+11,t)|\binom{t}{r}11^{t-r}\right)S(6,i).$$

By means of, for example, a computer algebra system such as *Mathematica*, we can readily compute this expression for $r = 1, 2, \ldots, 17$. Next we quote the results as follows:

$$
\begin{aligned}
c^1_{6,10} &= -96598656000/C, & c^2_{6,10} &= -203020963200/C,\\
c^3_{6,10} &= 90994289760/C, & c^4_{6,10} &= 709177112512/C,\\
c^5_{6,10} &= 1021675563656/C, & c^6_{6,10} &= 812536224500/C,\\
c^7_{6,10} &= 423402217056/C, & c^8_{6,10} &= 155027658357/C,\\
c^9_{6,10} &= 41338556974/C, & c^{10}_{6,10} &= 8177397800/C,\\
c^{11}_{6,10} &= 1208226448/C, & c^{12}_{6,10} &= 132902770/C,\\
c^{13}_{6,10} &= 10728564/C, & c^{14}_{6,10} &= 617100/C,\\
c^{15}_{6,10} &= 23936/C, & c^{16}_{6,10} &= 561/C,\\
c^{17}_{6,10} &= 6/C,
\end{aligned}
$$

where $C = 17!/5! = 2964061900800$, in accordance with (9). We note that $\sum_{r=1}^{17} c^r_{6,10} = 1$. This is an instance of the general relation

$$\sum_{r=1}^{k+m+1} c^r_{k,m} = 1,$$

which follows from the fact that $P_k^{(m)}(1) = 1$ for all $k \geq 0$ and $m \geq 0$.

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