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A Generalization of the Gcd-Sum Function

Ulrich Abel, Waqar Awan, and Vitaliy Kushnirevych
Fachbereich MND
Technische Hochschule Mittelhessen
University of Applied Sciences
Wilhelm-Leuschner-Straße 13
61169 Friedberg
Germany
Ulrich.Abel@mnd.thm.de
Waqar.Awan@stud.h-da.de
Vitaliy.Kushnirevych@mnd.thm.de

Abstract

In this paper we consider the generalization $G_d(n)$ of the Broughan gcd-sum function, i.e., the sum of such gcd's that are divisors of the positive integer d. Examples of Dirichlet series and asymptotic relations for G_d and related functions are given.

1 Introduction

In the recent article [5], Broughan studies the sum of the greatest common divisors of the first n positive integers with n, i.e., the arithmetic function

$$G(n) := \sum_{k=1}^{n} \gcd(k, n).$$

This function arises in deriving asymptotic estimates for a lattice point counting problem [5, Sect. 5]. The function G has polynomial growth as n tends to infinity. For $p \in \mathbb{P}$ (throughout the paper $\mathbb{P}$ denotes the set of prime numbers) and $\alpha \in \mathbb{N}$, it is not difficult to show that

$$G(p^\alpha) = \sum_{j=0}^{\alpha-1} \underbrace{(p-1)p^{\alpha-1-j}}_{\text{number of gcd's equal to } p^j} p^j + 1 \cdot p^\alpha = (\alpha+1)p^\alpha - \alpha p^{\alpha-1}.$$

(cf. [5, Th. 2.2]). Following [5, Cor. 2.1] G is a multiplicative function, i.e., $G(mn) = G(m)G(n)$ for coprime $m, n \in \mathbb{N}$, that is, $\gcd(m, n) = 1$. The corresponding Dirichlet series $\mathcal{G}(s)$ converges at all points of the complex plane, except at the zeros of the Riemann zeta function and the point $s = 2$, where it has a double pole. Moreover, Broughan derives asymptotic expressions for the partial sums of the Dirichlet series at all real values of s.

The following generalization of G (see [2]) arises in the study of distribution of determinant values in residue class rings.

For $d \in \mathbb{N}$, we introduce the function

$$G_d(n) := \sum_{\substack{k=1 \\ \gcd(k,n)|d}}^{n} \gcd(k, n).$$

Obviously, $G(n) = G_n(n)$ and $G_1(n) = \varphi(n)$, where φ is Euler's totient function.

The purpose of this note is to study the function G_d. In the next section we present some elementary properties of G_d. Furthermore, we study the corresponding Dirichlet series $\mathcal{G}_d(s)$. Some of the results will be applied in a forthcoming paper on the distribution of determinant values in residue class rings and finite fields. As an example we mention that in the residue class ring $\mathbb{Z}_n$ ($n \in \mathbb{N}$), for $r \in \mathbb{Z}_n$,

$$H_n(r) = |\{(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n \mid i \cdot j = r\}|,$$

the number of products equal to r having precisely two factors in $\mathbb{Z}_n$, is equal to

$$H_n(r) = \begin{cases} G_n(n) = G(n), & \text{if } r = 0; \\ G_d(n) = G_{\gcd(r,n)}(n), & \text{if } r \neq 0. \end{cases}$$

A similar problem as the calculation of the value $H_n(r)$ in the domain of positive integers is the so-called multiplication table problem posed by Erdős (see [7]): how many integers can be written as a product $i \cdot j$ for a given positive integer $n \in \mathbb{N}$ with positive integers $i \leqslant n$ and $j \leqslant n$? Erdős ([7, 8]) gave the first estimates of this quantity. Tenenbaum [13] had made the results of Erdős more precise. Ford ([9, 10]) derived the exact order of magnitude of the $n \times n$ multiplication table size completely. Koukoulopolous [11, 12] presents a perfect overview of the actual situation and the further development of Ford-Erdős results.

2 Properties of G_d

The following lemma gathers some elementary properties of $G_d(n)$.

Lemma 1.

(i) For $m, n \in \mathbb{N}$, we have $G_m(n) = G_{\gcd(m,n)}(n)$.
In particular, for $m, \alpha \in \mathbb{N}$, $p \in \mathbb{P}$, we have $G_{\gcd(m,p^\alpha)}(p^\alpha) = G_m(p^\alpha)$;

(ii) for coprime $d, n \in \mathbb{N}$, we have $G_d(n) = \varphi(n)$;

(iii) for $d = d_1 d_2$ with $\gcd(d_1, n) = 1$, we have $G_d(n) = G_{d_2}(n)$.

Proof. (i) Since $\gcd(k, n) \mid \gcd(m, n) \iff \gcd(k, n) \mid m$ for all $m, n, k \in \mathbb{N}$, the first formula follows from the definition. One obtains the second one by substituting $n = p^\alpha$.

(ii) If $\gcd(d, n) = 1$, using (i) we get

$$G_d(n) = G_1(n) = \varphi(n).$$

(iii) Since $\gcd(d_1, n) = 1 \Rightarrow \gcd(d_1 d_2, n) = \gcd(d_2, n)$, it follows that

$$G_d(n) = G_{d_1 d_2}(n) = G_{\gcd(d_1 d_2, n)}(n) = G_{\gcd(d_2, n)}(n) = G_{d_2}(n),$$

where we used (i) twice.

The proof of the lemma is completed. ☐

Let ρ_d denote the multiplicative function

$$\rho_d(w) = \begin{cases} w, & \text{if } w \mid d; \\ 0, & \text{if } w \nmid d. \end{cases}$$

Then we have the representation

$$G_d = \rho_d * \varphi, \tag{1}$$

where $*$ denotes Dirichlet product. Indeed,

$$G_d(n) = \sum_{\substack{k=1 \\ \gcd(k,n) \mid d}}^{n} \gcd(k, n) = \sum_{\substack{w \mid d \\ w \mid n}} w \varphi\left(\frac{n}{w}\right) = \sum_{w \mid n} \rho_d(w) \varphi\left(\frac{n}{w}\right) = (\rho_d * \varphi)(n).$$

Therefore, G_d is multiplicative as it is the Dirichlet product of multiplicative functions [1, Th. 2.5(c) and Th. 2.14].

Theorem 2. *G_d is a multiplicative function, i.e., for coprime $m, n \in \mathbb{N}$, we have*

$$G_d(mn) = G_d(m) G_d(n).$$

We also give a direct proof of the preceding theorem.

Proof. Let $d \mid n_1 n_2$ with coprime $n_1, n_2 \in \mathbb{N}$. This implies $d = d_1 d_2$ with $d_1 \mid n_1$ and $d_2 \mid n_2$, so that d_1 and d_2 are coprime. One has

$$G_d(n_1 n_2) = G_{d_1 d_2}(n_1 n_2) = \sum_{w \mid d_1 d_2} w \varphi\left(\frac{n_1 n_2}{w}\right) = \sum_{w_1 \mid d_1} \sum_{w_2 \mid d_2} w_1 w_2 \varphi\left(\frac{n_1 n_2}{w_1 w_2}\right).$$

Because φ is multiplicative and $\gcd\left(\frac{n_1}{w_1}, \frac{n_2}{w_2}\right) = 1$, one obtains

$$G_d(n_1n_2) = \sum_{w_1|d_1} w_1\varphi\left(\frac{n_1}{w_1}\right) \sum_{w_2|d_2} w_2\varphi\left(\frac{n_2}{w_2}\right) = G_{d_1}(n_1)\, G_{d_2}(n_2) = G_d(n_1)\, G_d(n_2).$$

This completes the proof. $\square$

Theorem 3. *For $n \in \mathbb{N}$ and for coprime $d_1, d_2 \in \mathbb{N}$, we have*

$$G_{d_1}(n) \cdot G_{d_2}(n) = \varphi(n) \cdot G_{d_1d_2}(n).$$

In particular, $G_{d_1d_2}(n) \mid G_{d_1}(n)G_{d_2}(n)$.

Proof. Let $d = d_1d_2$ with $\gcd(d_1, d_2) = 1$. By Equation (1) we have

$$G_d(n) = (\rho_d * \varphi)(n) = \sum_{w|n} \rho_{d_1d_2}(w)\varphi\left(\frac{n}{w}\right) = \sum_{w_1|n}\sum_{w_2|n} \rho_{d_1}(w_1)\rho_{d_2}(w_2)\varphi\left(\frac{n}{w_1w_2}\right).$$

Now, decompose $n = kn_1n_2$ in a product of three pairwise coprime factors k, n_1, n_2 such that $d_i \mid n_i$ $(i = 1, 2)$. If $w_i \mid d_i$ $(i = 1, 2)$ we conclude that

$$\varphi\left(\frac{n}{w_1w_2}\right) = \varphi(k)\varphi\left(\frac{n_1}{w_1}\right)\varphi\left(\frac{n_2}{w_2}\right) = \varphi(k)\,\frac{\varphi\left(\frac{n}{w_1}\right)\varphi\left(\frac{n}{w_2}\right)}{\varphi(kn_2)\varphi(kn_1)} = \frac{\varphi\left(\frac{n}{w_1}\right)\varphi\left(\frac{n}{w_2}\right)}{\varphi(n)}.$$

Hence, we obtain

$$\varphi(n)\, G_d(n) = \sum_{w_1|n} \rho_{d_1}(w_1)\varphi\left(\frac{n}{w_1}\right) \sum_{w_2|n} \rho_{d_2}(w_2)\varphi\left(\frac{n}{w_2}\right) = G_{d_1}(n) \cdot G_{d_2}(n)$$

which is the desired formula. $\square$

We close this section with the following nice formula.

Theorem 4. *For all $n \in \mathbb{N}$, we have*

$$\sum_{i=1}^{n} G_i(n) = n^2.$$

Proof. Analogously to the proof of Theorem 2 one has

$$\begin{aligned}\sum_{i=1}^{n} G_i(n) &= \sum_{i=1}^{n}\sum_{w|n} \rho_i(w)\varphi\left(\frac{n}{w}\right) = \sum_{w|n}\varphi\left(\frac{n}{w}\right)\sum_{i=1}^{n}\rho_i(w)\\ &= \sum_{w|n}\varphi\left(\frac{n}{w}\right) w \sum_{\substack{1\leqslant i\leqslant n\\ w|i}} 1 = \sum_{w|n}\varphi\left(\frac{n}{w}\right) w\frac{n}{w} = n\sum_{w|n}\varphi(w) = n^2,\end{aligned}$$

where we used that $\sum_{w|n}\varphi(w) = n$. $\square$

3 Evaluation of G_d at positive integers

In this section we consider the problem how to calculate the values of $G_d(n)$ for positive integers. We start with the special case of prime powers. In the following $\delta_{\alpha\beta}$ denotes the Kronecker symbol defined by $\delta_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$

Proposition 5. *For prime powers $n = p^\alpha$ ($\alpha \in \mathbb{N}$) and $d = p^\beta$, $\beta \leqslant \alpha$ ($\beta \in \mathbb{N} \cup \{0\}$), we have*

$$G_{p^\beta}(p^\alpha) = \varphi(p^\alpha)\left(1 + \beta + \frac{\delta_{\alpha\beta}}{p-1}\right).$$

For prime powers $n = p^\alpha$ ($\alpha \in \mathbb{N}$) and $d = p^\beta$, $\beta > \alpha$ ($\beta \in \mathbb{N}$), we have

$$G_{p^\beta}(p^\alpha) = G_{p^\alpha}(p^\alpha) = \varphi(p^\alpha)\left(1 + \alpha + \frac{1}{p-1}\right).$$

Proof. For $0 < \beta < \alpha$, we have

$$G_{p^\beta}(p^\alpha) = \sum_{j=0}^{\beta}(p^{\alpha-j} - p^{\alpha-j-1})p^j = (p^\alpha - p^{\alpha-1})(1+\beta) = \varphi(p^\alpha)(1+\beta),$$

and, for $\beta = \alpha$,

$$\begin{aligned} G_{p^\beta}(p^\alpha) = G_{p^\alpha}(p^\alpha) = G(p^\alpha) &= (\alpha+1)p^\alpha - \alpha p^{\alpha-1} \\ &= (\alpha+1)(p^\alpha - p^{\alpha-1}) + p^{\alpha-1} = \varphi(p^\alpha)(1+\beta) + p^{\alpha-1}. \end{aligned}$$

In the case $\beta = 0$ application of Lemma 1 (ii) leads to $G_1(p^\alpha) = \varphi(p^\alpha) = \varphi(p^\alpha)(1+\beta)$. Thus, for all $0 \leqslant \beta \leqslant \alpha$, one has

$$G_{p^\beta}(p^\alpha) = \varphi(p^\alpha)(1+\beta) + p^{\alpha-1}\cdot\delta_{\alpha\beta} = \varphi(p^\alpha)\left(1 + \beta + \frac{p^{\alpha-1}\cdot\delta_{\alpha\beta}}{\varphi(p^\alpha)}\right).$$

Taking into account that $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$ one obtains the first result.
For $\beta > \alpha$, we have $\gcd(k, p^\alpha) \mid p^\beta \iff \gcd(k, p^\alpha) \mid p^\alpha$. Hence,

$$G_{p^\beta}(p^\alpha) = \sum_{\substack{k=1 \\ \gcd(k,p^\alpha)\mid p^\beta}}^{p^\alpha} \gcd(k, p^\alpha) = \sum_{\substack{k=1 \\ \gcd(k,p^\alpha)\mid p^\alpha}}^{p^\alpha} \gcd(k, p^\alpha) = G_{p^\alpha}(p^\alpha)$$

and the second result follows by application of the first formula. $\square$

Remark 6. The result of Proposition 5 can be written in one single formula: for $p \in \mathbb{P}$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N} \cup \{0\}$, we have

$$G_{p^\beta}(p^\alpha) = \varphi(p^\alpha)\left(1 + \min(\alpha,\beta) + \frac{\delta_{\alpha,\min(\alpha,\beta)}}{p-1}\right).$$

Theorem 7. *For $n \in \mathbb{N}$ with prime powers decomposition $n = p_1^{\lambda_1} \cdot \ldots \cdot p_t^{\lambda_t}$ and positive integer $d = c \cdot p_1^{\kappa_1} \cdots p_t^{\kappa_t}$ with $p_j \nmid c$ for all $j = 1, \ldots, t$, and $0 \leqslant \kappa_j$ we have the representation*[1]

$$G_d(n) = \varphi(n) \cdot \prod_{j=1}^{t} \left(1 + \min(\kappa_j, \lambda_j) + \delta_{\lambda_j, \min(\kappa_j, \lambda_j)} \frac{1}{p_j - 1}\right).$$

Proof. Because G_d is multiplicative, by Theorem 2, and applying Lemma 1 (iii), we obtain

$$\begin{aligned} G_d(n) &= G_d\left(\prod_{j=1}^{t} p_j^{\lambda_j}\right) \\ &= \prod_{j=1}^{t} G_{c \cdot p_1^{\kappa_1} \cdots p_t^{\kappa_t}}\left(p_j^{\lambda_j}\right) \\ &= \prod_{j=1}^{t} G_{p_j^{\kappa_j}}\left(p_j^{\lambda_j}\right) \\ &= \prod_{j=1}^{t} \varphi\left(p_j^{\lambda_j}\right)\left(1 + \min(\kappa_j, \lambda_j) + \frac{\delta_{\lambda_j \min(\kappa_j, \lambda_j)}}{p_j - 1}\right), \end{aligned}$$

where the last equation is a consequence of Rem. 6. □

We note that under the notation of Theorem 7 the equation

$$\gcd(d, n) = p_1^{\kappa_1} \cdots p_t^{\kappa_t}$$

defines unique numbers κ_j ($j = 1, \ldots, t$) with $0 \leqslant \kappa_j \leqslant \lambda_j$, such that the result can be written in the form

$$G_d(n) = G_{\gcd(d,n)}(n) = \varphi(n) \cdot \prod_{j=1}^{t} \left(1 + \kappa_j + \delta_{\lambda_j, \kappa_j} \frac{1}{p_j - 1}\right).$$

4 Dirichlet series, averages and asymptotic properties

Some asymptotic formulas of the Broughan's gcd-sum function were derived by Broughan [5] and Bordellès [4]. The average order of the Dirichlet series of the Broughan's gcd-sum function was studied by Broughan [6] and Bordellès [3]. In this section we give some examples of Dirichlet series of arithmetic functions connected with $G_d(n)$. We calculate the average functions and derive some asymptotic formulas for these examples.

[1] $\kappa_j = 0$ means that p_j is not present in the decomposition of d, i.e., $p_j \nmid d$.

The Dirichlet series for an arithmetic function $f(n)$ is defined (see, e.g., [1, 11.1, p. 224]) by

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

The most prominent example is the Riemann ζ function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. It is clear, that $\zeta(s)$ is the Dirichlet series associated to $f(n) = 1$, for all $n \in \mathbb{N}$.

For any prime number p, the Bell series [1, Sect. 2.15, p. 42ff] of an arithmetic function f is the formal power series

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n.$$

If f is multiplicative the corresponding Dirichlet series is given by

$$\mathcal{F}(s) = \sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p f_p(p^{-s})$$

provided that the Dirichlet series converges absolutely for $\operatorname{Re} s > a$ (see, e.g., [1, Th. 11.7, p. 231]).

The number $e \in \mathbb{N} \cup \{0\}$ is called the *m-adic order of $n \in \mathbb{N}$* ($m \in \mathbb{N}$), if $m^e \mid n$ and $m^{e+1} \nmid n$. It is denoted by $e = \nu_m(n)$.

4.1 The arithmetic function G_d

4.1.1 Dirichlet series

Since $G_d = \rho_d * \varphi$ (see (1)) and

$$\mathcal{P}_d(s) := \sum_{n=1}^{\infty} \frac{\rho_d(n)}{n^s} = \sum_{n|d} \frac{1}{n^{s-1}};$$

$$\Phi(s) := \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} = \prod_p \frac{1-p^{-s}}{1-p^{1-s}},$$

([1, Ex. 4, p. 229 and p. 231]), we have according to [1, Th. 11.5]: for $\operatorname{Re} s > 2$,

$$\mathcal{G}_d(s) := \sum_{n=1}^{\infty} \frac{G_d(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\rho_d(n) * \varphi(n)}{n^s} = \mathcal{P}_d(s)\Phi(s),$$

so

$$\mathcal{G}_d(s) = \frac{\zeta(s-1)}{\zeta(s)} \cdot \sum_{n|d} \frac{1}{n^{s-1}} = \prod_p \frac{1-p^{-s}}{1-p^{1-s}} \cdot \sum_{n|d} \frac{1}{n^{s-1}}.$$

If $d=1$ one obviously has $\mathcal{G}_1(s)=\dfrac{\zeta(s-1)}{\zeta(s)}$ (cf. [1, Ex. 3, p. 231]). For $d\in\mathbb{P}$, one has

$$\mathcal{G}_d(s)=\frac{\zeta(s-1)}{\zeta(s)}\left(1+\frac{1}{d^{s-1}}\right).$$

4.1.2 Average functions

We study the asymptotic behaviour of the average function

$$\mathcal{G}_d^{[\alpha]}(x):=\sum_{n\le x} n^{-\alpha}G_d(n)$$

as n tends to infinity. Taking advantage of the representation $G_d=\rho_d*\varphi$ we obtain

$$\mathcal{G}_d^{[\alpha]}(x)=\sum_{n\le x}\sum_{w|n}\frac{\rho_d(w)}{w^\alpha}\frac{\varphi\left(\frac{n}{w}\right)}{\left(\frac{n}{w}\right)^\alpha}.$$

By application of [1, Th. 3.10, p. 65], we conclude that

$$\mathcal{G}_d^{[\alpha]}(x)=\sum_{n\le x} n^{-\alpha}\rho_d(n)\,\Phi^{[\alpha]}\left(\frac{x}{n}\right)=\sum_{w|d} w^{-(\alpha-1)}\Phi^{[\alpha]}\left(\frac{x}{w}\right),$$

where $\Phi^{[\alpha]}$ denotes the average

$$\Phi^{[\alpha]}(x):=\sum_{n\le x} n^{-\alpha}\varphi(n)$$

of Euler's totient function φ. We distinguish 3 cases. Because, for $\alpha\le 1$,

$$\Phi^{[\alpha]}(x)\sim\frac{x^{2-\alpha}}{2-\alpha}\zeta^{-1}(2)+O\left(x^{1-\alpha}\log x\right)\qquad(x\to\infty),$$

([1, Ex. 8, p. 71]), we have

$$\mathcal{G}_d^{[\alpha]}(x)\sim\frac{x^{2-\alpha}}{2-\alpha}\zeta^{-1}(2)\sum_{w|d}\frac{1}{w}+O\left(x^{1-\alpha}\log x\right)\qquad(x\to\infty).$$

Because, for $\alpha>1,\alpha\ne 2$,

$$\Phi^{[\alpha]}(x)\sim\frac{x^{2-\alpha}}{2-\alpha}\zeta^{-1}(2)+\frac{\zeta(\alpha-1)}{\zeta(\alpha)}+O\left(x^{1-\alpha}\log x\right)\qquad(x\to\infty),$$

([1, Ex. 7, p. 71]), we have

$$\mathcal{G}_d^{[\alpha]}(x)\sim\frac{x^{2-\alpha}}{2-\alpha}\zeta^{-1}(2)\sum_{w|d}\frac{1}{w}+\frac{\zeta(\alpha-1)}{\zeta(\alpha)}\sum_{w|d}w^{-(\alpha-1)}+O\left(x^{1-\alpha}\log x\right)\qquad(x\to\infty).$$

Finally, for $\alpha = 2$, we have

$$\Phi^{[2]}(x) \sim \frac{\log x}{\zeta(2)} + \frac{\gamma}{\zeta(2)} - A + O\left(\frac{\log x}{x}\right) \qquad (x \to \infty),$$

where γ is Euler's constant and $A = \sum_{n=1}^{\infty} \mu(n) n^{-2} \log n$ ([1, Ex. 6, p. 71]), and we conclude that

$$\mathcal{G}_d^{[2]}(x) \sim \frac{1}{\zeta(2)} \sum_{w|d} \frac{\log(x/w)}{w} + \left(\frac{\gamma}{\zeta(\alpha)} - A\right) \sum_{w|d} w^{-1} + O\left(\frac{\log x}{x}\right) \qquad (x \to \infty).$$

4.2 The arithmetic function $G_{n/\gcd(r,n)}(n)$

Let $r \in \mathbb{N}$ be given. Consider the arithmetic function

$$b^{(r)}(n) := G_{n/\gcd(r,n)}(n).$$

which is easily seen to be multiplicative. Let p be a prime number and put $\beta = \nu_p(r)$. According to Prop. 5 one has

$$b^{(r)}(p^n) = G_{p^n/\gcd(r,p^n)}(p^n) = G_{p^{n-\beta}}(p^n) = \varphi(p^n)\left(1 + n - \beta + \frac{\delta_{n,n-\beta}}{p-1}\right).$$

So, if $\beta = 0$ one has $\delta_{n,n-\beta} = 1$ and

$$b^{(r)}(p^n) = \varphi(p^n)\left(1 + n + \frac{1}{p-1}\right) = (n+1)p^n - np^{n-1}.$$

Therefore, for $\beta = 0$, the Bell series is given by

$$b_p^{(r)}(x) = \sum_{n=0}^{\infty} b^{(r)}(p^n)x^n = \sum_{n=0}^{\infty} \left((n+1)p^n - np^{n-1}\right)x^n = \frac{1-x}{(1-px)^2}.$$

If $\beta > 0$ one has $\delta_{n,n-\beta} = 0$ and

$$\begin{aligned} b_p^{(r)}(x) &= \sum_{n=0}^{\infty} b^{(r)}(p^n)x^n = \sum_{n=0}^{\infty} \varphi(p^n)(1+n-\beta)x^n \\ &= \sum_{n=0}^{\infty} (p^n - p^{n-1})(1+n-\beta)x^n = \frac{(p-1)(px\beta - \beta + 1)}{p(px-1)^2}. \end{aligned}$$

Hence, the Dirichlet series is given by

$$\mathcal{B}^{(r)}(s) := \frac{\zeta^2(s-1)}{\zeta(s)} \prod_{p|r} \frac{(p-1)(1-(1-p^{1-s})\beta(p))}{p - p^{1-s}} \qquad (\operatorname{Re} s > 2),$$

where $\beta(p) = \nu_p(r)$.

4.3 The arithmetic function $G_n(\gcd(r,n)n)$

Let $r \in \mathbb{N}$ be given. Consider the arithmetic function

$$a^{(r)}(n) := G_n(\gcd(r,n)\cdot n)$$

which is easily seen to be multiplicative. Let p be a prime number and put $\beta = \nu_p(r)$. According to Remark 6 one has

$$\begin{aligned} a^{(r)}(p^n) &= G_{p^n}(\gcd(r,p^n)p^n) = G_{p^n}\left(p^{n+\min(\beta,n)}\right) \\ &= \varphi\left(p^{n+\min(\beta,n)}\right)\left(1+n+\delta_{0,\min(\beta,n)}\frac{1}{p-1}\right). \end{aligned}$$

If $\beta = 0$, one has $\delta_{0,\min(\beta,n)} = 1$ and

$$a^{(r)}(p^n) = \varphi(p^n)\left(1+n+\frac{1}{p-1}\right) = (n+1)p^n - np^{n-1}.$$

Therefore, for $\beta = 0$, the Bell series is given by

$$a_p^{(r)}(x) = \sum_{n=0}^{\infty} a^{(r)}(p^n)x^n = \sum_{n=0}^{\infty}\left((n+1)p^n - np^{n-1}\right)x^n = \frac{1-x}{(1-px)^2}.$$

If $\beta > 0$ one has $\delta_{0,\min(\beta,n)} = 0$ and

$$\begin{aligned} a_p^{(r)}(x) &= 1 + \sum_{n=1}^{\infty}(1+n)\varphi(p^{n+\min(\beta,n)})x^n \\ &= 1 + \sum_{n=1}^{\beta}(1+n)\varphi(p^{2n})x^n + \sum_{n=\beta+1}^{\infty}(1+n)\varphi(p^{n+\beta})x^n \\ &= 1 + \sum_{n=1}^{\beta}(1+n)(p^{2n}-p^{2n-1})x^n + \sum_{n=\beta+1}^{\infty}(1+n)(p^{n+\beta}-p^{n+\beta-1})x^n \\ &= 1 + \frac{p-1}{p}\sum_{n=1}^{\beta}(n+1)(p^2x)^n + p^{\beta-1}(p-1)\sum_{n=\beta+1}^{\infty}(n+1)(px)^n \\ &= 1 + \frac{(p-1)px((\beta+1)(p^2x)^{\beta+1} - (\beta+2)(p^2x)^{\beta} - p^2x + 2)}{(p^2x-1)^2} \\ &\qquad - \frac{(p-1)p^{2\beta}x^{\beta+1}((\beta+1)px - (\beta+2))}{(px-1)^2}. \end{aligned}$$

Hence, the Dirichlet series is given by

$$\mathcal{A}^{(r)}(s) := \frac{\zeta^2(s-1)}{\zeta(s)}\prod_{p|r}\left(\frac{(1-p^{1-s})^2}{1-p^{-s}}a_p^{(r)}(p^{-s})\right) \qquad (\operatorname{Re} s > 2)\,.$$

where

$$a_p^{(r)}(p^{-s}) = 1 - \frac{(p-1)p^{2\beta}p^{-s(\beta+1)}((\beta+1)p^{1-s} - (\beta+2))}{(1-p^{1-s})^2} + \frac{(p-1)p^{1-s}((\beta+1)p^{(2-s)(\beta+1)} - (\beta+2)p^{(2-s)\beta} - p^2x + 2)}{(p^2x-1)^2}$$

and $\beta = \beta(p) = \nu_p(r)$.

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