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Equivalence Classes of Permutations under Various Relations Generated by Constrained Transpositions

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Abstract

We consider a large family of equivalence relations on the symmetric group of permutations of n that generalize those discovered by Knuth in his study of the Robinson-Schensted correspondence. In our most general setting, two permutations are equivalent if one can be obtained from the other by a sequence of pattern-replacing moves of prescribed form; however, we limit our focus to patterns where two elements are transposed, subject to the constraint that a third element of a suitable type be in a suitable position. For various instances of the problem, we compute the number of equivalence classes, determine how many n-permutations are equivalent to the identity permutation, or characterize this equivalence class. Although our results feature familiar integer sequences (e.g., Catalan, Fibonacci, and Tribonacci numbers) and special classes of permutations (layered, connected, and 123-avoiding), some of the sequences that arise appear to be new.

1 Introduction and motivation

We consider a family of equivalence relations on permutations in S_n in which two permutations are considered to be equivalent if one can be converted into the other by replacing a short subsequence of (not necessarily adjacent) elements by the same elements permuted in a specific fashion, or (extending by transitivity) by a sequence of such moves. These generalize the relations discovered by Knuth in his study of the Robinson-Schensted correspondence, though the original motivations for this project were unrelated. We begin the systematic study of such equivalence relations, connecting them with integer sequences both familiar and (apparently) new.

Consider the following three examples of turning one 7-permutation into another in which selected 3-subsequences (marked in **bold**) are re-ordered:

$$12\mathbf{34}56\mathbf{7} \rightarrow 12\mathbf{74}56\mathbf{3} \tag{1}$$
$$\mathbf{127}4563 \rightarrow \mathbf{721}4563 \tag{2}$$
$$721\mathbf{456}3 \rightarrow 721\mathbf{654}3 \tag{3}$$

In each of these examples, a subsequence of *pattern* 123 (i.e., a triple of not necessarily adjacent entries whose elements are in the same relative order as 123) is replaced by the same set of elements arranged in the pattern 321. Allowing replacements of a designated kind to be performed ad libitum, in reverse as well as forward, induces an equivalence relation on the symmetric group S_7. Accordingly we can say that the permutations 1234567, 1274563, 7214563, and 7216543 are all equivalent under the replacement $123 \leftrightarrow 321$.

Interesting enumerative questions arise when the elements being replaced are allowed to be in general position (Section 2), when the replacements are constrained to involve only adjacent elements (Section 3), and when replacements are constrained to affect only subsequences of consecutive elements representing a run of consecutive values (Section 4). Each of these respective *types* of replacements is illustrated in one of the three examples above. It will be convenient to group subsequences that are allowed to replace one another into sets, e.g., describing the three permutations above as being "$\{123, 321\}$-equivalent".

We may also wish to allow more than one type of (bi-directional) replacement, such as both $123 \leftrightarrow 321$ and $123 \leftrightarrow 132$. If the intersection of these sets is nonempty, the new relation can be described simply by the union of the two sets: $\{123, 132, 321\} = \{123, 321\} \cup \{123, 132\}$. To formalize this more generally we consider collections of disjoint replacement sets that form a set partition of S_3; any two patterns within the same set may replace one another within the larger permutation to give an equivalent permutation.

Let $\pi \in S_n$, and let $P = \{B_1, B_2, \ldots, B_t\}$ be a (set) partition of S_k, where $k \leq n$. Each block B_l of P represents a list of k-length patterns which can replace one another. We call two permutations $P^{::}$-equivalent if one can be obtained from the other by a sequence of replacements, each replacing a subsequence of pattern σ_i with the same elements in the pattern σ_j, where σ_i and σ_j lie in the same block B_l of P. Let $\mathrm{Eq}^{::}(\pi, P)$ denote the set of permutations equivalent to π under $P^{::}$-equivalence; e.g., 1234567, 7214563, and $7216543 \in \mathrm{Eq}^{::}\big(1274563, \big\{\{123, 321\}\big\}\big)$. Similarly we denote by $P^{||}$ the equivalence relation, and by $\mathrm{Eq}^{||}(\pi, P)$ the equivalence class of π, under replacement within P only of adjacent elements; e.g., 7214563 and $7216543 \in \mathrm{Eq}^{||}\big(1274563, \big\{\{123, 321\}\big\}\big)$. We use $P^{\square}$ and $\mathrm{Eq}^{\square}(\pi, P)$ when both positions and values are constrained, e.g., $7214563 \in \mathrm{Eq}^{\square}\big(7216543, \big\{\{123, 321\}\big\}\big)$. To refer to such classes generically we use the notation $\mathrm{Eq}^*(\pi, P)$. The automorphism $\pi \mapsto \pi^{-1}$ replaces adjacency of positions with adjacency of values; hence, for the enumerative questions we treat, there is no need to separately consider a fourth case where we only constrain values to be adjacent. The set of distinct equivalence classes into which S_n splits under an equivalence P^* is denoted by $\mathrm{Classes}^*(n, P)$.

The present paper begins the study of these equivalence relations by considering three types of questions:

(A) Compute the number of equivalence classes, $\#\mathrm{Classes}^*(n, P)$, into which S_n is partitioned.

(B) Compute the size, $\#\mathrm{Eq}^*(\iota_n, P)$, of the equivalence class containing the identity $\iota_n = 123 \cdots n$.

(C) Characterize the set $\mathrm{Eq}^*(\iota_n, P)$ of permutations equivalent to the identity.

Although the framework above allows for much greater generality, in this paper we will mainly restrict our attention to replacements by patterns of length $k = 3$, and usually to replacement patterns built up from pairs in which one permutation is the identity, and the other is a transposition (i.e., fixes one of the elements). This is both to keep the paper within reasonable bounds and so that the equivalence class of the identity $12 \cdots n$ emerges as a large component of specific interest. Omitting some cases by symmetry, we have the following possible partitions of S_3, where (as usual) we omit singleton blocks:

$$P_1 = \big\{\{123, 132\}\big\},$$
$$P_2 = \big\{\{123, 213\}\big\},$$
$$P_4 = \big\{\{123, 321\}\big\}.$$

We will also consider applying two of these replacement operations simultaneously, and we will number the appropriate partitions as

$$P_3 = \big\{\{123, 132, 213\}\big\},$$
$$P_5 = \big\{\{123, 132, 321\}\big\},$$
$$P_6 = \big\{\{123, 213, 321\}\big\},$$

following the convention $P_{i+j} := P_i \vee P_j$, the *join* of these two partitions [18, ch.3]. Indeed we can allow all three replacements: $P_7 = \big\{\{123, 132, 213, 321\}\big\}$. (In fact, the cases P_1 and P_2 are equivalent by symmetry, as are P_5 and P_6. We list P_1 and P_2 separately only in order to consider their join.)

Our motivation for focussing attention on pairs of this form is that we can then think of an operation, not in terms of replacing one pattern by another, but simply in terms of *swapping* two elements within the pattern, with the third serving as a catalyst enabling the swap. In a followup [16] to the current paper, the authors treat the remaining (non-swapping) cases for all partitions of S_3 consisting of exactly one non-singleton block which contains the identity 123.

By far the best-known example of constrained swapping in permutations is the *Knuth Relations* [11], which allow the swap of adjacent entries provided an intermediate value lies immediately to the right or left. In the notation of this paper, they correspond to $P_K^{||} = \big\{\{213, 231\}, \{132, 312\}\big\}$. Permutations equivalent under this relation map to the same first coordinate (P-tableau) under the Robinson-Schensted correspondence.

Mark Haiman introduced the notion of *dual equivalence* of permutations: π and τ are dual equivalent if one can be obtained from the other by swaps of adjacent *values* from the above P_K, i.e., if their inverses are Knuth-equivalent, or (equivalently) if they map to the same second coordinate (Q-tableau) under the Robinson-Schensted correspondence [10]. For the enumerative problems in this paper, we get the same answers for Knuth and dual equivalence.

In her dissertation [3] Sami Assaf constructed graphs (with some extra structure) whose vertices are tableaux of a fixed shape (which may be viewed as permutations via their "reading words"), and whose edges represent (elementary) dual equivalences between vertices. For this particular relation (equivalently for the Knuth relations), she was able to characterize the local structure of these graphs, which she later used to give a combinatorial formula for the Schur expansion of LLT polynomials and Macdonald polynomials. She also used these, along with crystal graphs, to give a combinatorial realization of Schur-Weyl duality [2].

Sergey Fomin has a very clear elementary exposition of how Knuth and dual equivalence are related to the Robinson-Schensted correspondence, Schützenberger's *jeu de taquin*, and the Littlewood-Richardson rule in [19, Ch. 7, App. 1]. For the problems considered above, the answers for $P_K^{||}$ are well known to be: (A) the number of involutions in S_n; (B) 1; and (C) $\{\iota_n\}$. In fact one can compute $\#\mathrm{Eq}^{||}(\pi, P_K)$ for any permutation π by using the Frame-Robinson-Thrall hook-length formula to compute the number of standard Young tableaux of the *shape* output by the Robinson-Schensted correspondence applied to π.

Any of the relations we consider can be naturally generalized to operate on $\mathcal{W}([n])$, the set of **words** (with repeated entries allowed) on the alphabet $[n]$: for example, the relation $123 \leftrightarrow 321$ would imply also moves of the form $112 \leftrightarrow 211$ and $122 \leftrightarrow 221$, treating letters with the same label within a word as increasing from left to right. In the case of the Knuth relations, the equivalence classes are simply the *elements* of the well-known *plactic monoid* of Lascoux and Schützenberger: $\mathcal{W}([n])/P_K$ [13, 12]. In [6] the authors study the analogous *Chinese monoid*, which is $\mathcal{W}([n])/P_3$ (up to the involution that reverses all words), for which they develop an analogue of the Robinson-Schensted correspondence and count some of the equivalence classes. It would be interesting to find variants of RSK connected with the other moves we study, but we have not managed to do this.

In recent work that was independently motivated [20], Richard Stanley studies the equivalence given by allowing the transposition of *any* adjacent pair of elements whose values are also adjacent. The corresponding partition of S_n is a refinement of all the classes we consider in Section 4. He counts the number of equivalence classes and all their sizes, and also connects $\mathrm{Eq}(\iota)$ to the set of linear extensions of a certain poset, whose corresponding distributive lattice has a multiplicity-free flag h-vector.

Given that the Knuth relations act on adjacent elements, and lead to some deep combinatorial results, it is perhaps not surprising that the most interesting problems and proofs in this paper are to be found in Section 3. A summary of our numbers and results is given in Figure 1.

An extended abstract of this paper appeared in the proceedings of FPSAC10 [14].

Figure 1: Summary of Results

These tables give numerical values and names (when available) of the sequences associated with enumerative questions (A) and (B). All sequences begin with the value for $n = 3$. Results proven in this paper have a gray background; for other cases we lack even conjectural formulae. Six-digit codes preceded by "A" cite specific sequences in Sloane [15]; those set in italic type were added to OEIS in conjunction with this paper.

Number of classes

Transpositions		§ 2 general	§ 3 only indices adjacent	§ 4 indices and values adjacent
(1)	$123 \leftrightarrow 132$	[5, 14, 42, 132, 429] Catalan A000108	[5, 16, 62, 284, 1507, 9104] *A210667*	[5, 20, 102, 626, 4458, 36144] *A212580*
(2)	$123 \leftrightarrow 213$			
(4)	$123 \leftrightarrow 321$	[5, 10, 3, 1, 1, 1] trivial	[5, 16, 60, 260, 1260, 6744] *A210668*	[5, 20, 102, 626, 4458, 36144] *A212580*
(3)	$123 \leftrightarrow 132 \leftrightarrow 213$	[4, 8, 16, 32, 64, 128] powers of 2 A000079	[4, 10, 26, 76, 232, 764] involutions A000085	[4, 17, 89, 556, 4011, 32843] *A212581*
(5)	$123 \leftrightarrow 132 \leftrightarrow 321$	[4, 2, 1, 1, 1, 1] trivial	[4, 8, 14, 27, 68, 159, 496] *A210669*	[4, 16, 84, 536, 3912, 32256] *A212432*
(6)	$123 \leftrightarrow 213 \leftrightarrow 321$			
(7)	$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$	[3, 2, 1, 1, 1, 1] trivial	[3, 4, 5, 8, 11, 20, 29, 57] *A210671*	[3, 13, 71, 470, 3497] *A212433*

Size of class containing identity

Transpositions		§ 2 general	§ 3 only indices adjacent	§ 4 indices and values adjacent
(1)	$123 \leftrightarrow 132$	[2, 6, 24, 120, 720] $(n-1)!$ A000142	[2, 4, 12, 36, 144, 576, 2880] prod. of adj. fact. A010551	[2, 3, 5, 8, 13, 21, 34, 55] Fibonacci A000045
(2)	$123 \leftrightarrow 213$			
(4)	$123 \leftrightarrow 321$	[2, 4, 24, 720] trivial	[2, 3, 6, 10, 20, 35, 70, 126] central bin. coeff. A001405	[2, 3, 4, 6, 9, 13, 19, 28] Comp. into {1,3} A000930
(3)	$123 \leftrightarrow 132 \leftrightarrow 213$	[3, 13, 71, 461] connected A003319	[3, 7, 35, 135, 945, 5193] Chinese Monoid *A212417*	[3, 4, 8, 12, 21, 33, 55, 88] Fib. or Fib.−1 A052952
(5)	$123 \leftrightarrow 132 \leftrightarrow 321$	[3, 23, 120, 720] trivial	[3, 9, 54, 285, 2160, 15825] *A212419*	[3, 5, 9, 17, 31, 57, 105, 193] tribonacci numbers A000213
(6)	$123 \leftrightarrow 213 \leftrightarrow 321$			
(7)	$123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$	[3, 23, 120, 720] trivial	[4, 21, 116, 713, 5030] *A212419*	[4, 6, 13, 23, 44, 80, 149, 273] tribonacci A000073 $-[n \text{ even}]$

If $\tau \in \mathrm{Eq}^*(\pi, P)$ we will say that τ is *reachable* from π (under P). If $\mathrm{Eq}^*(\iota_n, P) = S_n$, then every permutation in S_n is reachable from every other, and we will say that S_n is *connected* by P. If $\mathrm{Eq}^*(\pi, P) = \{\pi\}$ we will say that π is *isolated* (under P).

It is obvious that if P_i refines P_j as partitions of S_k (i.e., $P_i \leq P_j$ in the lattice of partitions of S_k), then the partition of S_n induced by P_i refines the one induced by P_j, because a permutation reachable from π under P_i is also reachable under P_j. This enables the following simple observations:

Proposition 1. *If P_i refines P_j (as partitions of S_k), then for all $\pi \in S_n$ with $n \geq k$*

$$\begin{aligned} \mathrm{Eq}^*(\pi, P_i) &\subseteq \mathrm{Eq}^*(\pi, P_j) \\ \#\mathrm{Eq}^*(\pi, P_i) &\leq \#\mathrm{Eq}^*(\pi, P_j) \\ \#\mathrm{Classes}^*(n, P_i) &\geq \#\mathrm{Classes}(n, P_j) \end{aligned}$$

2 General pattern equivalence

In this section, we allow moves within an equivalence relation with no adjacency restrictions. This case is closely related to the theory of *pattern avoidance* in permutations: replacing one pattern with another repeatedly leads eventually to a permutation which avoids the first pattern.

Some of the equivalence relations in this section are trivial, following immediately from the following observation. The others lead to familiar combinatorial numbers and objects.

Proposition 2. *Fix k with $2 \leq k \leq n-1$, and let P be any partition of S_k. If $\#\mathrm{Classes}^{::}(n-1, P) = 1$, then $\#\mathrm{Classes}^{::}(n, P) = 1$.*

Proof. We will show that any $\pi \in S_n$ can be reached from the identity, ι_n, under the supposition that any two permutations in S_{n-1} are equivalent, in two stages. If $\pi(1) \neq n$, simply apply the supposition to the elements/positions $1 \ldots n-1$ in ι_n to obtain any permutation beginning with $\pi(1)$; if $\pi(1) = n$, we use instead the elements/positions $1, 3, 4, 5, \ldots n$ (omitting 2, which is $\leq n-1$ by hypothesis) to move $\pi(1) = n$ to the front of a permutation equivalent to ι_n. Then in stage 2 we apply the supposition to the elements now occupying *positions* $2, \ldots, n$ to complete the construction of π. $\square$

The following results follow.

Proposition 3. $\#\mathrm{Classes}^{::}\big(n, \big\{\{123, 321\}\big\}\big) = 1$ *for $n \geq 6$. While for $n \geq 5$, we have* $\#\mathrm{Classes}^{::}\big(n, \big\{\{123, 132, 321\}\big\}\big) = 1$ *and* $\#\mathrm{Classes}^{::}\big(n, \big\{\{123, 132, 213, 321\}\big\}\big) = 1$

Proof. It is easy to verify by hand, or by computer, that all permutations in S_5 are reachable from 12345 by moves in $P_5 = \big\{\{123, 132, 321\}\big\}$. (Indeed, all permutations in S_4 are reachable from 1234 except for 3412, which is isolated.) As S_5 is connected, it follows (by induction) from the preceding proposition that S_n is connected for all $n \geq 5$.

Proposition 1 tells us that S_n is connected under $P_7 = \big\{\{123, 132, 213, 321\}\big\}$ whenever it is connected under P_5 since $P_7 \geq P_5$. (In S_4, the permutation 3412 remains isolated.) Finally, we can check by computer that under $P_4 = \big\{\{123, 321\}\big\}$ S_6 is connected; whence, S_n is connected for $n \geq 6$. $\square$

We remark that under P_4, S_4 splits into 10 equivalence classes, and S_5 into three classes. The class containing 12345 contains 24 elements. This suggests a possible bar bet. Hand your mark six cards numbered 1 through 6 and invite him or her to lay them out in any sequence. By applying moves of the form $123 \leftrightarrow 321$ ("Interchange two cards if and only if an intermediate (value) card lies (in any position) between them.") you will always be able to put the cards in order. (It may take some practice, however, to become proficient at doing this quickly.) Now "go easy" on your mark by reducing the number of cards to 5. Even from a random sequence, the mark has only one chance in five of being able to reach the identity.

Of course from Proposition 3 it immediately follows that:

Corollary 4. *$\#\mathrm{Eq}^{::}\left(\iota_n, \left\{\{123, 132, 321\}\right\}\right) = n!$, $\#\mathrm{Eq}^{::}\left(\iota_n, \left\{\{123, 132, 213, 321\}\right\}\right) = n!$ for $n \geq 5$; and $\#\mathrm{Eq}^{::}\left(\iota_n, \left\{\{123, 321\}\right\}\right) = n!$ for $n \geq 6$.*

Proposition 5. *$\#\mathrm{Eq}^{::}\left(\iota_n, \left\{\{123, 213\}\right\}\right) = (n-1)!$ for $n \geq 2$.*

Proof. Obviously the largest element n cannot be moved away from the end of the permutation. Equally obviously the n, remaining at the far right, enables the other elements to be freely pairwise transposed, thereby generating any permutation in S_{n-1}.

□

Proposition 6. *For $n \geq 1$, $\#\mathrm{Classes}^{::}\left(n, \left\{\{123, 213\}\right\}\right) = c_n = \frac{2n!}{n!(n+1)!}$, the nth Catalan number.*

Proof. Let $\pi \in S_n$. If $i < j < k$, and $\pi(i) < \pi(j) < \pi(k)$, then $\pi(k)$ enables the swapping of $\pi(i)$ and $\pi(j)$ to arrive at a permutation $\pi^{(1)}$ with a strictly larger number of inversions. We can continue in this way to obtain a sequence $\pi = \pi^{(0)} \to \pi^{(1)} \to \cdots \to \pi^{(n)}$, where $\pi^{(n)}$ has no such triples, i.e., $\pi^{(n)}$ is 123-avoiding. It remains to show that no matter which sequence of moves we make, the final permutation $\pi^{(n)}$ is unique.

Call an element in a permutation $\sigma \in S_n$, a *right-to-left maximum* if it is greater than every element that occurs to its right. (More formerly, σ_k is a right-to-left maximum if $\sigma_k > \sigma_l$ for all $k < l \leq n$.) Denote the set of right-left maxima by $\mathcal{M}(\pi)$. Each such element remains positionally fixed by the relation because there is no element to the right to play the role of "3", so the elements in $\mathcal{M}(\pi)$ form a decreasing subsequence in the same positions in each permutation $\pi^{(i)}$. Now a permutation is 123-avoiding if and only if it is a union of at most two decreasing subsequences. So $\pi^{(n)}$ must be the unique permutation formed by rearranging the elements of π not in $\mathcal{M}(\pi)$ in decreasing order. It is clear that the elements of $\mathcal{M}(\pi)$ are positioned so as to enable all the necessary transpositions.

Thus the "largest" (by number of inversions) elements in each equivalence class are exactly the 123-avoiding permutations, of which there are c_n [5, ch. 14] or [4, Sec. 4.2]. (Similarly one can show that the "smallest" elements are the 213-avoiding permutations.) □

Example 7. Working within S_9, we have the following sequence of equivalences, where elements in the pattern about to be transformed are indicated in **bold**. The subsequence of right-to-left maxima is 976 (always in positions 4, 8, and 9, respectively).

382941**57**6 → 58**2**9**4**13**7**6 → 5849**2**1**37**6 → 58493**127**6 → **58**4**9**32176 → 854932176

The reader is encouraged to draw corresponding permutation matrices or diagrams, which clarify visually how the right-to-left maxima facilitate the transformation of the other elements into a decreasing subsequence.

The next two propositions study an equivalence relation and class whose enumeration is equivalent under symmetry (reversal or complementation) to $\mathrm{Eq}^{::}(\iota_n, P_3)$. The first leads to *connected* or *indecomposable permutations* [8, 9, 15, A003319], namely those not fixing $\{1, 2, \ldots j\}$ for any $1 \leq j < n$. If we define the *direct sum* of two permutations so that it corresponds to the direct sum of the corresponding permutation matrices, then these are simply the permutations which are indecomposable as direct sums in the usual matrix sense. Some authors use the term *plus-indecomposable* [1] to describe this class. The second leads to the *layered permutations*, namely those which are a direct sum of decreasing permutations, introduced by W. Stromquist [22], and studied carefully by A. Price in his thesis [17].

Proposition 8. *Let ρ_n denote the "reverse word" permutation $n, n-1, \ldots 1$. Then*

$$\mathrm{Eq}^{::}(\rho_n, \{\{321, 312, 231\}\})$$

is the set of indecomposable permutations.

Proof. When viewed as a $(0, 1)$-matrix, any permutation decomposes as a direct sum of irreducible blocks along the main diagonal; in particular, the identity ι_n decomposes into n singleton blocks, while ρ_n is indecomposable and is one large block. A permutation is connected if and only if it decomposes as a single block.

First note that if any transformation of entries $(a_1, a_2, a_3) \to (b_1, b_2, b_3)$ applied within a block causes it to split into more than one block, then b_1 must be in the leftmost/lowest of the new blocks, and b_3 in the rightmost/highest. Therefore b_1 must be less than b_3, which is exactly what does not happen with any of our possible transformations, because the first element is larger than the third in each of 321, 312 and 231. Since all of our transformations are reversible, this shows also that we cannot combine blocks. Thus, *the irreducible block structure of a permutation does not change under these transformations.* In particular, if we start with an indecomposable permutation such as ρ_n, successive applications of the permitted operations will always produce indecomposable permutations.

Next we have to show that all indecomposable permutations are in fact reachable from ρ_n. Remembering that our replacement operations are all reversible, we will instead show that we can always return to ρ_n from an arbitrary indecomposable permutation. Take $n \geq 3$, and let $\tau = \tau_1 \tau_2 \cdots \tau_n$ be an arbitrary indecomposable permutation other than ρ_n. We will show that τ always contains at least one of 312 or 231. It's easy to see that τ must have an *ascent*, i.e., there exists k such that $\tau_k < \tau_{k+1}$. Now if any element to the right of τ_{k+1} is less than τ_k we have a 231, so assume there are none such. Similarly, assume there is no element to the left of τ_k and greater than τ_{k+1} (avoiding 312). But there must be some y to the left of τ_k which is greater than some x to the right of τ_{k+1}, or otherwise the permutation decomposes between τ_k and τ_{k+1}. These four elements y, τ_k, τ_{k+1}, x form a 3142, which contains both a 312 (y, τ_k, x) and a 231 (y, τ_{k+1}, x).

Having now located a 312 or 231, we can then apply either $312 \to 321$ or $231 \to 321$, as appropriate. Each of these operations simply switches a pair of elements, and (as we have seen in the proof of Proposition 6) strictly increases the number of inversions, progressing us toward ρ_n. This completes the proof that all indecomposable permutations are reachable, and therefore the proof that the reachable permutations are exactly the indecomposable permutations. $\square$

Proposition 9. $\#\text{Classes}^{\because}\left(n, \left\{\{321, 312, 231\}\right\}\right) = 2^{n-1}$ *for* $n \geq 1$.

Proof. As we saw in the proof of the previous proposition, the irreducible block structure of a permutation does not change under the transformations we are considering here. By the arguments already given, we can work within any indecomposable block to restore it to an anti-identity. Therefore each equivalence class consists of all the permutations with a given block structure, and contains exactly one permutation which is a direct sum of anti-identities.

These are exactly the layered permutations, and there are clearly 2^{n-1} of them, with a factor of 2 according to whether each consecutive pair of elements is or is not in the same layer. (Equivalently, any such permutation is determined by the composition of n representing its block sizes, of which there are 2^{n-1}.) $\square$

Finally we apply the reversal (or complementation) involution on S_n to the above result to get our result for the partition P_3.

Theorem 10. $\#\text{Classes}^{\because}\left(n, \left\{\{123, 132, 213\}\right\}\right) = 2^{n-1}$ *for* $n \geq 1$.

3 Adjacent transformations

As mentioned in the introduction, this section contains our most interesting results and proofs. The first rediscovers sequence A010551 from Sloane [15].

Theorem 11. $\#\text{Eq}^{||}\left(\iota_n, \left\{\{123, 213\}\right\}\right) = \lfloor n/2 \rfloor ! \lceil n/2 \rceil !$ *for* $n \geq 1$.

Proof. Generically stated, our rules in this case allow the transposition of any two adjacent elements if the element immediately to their right is bigger than both of them. Applying these successively to ι_n, we note first that the largest element, n, never comes unglued from the right end, because there is nothing to enable it; therefore, $n-1$ must stay somewhere in the last three positions (as only n can enable its movement). Similarly, $n-2$ remains somewhere in the last five, $n-3$ within the last seven and so on; such restrictions apply to the largest $\lfloor n/2 \rfloor$ of the elements. This limits the number of permutations potentially reachable to $\lfloor n/2 \rfloor ! \lceil n/2 \rceil !$: placing the elements from largest to smallest, one has a choice of $1, 2, 3, \ldots, \lfloor n/2 \rfloor, \lceil n/2 \rceil, \ldots, 3, 2, 1$ positions to put each element.

Next we will show that all permutations conforming to these restrictions are indeed reachable from ι_n. We will do this in two stages. In Stage 1 we move each of the large, constrained elements as far left as it can go. (In the most natural way to achieve this, the smaller, unconstrained elements remain in their natural increasing order, although we shall see that this does not matter as they can then be permuted freely.) In Stage 2 we construct the target permutation two elements at a time, working from left to right.

Stage 1: Maximally spread out the $\lfloor n/2 \rfloor$ *largest elements.* First move $\lfloor n/2 \rfloor + 1$ one step left, using a move of type $123 \to 213$, in which $\lfloor n/2 \rfloor + 2$ plays the role of the facilitating

“3”. In the same way, move the element $\lfloor n/2 \rfloor + 2$ to the left, continuing until the entire block $\lfloor n/2 \rfloor, \ldots, n-1$ has been shifted one to the left. The element $n-1$ has now reached its leftmost permitted position, and will remain in place as we now similarly transform the block $\lfloor n/2 \rfloor, \ldots, n-2$. This moves $n-2$ as far left as it will go, and we now transform the next smaller block, etc. Continue until reaching a permutation which alternates the subsequences $1, 2, \ldots \lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor + 1, \ldots, n$ (e.g., $15263748 \in S_8$ or $516273849 \in S_9$). This places each constrained element (in the latter subsequence) as far left as possible. These elements will now serve as a “skeleton” enabling the construction of the target permutation.

Stage 2: Construct the target permutation. The key observation making this stage possible is that the small, unconstrained elements can be freely moved about while leaving the large elements in the skeleton fixed. For if $\{a, b\} < X < Y$, we can always execute the following sequence of moves: $a\mathbf{XbY} \to \mathbf{abX}Y \to b\mathbf{aXY} \to bXaY$. In the case where n is odd, we may consider the leftmost element in the skeleton to be in position 3, and the two small elements in positions 1 and 2 can be interchanged if desired.

Now we examine the target permutation and move the required element(s) into the first position (if n is even), or the first two positions (if n is odd). At this point, the elements occupying the next two positions are reclassified as small, so that the skeleton terminates two positions further to the right, and we continue by placing and ordering the next pair of elements. By continuing two elements at a time, we can build the entire target permutation. $\square$

Example 12. To reach the target permutation 452637819 according to the above scheme we would apply the following moves. The numbers indicated in **bold** are about to be transposed, either by a standard **12**3 $\leftrightarrow$ **21**3 move, or by the move (suppressing intermediate steps) $\mathbf{a}X\mathbf{b}Y \leftrightarrow \mathbf{b}X\mathbf{a}Y$ described above.

$$\begin{array}{rlllll}
123\mathbf{45}6789 & \to 1235\mathbf{46}789 & \to 12356\mathbf{47}89 \to 123567\mathbf{48}9 & \to 12\mathbf{35}67849 \to 125\mathbf{36}7849 \\
& \to 1256\mathbf{37}849 & \to 1\mathbf{25}673849 \to 15\mathbf{26}73849 & \to \mathbf{15}6273849 \to 516273849 \\
51627\mathbf{3}8\mathbf{4}9 & \to 516\mathbf{2}7\mathbf{4}839 & \to 5\mathbf{1}6\mathbf{4}72839 \to \mathbf{54}6172839 & \to 456\mathbf{1}7\mathbf{2}839 \\
& \to 45\mathbf{62}71839 & \to 45267\mathbf{1}8\mathbf{3}9 \to 4526\mathbf{73}819 & \to 452637819
\end{array}$$

Theorem 13. *Let n be an integer ≥ 3, and for any odd positive integer m set $m!! = 1 \cdot 3 \cdot \cdots \cdot m$, the product of odd natural numbers less than or equal to m. Then*

$$\#\mathrm{Eq}^{\mid\mid}\big(\iota_n, \big\{\{123, 132, 321\}\big\}\big) = \begin{cases} \frac{3}{2}(k)(k+1)(2k-1)!\,, & \textit{for } n = 2k+1 \textit{ odd;} \\ \frac{3}{2}(k)(k-\frac{1}{3})(2k-2)! - (2k-3)!!\,, & \textit{for } n = 2k \textit{ even.} \end{cases}$$

Proof. As in the previous proof, we begin by giving a set of necessary conditions for the a permutation to be reachable from ι_n, then show how to reach each such permutation, thereby proving that our conditions in fact characterize $\mathrm{Eq}^{\mid\mid}(\iota_n)$.

The first restriction is that the element 1 must occupy a position of odd index, because it can only participate in a move as a “1”, and every move either leaves it fixed or moves it by two positions. The second restriction is that the element 2 cannot occupy a position of odd index to the left of 1, because if it winds up to the left of 1, its last move there was $132 \to 321$, and since 1 is always in a position of odd index, this places 2 into a position of even index. Then it stays on the left of the 1, so it must play the role of "1" in any future swaps, again preserving the parity of its position. Let us call the class of permutations thus described $\mathcal{A}_n$, the class of *admissible* permutations.

Now in the case where $n = 2k + 1$ is odd, this characterization is exact, so we will first complete the proof for odd n. In the case where $n = 2k$ is even, there are a small number of exceptional permutations which must be excluded; we will turn to these at the end of the proof.

Case 1: n is odd. First we count the number of admissible permutations: If the 1 is in position 1, then the 2 can be in any of $n-1$ positions, and the remaining $n-2$ elements can be arranged in $(n-2)!$ ways. If the 1 is in position 3, then the 2 can be in any of $n-2$ positions; if the 1 is in position 5, then in any of $n-3$ positions, and so forth, while in each case, the remaining $n-2$ elements can be placed freely. Summing over the possible locations for the element 1, we arrive at the given formula for odd n, and also at the formula for even n upon suppression of the double-factorial correction term. For example $\mathcal{A}_5$ consists of 54 permutations: all 24 of the form $1****$, 18 of the form $**1**$ (all but the six of the form $2*1**$), and 12 of the form $****1$ (all but those of the form $2***1$ or $**2*1$).

It remains to show that all admissible permutations are in fact reachable. We do this in two stages.

Stage 1: First we will show that all permutations beginning with a 1 are reachable from the identity. We proceed in steps; after each, we will have a monotonically increasing initial segment, followed by a segment that matches the target permutation. This segment gets created from right to left, each step increasing the length of the completed segment by 1 by selecting and moving one element from the increasing segment to the left end of the completed segment.

Note that within an increasing segment, the concatenation of moves $a\mathbf{bxy} \to \mathbf{ayx}b \to axyb$ allows a selected element $b > 1$ to move two positions rightward while maintaining that the segment to its left is increasing. So if the target position for b is an even number of positions away, an appropriate number of such moves will suffice. If b is an odd number of positions away, first apply the move $\mathbf{abx}y \to axby$, then proceed as before. This shows that we can reach any permutation that starts with a 1 from ι_n.

Stage 2: To show that we can get to the identity from an arbitrary admissible permutation, it remains to show that the element 1 can always be moved to the front of such a permutation. In fact we only need show that the element 1 can always be moved *toward* the front (necessarily by two positions), and then we can just move it repeatedly until it is *at* the front.

If the 1 is at the very end of the permutation, the 2 must be to its left and in a position of opposite parity. Move the 2 rightward using moves $123 \to 321$ or $132 \to 321$ (the 2 functioning as a "1") until it is adjacent to the 1; the 1 can then be moved leftward.

We use the term *k-factor* here and in later proofs as shorthand for "length k subsequence of adjacent elements" of a permutation. If the 1 is not at the very end of the permutation, then consider the 5-factor centred on the 1. (At this point, we are relying on the assumption that n is odd, because the largest element occupies a position of odd index, and therefore if it is not at the end of an odd permutation, it must be at least two positions away from the end, guaranteeing the 5-factor which we need. We will return to this point when we consider the even case below.) There are 24 cases. For 18 of these cases, we know that we can convert this segment to an increasing one (or to any other permutation beginning with a 1) using the analysis for $n = 5$, which is easy to check by hand. The cases which cannot be handled are those of the form 2*1**. We will add a preprocessing step to make sure that we are not in such a case. Namely, we will locate the element 2 (the actual 2) and move into one of the spaces indicated by a $*$.

If the 2 is somewhere to the left of the 1 then the same argument used above in the case of permutations ending in 1 again shows that the 1 can be moved leftward.

If the 2 is somewhere to the right of the 1, we will go and fetch it as follows. Move the 1 to the *right* until it is either one or two positions left of the 2. We do this by moving it two positions at a time, using either $123 \to 321$ or $132 \to 321$, as required. This leaves behind a consecutive trail of elements in which each odd-position element is larger than the even-position element which follows it. We will call these "odd/even descents".

If the 2 was in an odd position, we will arrive at $1x2$, which we correct to $12x$. If the 2 was in an even position, we will arrive at $12x$ directly.

Now we pull both the 1 and the 2 back through the odd/even descents by a sequence of consecutive moves of the form $\mathbf{yx1}2 \to \mathbf{1xy}2 \to 1\mathbf{yx2} \to 12yx$, where $y > x > 2$. (We may also apply $12\mathbf{yx} \to 12xy$ if we wish, but this isn't necessary.) This brings us to a permutation in the same equivalence class, where the 5-factor has been modified to **12*. But we know that just working within this 5-factor, we can use the $n = 5$ case to modify it to the form 12345 (where the 1 and 2 are actual values, the others relative values). In particular, we have moved the actual 1 two spaces to the left. Doing this repeatedly gets us to a permutation beginning with 1, which we have seen in Stage 1 is equivalent to ι_n.

Case 2: $n = 2k$ is even. In the even case, we need to describe an additional class of permutations that not reachable from ι_n. Let $\mathcal{X}_n$ consist of all permutations obtainable as follows: Fill the positions in order $n-1, n, n-3, n-2, n-5, n-4 \ldots 3, 4, 1, 2$, according to the following rule. When filling positions of odd index, the smallest available element must be chosen; the subsequent selection of an element to place to its right is then unconstrained. Thus 1 must be placed in position $n-1$, and the element placed in position n could be any other number; however, if it is not 2, then the 2 is immediately placed in position $n-3$; otherwise 3 is placed in this position. For example, $\mathcal{X}_4 = \{3412, 2413, 2314\}$ and

$$\mathcal{X}_6 = \begin{Bmatrix} 563412 & 562413 & 562314 & 462315 & 452316 \\ 463512 & 462513 & 362514 & 362415 & 352416 \\ 453612 & 452613 & 352614 & 342615 & 432516 \end{Bmatrix}.$$

The number of permutations in the class $\mathcal{X}_n$ just described is $(n-1)!!$. As we will see next, none of them is reachable. However, it is also true that most of them are not in $\mathcal{A}_n$, and therefore have not been included in the enumeration; this is because most of the permutations in $\mathcal{X}_n$ have the 2 in position $n-3$, which is a position of odd index to the left of the 1. The only permutations in $\mathcal{X}_n$ which we have counted, and which therefore must be subtracted off, are the ones where the 2 is in position n, of which there are $(n-3)!!$.

To see that none of the permutations in $\mathcal{X}_n$ is reachable, consider their 3-factors. Every 3-factor centred on a position of odd index is either a 213 or a 312, because the middle element was placed before either of its neighbors, and was the minimal available element at the time it was placed. And every 3-factor centred on a position of even index is a 231, because the elements in positions of odd index, which are the minimal elements, descend from left to right. Therefore permutations belonging to $\mathcal{X}_n$ contain *no* factors of form 123, 132, or 321, and are therefore isolated by the relation, each one being a singleton equivalence class. In particular they are not in the equivalence class of the identity.

Now we have to consider which permutations in $\mathcal{A}_n$ are not in fact reachable. The proof for odd n almost carries through completely; indeed, as remarked, it only fails when the element 1 lies in the penultimate position $n-1$. We have already seen that the permutations belonging to $\mathcal{X}_n \cap \mathcal{A}_n$ are not reachable; we will show that all others are. Take any permutation $\pi \notin \mathcal{X}_n$, but with the minimal element 1 placed in position $n-1$. Checking the conditions from right to left, suppose the element $\pi_j = y$ represents the last time that we were in compliance with the conditions, and suppose $\pi_i = x$ is the first minimal element which has not gone where it should go. That is, all odd positions from j to $n-1$ are occupied by elements which are left-to-right minima, but the smallest element situated in positions 1 through $j-1$ is not in position $j-2$, as expected, but in position i with value x.

As before, all we need to do is show that we can move the element 1 to the left. This exploits two facts: that x is the minimal element in a lefthand region, and the righthand region is alternating.

Because the righthand portion of π, from position j onward, is alternating, with every step from an odd to an even position being an ascent, and every step from even to odd being a descent, we will have a particular interest in a certain type of 3-factor beginning in a position of odd index. Namely, we will refer to a 3-factor $\pi_h, \pi_{h+1}, \pi_{h+2}$ as an *odd 321* if $\pi_h > \pi_{h+1} > \pi_{h+2}$ and if h is odd. Note that an odd 321 beginning in position $n-3$ is exactly what we need, because either option for replacing it shifts the element 1 from position $n-1$.

First, take the element x and use moves $\to 321$ to shift it rightward, two positions at a time, until it arrives in position $j-2$ or $j-1$. This is possible because x is moving through a region in which it is itself the minimal element.

Now $j-2$ is an odd position, so if x has reached position $j-2$ then positions $j-4, j-3, j-2$ now form an odd 321. Alternatively, if x has reached position $j-1$ then positions $j-2, j-1, j$ now form an odd 321, because the second and third of these positions are occupied by x and y and $y < x$. We will show that we can propagate either of these odd 321s rightward until they capture the smallest element, which can then be moved.

In either case, we have an odd 321, followed, two positions later, by an element which is smaller than everything to its left. This gives us, in other words, a configuration $432-1$, which, filling in the blank, might actually be (a) a 54231, (b) a 53241, or (c) a 43251. Check that the following moves are available in each case: (a) $54231 \to 24531$; (b) $53241 \to 23541$; (c) $43251 \to 23451 \to 25431$.

Note that these moves each replace a configuration which begins with an odd 321 by one which *ends* with an odd 321. And, because of the placement of the left-to-right minima, this new odd 321 either terminates with the smallest element 1, or again has another left-to-right minimum two positions to its right.

Therefore we can propagate the 321 rightward until it reaches the smallest element; therefore we can move the smallest element; therefore the permutations not belonging to $\mathcal{X}_n$ are in fact reachable.

This completes the missing step in the proof for even n. $\square$

Theorem 14. $\#\mathrm{Eq}^{||}\big(\iota_n, \{\{123, 321\}\}\big) = \dbinom{n-1}{\lfloor (n-1)/2 \rfloor}$.

Proof. We claim that the permutations in this class are direct sums of singletons and of blocks of odd size greater than one, where within each block the even elements (with respect to the block) are on the diagonal, and the odd elements form an *indecomposable* 321-avoiding permutation.

Let us call the set that we have just described $\mathcal{A}_n$. Because all the even elements within a block are fixed points of the permutation, the indecomposability of the odd elements is equivalent to the indecomposability of the entire block.

First we will show that $\mathcal{A}_n$ is closed under $123 \leftrightarrow 321$; since the identity is in $\mathcal{A}_n$ this will establish that the equivalence class of the identity is a subset of $\mathcal{A}_n$. Then we will show that we can return to the identity from any permutation in $\mathcal{A}_n$, which will establish that the two sets are identical. Finally we will use generating functions to enumerate $\mathcal{A}_n$.

Let π be an arbitrary permutation belonging to $\mathcal{A}_n$. By definition, π is a direct sum of singleton blocks and of larger blocks having a specific form. We will call any non-singleton block of π *large*. Unless π is the identity, it contains at least one large block. Note that large blocks always begin with descents: for if the first element of the block were on the diagonal, we could split the block immediately after it to obtain a direct sum decomposition; therefore, the first element is below the diagonal (i.e., is an excedance) but the second element is on the diagonal. For symmetric reasons, large blocks end with descents as well.

First we show that any application of $123 \to 321$ to π produces an element π' in $\mathcal{A}_n$. Consider the different ways that a 3-factor $\pi_i, \pi_{i+1}, \pi_{i+2}$ of form 123 might occur within π.

Case (a) All three elements are in singleton blocks; then the result is the unique large block of size 3 permissible within elements of $\mathcal{A}_n$, namely $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Case (b) Exactly two of the elements (necessarily the first two or the last two) are in singleton blocks. Assume without loss of generality that it is the last two, i.e., $\pi_{i+1} = i+1$ and $\pi_{i+2} = i+2$, and that $\pi_i < i$ belongs to a large block B of size $2k+1$. Since π_i is the last element of B, it must be an odd element within the block. The replacement produces a larger block B' of size $2k+3$. The k even elements of B, along with π_{i+1} are diagonal elements that remain unchanged by the transformation, so all the even elements of B' lie on the diagonal. The block B' must be indecomposable, because any breakpoint before π'_i would already have been a breakpoint for B itself, and no breakpoint can occur thereafter since $\pi'_i > \pi'_{i+1} > \pi'_{i+2}$.

Finally, π'_i is the largest element of the block, so could only play the role of "3" in a 321 pattern, but there is only one odd element to its right in the block. So any 321 pattern of odd elements in B' that did not already exist in B must use π'_{i+2} as "1" and odd elements to the left of π_i' for "3" and "2". But then these elements (which haven't moved) together with π_i would have formed a 321 in B, which wasn't allowed.

Case (c) Just one element is in a singleton block. This can't be the third (or, symmetrically, the first) element, because if π_i and π_{i+1} are the final two elements of a large block, then $\pi_i > \pi_{i+1}$, so our 3-factor is not a 123. So it must be π_{i+1} which is the singleton, while the other two elements belong to two large blocks. The replacement merges these three blocks into one; the even elements, including π_{i+1}, remain on the diagonal, and as in the previous case any point at which the new block split would also imply a decomposition of one of the old blocks at the same position. The odd elements of π' are 321-avoiding because if a 321 contained just one of π_i' or π_{i+2}' then it would be pre-existing (with π_{i+2} or π_i respectively). If it contained both, then the third element in the pattern would be either on the left and too large for the old lefthand block, or on the right and too small for the old righthand block.

Case (d) The three elements are all within a single large block $\mathcal{B}$.

First we claim that the middle element, π_{i+1} must be in an even position (within $\mathcal{B}$). Otherwise, π_i and π_{i+2} would be in even positions, hence on the diagonal, and the 123 form of the 3-factor would mean π_{i+1} was also on the diagonal; thus, $\mathcal{B}$ would have to be of size at least 5. Now if all elements to the left of π_i were smaller than it, $\mathcal{B}$ would split into summands before position i. But if some π_j is greater than π_i (for some odd $j < i$) it must be greater than π_{i+2}, forcing a compensatory $\pi_k < \pi_i$ for some odd $k > i+2$. But then π_j, π_{i+1}, π_k formed a 321-pattern of odd positions within the block, contrary to hypothesis. The claim follows.

Now the replacement $\pi_i\pi_{i+1}\pi_{i+2} \to \pi_{i+2}\pi_{i+1}\pi_i$ cannot create a new direct sum decomposition since it is increasing the left element and decreasing the right one. Suppose that somehow this move created a 321 among the odd elements (within $\mathcal{B}$). If it only used one of π_i, π_{i+2}, then it must have been pre-existing with the other one, contrary to hypothesis. If it used both, then without loss of generality assume $\mathcal{B}$ contains an element x to the left of the replaced 3-factor, but larger than π_{i+2}. Because x is also greater than π_{i+1}, it uses up one of the odd values greater than the diagonal element π_{i+1}, meaning that there must be a y to the right of π_{i+1} but smaller than it, and then x, π_{i+2}, y is a pre-existing 321. The case where $\mathcal{B}$ contains an element y to the right of the replaced 3-factor, but smaller than π_i follows by symmetry.

Non-Case (e) The last possibility to consider is that the 3-factor is split across two adjacent large blocks, necessarily with two elements at the start or end of one of the large blocks. But this is ruled out because large blocks begin and end with descents.

Note that in each of these cases the replacement $123 \to 321$ winds up gluing together all the blocks of π which it straddles, leaving the same number or fewer blocks in π'. In particular, the replacement may glue together blocks, but never splits any apart.

Now consider applications of $321 \to 123$ within a permutation $\rho \in \mathcal{A}_n$ to obtain a new permutation $\hat{\rho}$. Clearly, any adjacent 321 must lie within a single block, as in any two blocks, all the elements in the block to the right are larger than all the elements in the block to the left. Because the even elements within a block increase monotonically, the 321 is composed of odd, even, odd elements. An analysis similar to that given above shows that any such transformation is simply the reverse of one of the cases (a–d) described above, so $\hat{\rho}$ is always in $\mathcal{A}_n$.

Now we need to show that we can use these transformations to return to the identity from any permutation σ in $\mathcal{A}_n$. We first claim that every large block of σ contains a 321 as a factor. For the first element of the block must lie below the diagonal and the last element must lie above it; therefore, there are two consecutive odd elements in the block with the first below and the second above the diagonal. Together with the even element which separates them, and which lies *on* the diagonal, this forms a 321.

Unless ρ is itself the identity, it contains a large block, and therefore a 321. Replacing this 321 with a 123 yields a permutation $\hat{\rho}$ having strictly fewer inversions than ρ. But as $\mathcal{A}_n$ is closed under such replacements, we know that $\hat{\rho}$ also belongs to $\mathcal{A}_n$, and therefore is either the identity or else contains a 321. By iterating this process, we must eventually arrive at a permutation having no inversions, namely the identity.

This establishes that $\mathrm{Eq}^{\sqcup}\big(\iota_n, \big\{\{123, 321\}\big\}\big) = \mathcal{A}_n$, so all that is left is the enumeration of these classes. It is an easy exercise [21, (n^6)] or [7, p. 15] that the number of *indecomposable* 321-avoiding permutations on $m+1$ elements is the Catalan number $c_m = \frac{1}{m+1}\binom{2m}{m}$. This is also the number of possible blocks of size $2m+1$. We define the following three generating functions, which enumerate central binomial coefficients of even order (E), of odd order (O), and the Catalan numbers (C).

$$\begin{array}{llll} E(x) & = \frac{1}{\sqrt{1-4x}} & = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \dots \\ O(x) & = \frac{\frac{1}{\sqrt{1-4x}}-1}{2x} & = 1 + 3x + 10x^2 + 35x^3 + 126x^4 + \dots \\ C(x) & = \frac{1-\sqrt{1-4x}}{2x} & = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots \end{array}$$

The statement of the theorem is equivalent to showing that $E(x) = \sum_{n\geq 0} A_{2n+1}x^n$ and $O(x) = \sum_{n\geq 0} A_{2n+2}x^n$, where we set $A_n := \#\mathcal{A}_n$.

Now a reachable permutation of even size $2n+2$ is the direct sum of an indecomposable block of size $2i+1$ ($i \geq 0$) and a reachable permutation of odd size $2(n-i)+1$. This translates into the recursion/convolution

$$A_{2n+2} = \sum_{i=0}^{n} c_k A_{2(n-i)+1}$$

which is equivalent to $O(x) = E(x)C(x)$, and which is also easily verified from the closed-form expressions for these generating functions. Similarly, a reachable permutation of odd-size $2n+1$ is the direct sum of an indecomposable block of size $2i+1$ and a reachable permutation of even size $2(k-i)$, corresponding to the easily-verified equality of generating functions $E(x) = (1 + xO(x))C(x)$. This completes the proof. $\square$

Although the above proof seems natural enough from the structure of the equivalence class $\mathcal{A}_n$, the simple form of the enumeration as a single binomial coefficient begs the question of whether there is a more direct (perhaps bijective) argument.

The next theorem provides independent proofs of two results which appeared 10 years ago in [CEHKN].

Theorem 15. *(a)* $\#\mathrm{Classes}^{||}\big(n, \big\{\{123, 132, 213\}\big\}\big) = \mathrm{inv}_n$*, the number of involutions of order* n*.*

(b) $\#\mathrm{Eq}^{||}\big(\pi, \big\{\{123, 132, 213\}\big\}\big)$ *is odd for all* n *and for each* $\pi \in S_n$*.*

Proof. Write each involution in $\tau \in \mathrm{Inv}_n \subseteq S_n$ canonically as a product of 1-cycles and 2-cycles, with the elements increasing within each 2-cycle, and with the cycles in decreasing order of largest element. Omitting the parentheses, we view the resulting word $D(\tau)$ as a permutation. Let $\mathcal{D}_n := D(\mathrm{Inv}_n)$ be the image of this map (which is easily reversible by placing parentheses around the ascents of $\sigma \in \mathcal{D}_n$). We claim that this is a canonical set of representatives for the equivalence classes of S_n under $P_3 = \{\{123, 132, 213\}\}$ transformations.

Each permutation $\pi \in S_n$ can be transformed to an element of $\mathcal{D}_n$ as follows: if n is at the front of π, it must stay there. (This corresponds to having n as a fixed point of the involution.) Otherwise, use $123 \to 132$ and $213 \to 132$ (at least one of which is possible at each step) to push n leftward into position 2, which is as far as it will go. The element which is thus pushed into position 1 is the minimal element m which was to the left of n to begin with. This is because m can never trade places with n under the given operations, as 1 is left of 3 in all of 123, 132 and 213. Leaving the leftmost 1-factor n or 2-factor mn fixed, proceed inductively among the remaining elements, at each step moving the maximal remaining element as far left as possible. The end result of this deterministic procedure is a permutation $L(\pi) \in \mathcal{D}_n$. This shows that the number of P_3-equivalence classes is at most $\mathrm{inv}_n = \#\mathcal{D}_n$.

To show that they are the same, it remains to show that each π can be transformed to a *unique* member of $\mathcal{D}_n$, or equivalently that it is not possible to move from one member of $\mathcal{D}_n$ to another using P_3-moves. We will prove this by induction on n. At the same time we will prove statement (b) of the theorem. Assume as an induction hypothesis that both statements have been demonstrated for $n-1$ and $n-2$. It is straightforward to check the base cases by hand. For $n = 3$ the four equivalence classes are P_3 and three singleton classes. For $n = 4$ the classes are $\{1234, 1243, 1324, 2134, \mathbf{1423}, 1342, 2143, 3142, 2314\}$, $\{1342, 3124, \mathbf{1432}, 3142, 3214\}$, $\{4123, \mathbf{4132}, 4213\}$, $\{2341, \mathbf{2431}, 3241\}$, and six singletons: $\{\mathbf{2413}\}, \{\mathbf{3412}\}$, $\{\mathbf{4312}\}$, $\mathbf{4231}$, $\mathbf{3421}$, $\mathbf{4321}$. (The elements in **bold** are the class representatives within $\mathcal{D}_n$.)

First note that if the largest element, n, is at the front of a permutation, then it is immobile under P_3-moves. Thus the equivalence classes split into two kinds: *special* equivalence classes, in which n is at the front of each permutation in the class, and *ordinary* equivalence classes, in which n is never at the front. Moreover it is obvious that the special equivalence classes for S_n correspond exactly to *all* the equivalence classes for S_{n-1} upon deletion of the first elements; therefore, the truth of both (a) and (b) as they apply to the *special* equivalence classes follows by induction.

Next we will look at the *ordinary* equivalence classes. For convenience of exposition, consider a (directed) graph in which the vertices correspond to the permutations in S_n, and there is a blue (directed) edge from π to π' if π' can be obtained from π by applying $123 \to 132$, and similarly a red edge for each $213 \to 132$, and a green edge for each $123 \to 213$. A blue edge just corresponds to a green edge followed by a red one, and indeed the edges always appear in matched sets: the appearance of a 213 in a permutation implies an incident green edge pointing in and a red edge pointing out, and also a blue edge making the chord of this triangle (and similarly for appearances of 123 and 132). The equivalence classes in which we are interested are the (undirected) connected components of this graph.

Now consider the forest of rooted trees which one obtains by taking only those red and blue edges in which the element n plays the role of the "3". The roots (i.e., sinks) of these trees are exactly the permutations in which the n has moved to position 2, which is as far left as it will go within an ordinary equivalence class. More generally, if $\pi_k = n$, then π lies on level $k-2$ of the tree. (We can say that it has *energy* $E(\pi) = k - 2 = \pi^{-1}(n) - 2$.) Note that blue and red edges reduce the energy by one, while green edges leave it unchanged. Each vertex in this forest has either zero or two children, because if it has a blue child (obtained by travelling backwards along a blue edge) then it also has a red child, and vice versa.

Each permutation π lies on a unique directed path to the root of its tree, which we will call the *ground state* of π, $g(\pi)$. Note that $g(\pi)_2 = n$, while $g(\pi)_1$ is the smallest element m to the left of n in π. Because each node has either zero or two children, each rooted tree has an odd number of nodes; indeed all of its level-sums are even except the zeroth level sum, which corresponds to the root vertex (i.e., ground state).

Now we will create larger classes as follows: declare two ground states τ and σ *similar* if $\tau_1 = \sigma_1$ and $\tau_3 \cdots \tau_n$ is P_3-equivalent to $\sigma_3 \cdots \sigma_n$ regarded (in the obvious way) as members of S_{n-2}. For $m \in [n-1]$ and $\nu \in \mathcal{D}_{n-2}$, let $\mathcal{K}(m, \nu)$ be the (disjoint!) union of all trees with similar ground states τ, where $\tau_1 = m$ and $\tau_3 \cdots \tau_n$ is P_3-equivalent to ν. Note that this gives us a total of $(n-1)\mathrm{inv}_{n-2}$ equivalence classes, in agreement with the well-known recursion: $\mathrm{inv}_n = \mathrm{inv}_{n-1} + (n-1)\mathrm{inv}_{n-2}$. (The special equivalence classes account for the first summand.)

We claim that these larger classes $\mathcal{K}(m, \nu)$ are exactly (the vertex sets of) the connected components of our directed graph; that is, there are no directed edges in the graph which escape from one class to another. Once this is shown, then by induction there is a unique member of the class $\mathcal{D}_{n-2}$ of canonical permutations among the ground states in a large component, to which we prepend mn to obtain the unique representative of $\mathcal{K}(m, \nu)$ within $\mathcal{D}_n$.

Furthermore, each $\mathcal{K}(m, \nu)$ will then be of odd size, because each rooted tree has odd size, having all level-sums even except the one corresponding to the ground states, and because the number of rooted trees in the union is odd by the induction hypothesis for $n-2$.

So suppose there is an edge (of any colour) from a $\pi \in \mathcal{K}(m, \nu)$ to $\pi' \in \mathcal{K}(m', \nu')$. Since this move does not involve moving the largest element n, π and π' have the same energy. Our goal is to show that $m = m'$ and ν is P_3-equivalent to ν'. The former follows from our earlier description of m as the minimum element lying to the left of n in π, because π and π' have the same set of elements to the left of n. The latter requires an analysis of the cases that can arise as π and π' move towards their ground states in their respective trees.

As the n moves leftward through each of the two permutations (following red and/or blue edges toward their respective ground states) then it sometimes encounters identical elements and therefore has the same effect; eventually it encounters the positions where the difference lies, having swept before it the minimal intervening element, b. What happens from this point forward depends on how π and π' differ, and the relative value of b.

To clarify the cases, let the three values where the difference was applied be $d < f < h$, and designate b by one of C, E, G or I (where $C < d < E < f < G < h < I$), depending on its relative order within the factor. For example, at some point along the path from π to $g(\pi)$ we may see a permutation containing the factor $dfhCn$, while at the same energy level on the path from π' to $g(\pi')$ we see instead $dhfCn$ (having followed a blue edge), the two permutations being otherwise identical. Advancing the element n three further steps to the left, we arrive in the first instance at $Cndfh$ and in the second instance at $Cndhf$; the n then continues forward all the way to position 2 (zero energy), making identical moves in each case. The resulting ground states $g(\pi)$ and $g(\pi')$ differ only by a (blue) move $dfh \to dhf$, so $\nu = \nu'$.

Here is a table of the cases that arise given the four possible relative values of b; blue edges are the composition of green with red.

$$\begin{array}{rcccc}
\text{Input} : dfhCn & \to & fdhCn & \to & dhfCn \\
\text{Output} : Cndfh & \to & Cnfdh & \to & Cndhf \\
\\
\text{Input} : dfhEn & \to & fdhEn & \to & dhfEn \\
\text{Output} : dnEfh & \to & dnfEh & \to & dnEhf \\
\\
\text{Input} : dfhGn & \to & fdhGn & \to & dhfGn \\
\text{Output} : dnfGh & = & dnfGh & \to & dnfhG \\
\\
\text{Input} : dfhIn & \to & fdhIn & \to & dhfIn \\
\text{Output} : dnfhI & = & dnfhI & = & dnfhI
\end{array}$$

Examining this table shows that the classes $\mathcal{K}(m, \nu)$ containing π and $\mathcal{K}(m', \nu')$ containing π and π' have ν P_3-equivalent to ν', which completes the proof. $\square$

This result is particularly striking because the equivalence relation has the same number of classes as Knuth equivalence, yet the two relations are materially different. For example, for $n = 3$, the equivalence classes for P_K have sizes 1,1,2,2, whereas for $P_3 = \big\{\{123, 132, 213\}\big\}$ the sizes are 1,1,1,3. In fact the authors in [6] show that the corresponding monoids (plactic and Chinese) share the same graded Hilbert series, and they obtain a partial recurrence for the numbers $\#\mathrm{Eq}^{||}(\iota_n, P_3)$.

Proposition 16 ([6], Cor. 4.3)**.** *For n odd, $\#\mathrm{Eq}^{||}(\iota_n, P_3) = n \cdot \#\mathrm{Eq}^{||}(\iota_{n-1}, P_3)$.*

The recurrence for n even appears still to be open.

4 Doubly adjacent transformations

For completeness we include a brief treatment of the situation where both indices and values are simultaneously constrained to be adjacent. In this highly constrained situation, the permutations reachable from the identity are easy to classify and enumerate in all cases. Since all the treatments are similar, we can wrap them up in one proposition.

As in the previous section, we have as yet no results related to the enumeration of equivalence classes.

The statement of this proposition makes use of the *Iverson bracket*: [S] is equal to 1 if the statement S is true, and 0 otherwise.

Proposition 17. $\#\mathrm{Eq}^{\square}(\iota_n, P_1)$ *obeys the recurrence* $a(n) = a(n-1) + a(n-2)$ *with* $a_1 = a_2 = 1$. *(Fibonacci numbers* $F(n)$*, [15, A000045]).*

$\#\mathrm{Eq}^{\square}(\iota_n, P_4)$ *obeys the recurrence* $a(n) = a(n-1) + a(n-3)$ *with* $a_0 = 0$, $a_1 = a_2 = 1$ *([15, A000930]).*

$\#\mathrm{Eq}^{\square}(\iota_n, P_3) = F(n+1) - [n \text{ is even}]$.

$\#\mathrm{Eq}^{\square}(\iota_n, P_5)$ *obeys the recurrence* $a(n) = a(n-1) + a(n-2) + a(n-3)$ *with* $a(0) = a(1) = a(2) = 1$ *(Tribonacci numbers, [15, A000213]).*

$\#\mathrm{Eq}^{\square}(\iota_n, P_7) = T(n+2) - [n \text{ is even}]$, *where* $T(n)$ *obeys the recurrence* $T(n) = T(n-1) + T(n-2) + T(n-3)$ *with* $T(0) = T(1) = 0, T(2) = 1$. *(Tribonacci numbers (with different initial conditions), [15, A000073]).*

Proof. We begin by characterizing the various equivalence classes of ι_n. In each case, no element can move any further from its starting position then it could via a single move. The resulting classes are subsets of those layered permutations which are direct sums of anti-identities of dimensions either 1, 2 or 3, as follows:

P_1 ($123 \leftrightarrow 132$): any direct sum of ρ_1 and ρ_2 beginning with ρ_1;

P_4 ($123 \leftrightarrow 321$): any direct sum of ρ_1 and ρ_3;

P_3 ($123 \leftrightarrow 132 \leftrightarrow 213$): any direct sum of ρ_1 and ρ_2 including at least one ρ_1;

P_5 ($123 \leftrightarrow 132 \leftrightarrow 321$): any direct sum of ρ_1, ρ_2 and ρ_3 not beginning with ρ_2;

P_7 ($123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$): any direct sum of ρ_1, ρ_2, ρ_3 with at least one of odd dimension;

In each case it is easy to see that the given class remains closed under application of the appropriate operations. It is also easy in general to see how to reach a given target permutation from ι_n, especially if we cast the block sizes in the language of regular expressions. The notation $\{xy\}$ means a single block of size either x or y. An asterisk following a number means zero or more copies of that number. An asterisk following a string within [ ] (not to be confused with the Iverson brackets in the statement of the proposition) indicates zero or more copies of that string.

P_1: The block sizes are $1\{12\}* = [12*]*$. Build each string of blocks of the form $12*$ from right to left.

P_4: The block sizes are $\{13\}*$; build each block freely.

P_3: From any non-identity permutation with block sizes as described, there is at least one instance of 21 or 12, which can be transformed by one of the rules into a 111. The resulting permutation has one fewer ρ_2. Proceed inductively to transform all the ρ_2's (i.e., 2-blocks) to consecutive 1-blocks until the identity is reached.

P_5: The block sizes are $[\{13\}2*]*$. First use $123 \to 132$ to build all the 2-blocks from right to left. Then use $123 \to 321$ to place the 3-blocks.

P_7: Build the 2-blocks first, as in the case of P_3, and then place the 3-blocks.

One now verifies all the necessary base cases, as trivially $a_1 = 1$, $a_2 = 1$, and $a_3 =$ the size of the non-singleton block of P_j.

As for the recurrences, for $n > 3$:

P_1: $a_n = a_{n-1} + a_{n-2}$, by appending respectively a ρ_1 or a ρ_2.

P_4: $a_n = a_{n-1} + a_{n-3}$, by appending respectively a ρ_1 or a ρ_3.

P_5: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, by appending ρ_1, ρ_2 or ρ_3.

P_3: Count all direct sums of ρ_1 and ρ_2 (obviously Fibonacci) and then subtract 1 from the even terms to remove the special case $2*$ (all blocks of size 2).

P_7: Count all direct sums of ρ_1, ρ_2, ρ_3 to get tribonacci numbers [A000073], and subtract 1 from the even terms because block structure $2*$ is disallowed. Alternatively, verify the recurrence $a_n = a_{n-2} + U_n$, where U_n is the P_5-recurrence [A000213], by noting that a permutation in $\mathrm{Eq}^{\square}(\iota_n, P_7)$ is either a ρ_2 prepended to a permutation in $\mathrm{Eq}^{\square}(\iota_{n-2}, P_7)$, or else belongs to $\mathrm{Eq}^{\square}(\iota_n, P_5)$. $\square$

5 Final remarks and open questions

Our results in this paper are just a tractable subset of questions that could be explored within these families of equivalence relations. We created the framework to easily allow for a number of extensions. The connections with familiar integer sequences, pattern-avoidance in permutations, and important combinatorial bijections indicate the value of further work. Possible directions for further study include the following:

1. Study the sizes (and characterize if possible) all equivalence classes $\mathrm{Eq}^*(\pi, P)$, not just for the case $\pi = \iota_n$. Corresponding to each equivalence relation is the multiset of sizes of the equivalences classes, perhaps best considered as an integer partition of $n!$. Is the study of these of interest?

2. Allow for more generality among the (set) partition P of S_3 which defines our relations. The authors in [16] allow substitution of patterns in S_3 where no element is fixed, but still restrict to partitions P consisting of exactly one non-singleton block containing the identity 123. Although it seems unwieldy to work with all $B(6) = 203$ possible partitions of S_4, perhaps a different restriction that forces greater symmetry among the relations would be useful. For example, the Knuth relations $P_K^{||} = \{\{213, 231\}, \{132, 312\}\}$ are closed under reversal and complementation.

3. Consider relations generated by partitions P of S_k for $k > 3$. Here one definitely needs some conditions to restrict focus to relations of particular interest, since the Bell number $B(4!)$ is already far too large to handle all cases.

4. Study in greater detail the structure of the graphs defined by these relations. What can one say about their degree sequences or diameters? How many moves are necessary in order to transform a given π to the identity?

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