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Enumerating Triangulations by Parallel Diagonals

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Abstract

The triangulations of a regular polygon are enumerated according to the number of diagonals parallel to a fixed edge. We also point out the connection to Dyck paths avoiding certain points on the x-axis.

1 Introduction

The purpose of this note is to enumerate triangulations of a regular convex polygon according to the number of diagonals parallel to a fixed edge. This enumeration is of interest because it provides insight into the "shape of a typical triangulation" and because of its connection to the Shapiro convolution identity (see (4) below).

We consider a triangulation of an n-gon as a labeled graph with vertices $0, 1, \ldots, n-1$[1] and edges denoted xy for distinct vertices x and y. The edges include n sides $01, 12, \ldots, (n-1)0$ and $n-3$ diagonals.

Definition 1. Let $f_{xy}(n, k)$ be the number of triangulations of a regular n-gon which include exactly k diagonals parallel to the edge xy. Also let $f_{xy}(n)$ denote $f_{xy}(n, 0)$.

For example, there are 14 triangulations of a hexagon, 4 of which include a diagonal parallel to 01 (see Figure 1). The remaining 10 triangulations all have zero diagonals parallel to 01. Therefore $f_{01}(6, 1) = 4$ and $f_{01}(6) = 10$.

[1]For convenience, the vertex 0 of an n-gon is sometimes also labeled n.

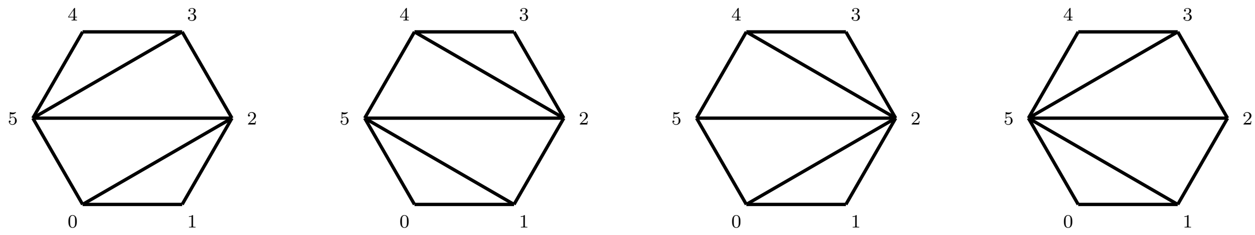

Figure 1: The triangulations of a hexagon that include one diagonal parallel to 01.

Given n and k, by symmetry $f_{xy}(n,k)$ depends only on the value of $y-x$ modulo n and not on the specific choice of x and y. Furthermore, two edges ab and cd in a triangulation of an n-gon are parallel if and only if $a+b$ and $c+d$ are congruent modulo n. It follows that for all n, x and y,

$$f_{xy}(2n,k) = \begin{cases} f_{01}(2n,k), & \text{if } x+y \text{ is odd;} \\ f_{02}(2n,k), & \text{if } x+y \text{ is even,} \end{cases}$$

and

$$f_{xy}(2n+1,k) = f_{01}(2n+1,k).$$

The question at hand is thus reduced to finding $f_{01}(2n,k)$, $f_{02}(2n,k)$ and $f_{01}(2n+1,k)$. Theorems 2 and 4 below provide explicit formulas for these functions when $k=0$ and when $k>0$, respectively. These formulas are given in terms of the Catalan numbers [5, A000108]

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Recall that there are C_{n-2} triangulations of an n-gon. Therefore for all n, x and y,

$$\sum_{k\geq 0} f_{xy}(n,k) = C_{n-2}. \tag{1}$$

The recursion relation

$$\sum_{i=0}^{n} C_i C_{n-i} = C_{n+1} \tag{2}$$

implies the identity

$$\sum_{i=0}^{n} C_{2i} C_{2n+1-2i} = \frac{1}{2} C_{2n+2}, \tag{3}$$

which is used below. We also make use of the Shapiro convolution identity:

$$\sum_{j=0}^{n} C_{2j} C_{2n-2j} = 4^n C_n \tag{4}$$

(see [5, A151403]). Andrews [1] recently gave several proofs of (4) and its q-analogs, with one of these proofs being purely combinatorial (however, finding a direct bijective proof of (4) is still an open problem).

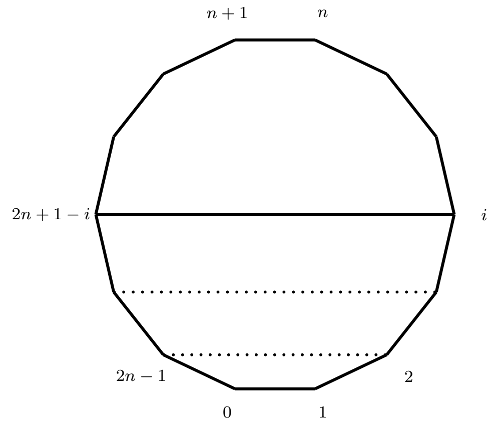

Figure 2: Illustration of the proof of (5). The dotted lines represent the avoided diagonals.

2 Avoiding diagonals of a fixed direction

We begin by enumerating the triangulations that avoid all diagonals parallel to a fixed edge.

Theorem 2. *For any $n \geq 2$,*

$$f_{01}(2n) = 2C_{2n-3} \tag{5}$$

and

$$f_{02}(2n) = C_{2n-1} + 2C_{2n-2} - 2^{2n-1}C_{n-1}. \tag{6}$$

For any $n \geq 1$,

$$f_{01}(2n+1) = 2^{2n-1}C_{n-1} - C_{2n-1}. \tag{7}$$

Equations (5), (6) and (7) can be proved by induction on n as follows; the base cases are easily verified.

Proof of (5):

Enumerate the triangulations of a $2n$-gon that include at least one diagonal parallel to 01 according to the minimal number i, with $2 \leq i \leq n-1$, such that $i(2n+1-i)$ is a diagonal of the triangulation (see Figure 2). The $(2n-2i+2)$-gon with vertices $i, i+1, \ldots, 2n+1-i$ can be triangulated in C_{2n-2i} ways. By induction the $2i$-gon with vertices $0, 1, \ldots, i, 2n+1-i, 2n+2-i, \ldots, 2n-1$ can be triangulated in $2C_{2i-3}$ ways. Subtracting these from the

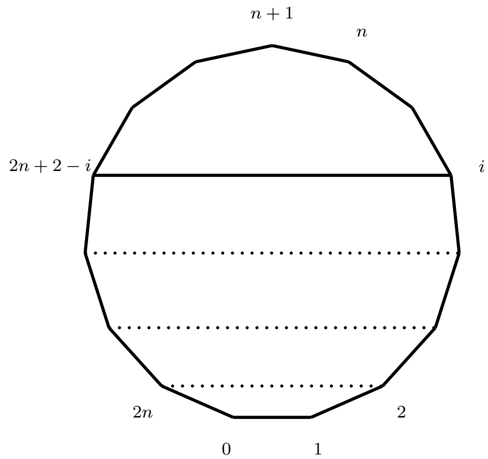

Figure 3: Illustration of the proof of (7).

total number of triangulations of a $2n$-gon gives

$$\begin{aligned} f_{01}(2n) &= C_{2n-2} - \sum_{i=2}^{n-1} C_{2n-2i} \cdot 2C_{2i-3} \\ &= C_{2n-2} - 2\sum_{i=2}^{n} C_{2n-2i}C_{2i-3} + 2C_0C_{2n-3} \\ &= 2C_{2n-3}, \end{aligned}$$

where in the last equality we have used (3). Another proof of (5), using a result of Callan on Dyck paths, is outlined in Section 4.

Proof of (7):

Enumerate the triangulations of a $(2n+1)$-gon that include at least one diagonal parallel to 01 according to their diagonal $i(2n+2-i)$ with minimal i (see Figure 3). The $(2n-2i+3)$-gon with vertices $i, i+1, \ldots, 2n+2-i$ can be triangulated in $C_{2n-2i+1}$ ways. By (5), the $2i$-gon with vertices $0, 1, \ldots, i, 2n+2-i, 2n+3-i, \ldots, 2n$ can be triangulated in $2C_{2i-3}$

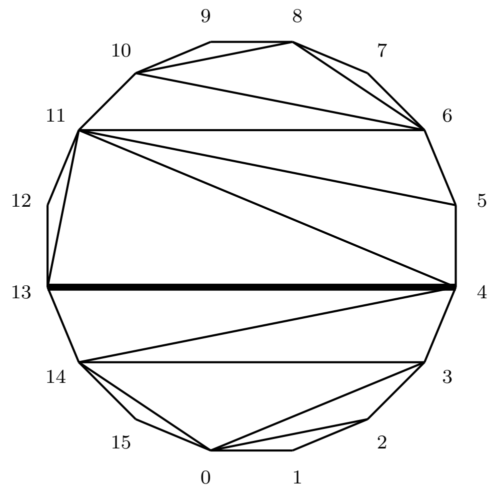

Figure 4: Example illustrating the proof of Lemma 3.

ways. Therefore

$$
\begin{aligned}
f_{01}(2n+1) &= C_{2n-1} - \sum_{i=2}^{n} C_{2n-2i+1} \cdot 2C_{2i-3} \\
&= \sum_{j=0}^{2n-2} C_j C_{2n-2-j} - 2\sum_{i=2}^{n} C_{2n-2i+1} C_{2i-3} \\
&= \sum_{j=0}^{2n-2} (-1)^j C_j C_{2n-2-j}.
\end{aligned}
$$

Thus by (2) and (4),

$$
f_{01}(2n+1) = 2\sum_{j=0}^{n-1} C_{2j} C_{2n-2-2j} - \sum_{j=0}^{2n-2} C_j C_{2n-2-j} = 2^{2n-1} C_{n-1} - C_{2n-1}.
$$

The following lemma will be used in the proof of (6).

Lemma 3. *For any $n \geq 2$,*

$$
\sum_{i=1}^{n-1} 2^{2i-1} C_{i-1} C_{2n-1-2i} = 4^{n-1} C_{n-1} - C_{2n-2}. \tag{8}
$$

Proof. Let $h(n)$ be the number of triangulations of a $2n$-gon together with a marking either on one of the sides 01, $n(n+1)$ or on one of the diagonals $k(2n+1-k)$, with $2 \leq k \leq n-1$, if any such diagonals are present. For example, Figure 4 shows a triangulation of a 16-gon with the diagonal 4(13) marked. Consider the following two ways to enumerate these marked triangulations.

1. First mark the edge $(j+1)(2n-j)$, with $0 \le j \le n-1$. For example, the marked triangulation in Figure 4 corresponds to $n=8$ and $j=3$. Then choose one of the C_{2j} triangulations of the $(2j+2)$-gon with vertices $0,1,\ldots,j+1,2n-j,2n-j+1,\ldots,2n-1$ and one of the $C_{2n-2j-2}$ triangulations of the $(2n-2j)$-gon with vertices $j+1,j+2,\ldots,2n-j$. Thus there are $C_{2j-2}C_{2(n-1)-2j}$ such marked triangulations for each j. By (4),

$$h(n) = \sum_{j=0}^{n-1} C_{2j}C_{2(n-1)-2j} = 4^{n-1}C_{n-1}. \tag{9}$$

2. There are C_{2n-2} marked triangulations whose edge $n(n+1)$ is the one marked. The remaining marked triangulations can be enumerated according to the maximal i, with $1 \le i \le n-1$, such that $i(2n+1-i)$ is one of the diagonals in the triangulation (where the case $i=1$ corresponds to triangulations avoiding all diagonals parallel to 01.) For example, in Figure 4 we have $n=8$ and $i=6$. For each such i, there are $h(i)$ marked triangulations of the $2i$-gon with vertices $0,1,\ldots,i,2n-2i+1,\ldots,2n-1$, and there are $f_{01}(2n-2i+2)$ triangulations of the $(2n-2i+2)$-gon with vertices $i,i+1,\ldots,2n+1-i$ which avoid the diagonals $(i+1)(2n-i),\ldots,n(n+1)$. Thus by (5) and (9),

$$\begin{aligned} h(n) &= C_{2n-2} + \sum_{i=1}^{n-1} h(i) f_{01}(2n-2i+2) \\ &= C_{2n-2} + \sum_{i=1}^{n-1} 4^{i-1}C_{i-1} \cdot 2C_{2n-2i-1}. \end{aligned}$$

Comparing this with (9) completes the proof. □

Proof of (6):

For convenience we calculate $f_{1(2n-1)}(2n) = f_{02}(2n)$. Enumerate the triangulations of a $2n$-gon that include at least one diagonal parallel to $1(2n-1)$ according to their diagonal $i(2n-i)$ with minimal i, where $1 \le i \le n-1$ (see Figure 5). By (7), the $(2i+1)$-gon with vertices $0,1,\ldots,i,2n-i,2n+1-i,\ldots,2n-1$ can be triangulated in $2^{2i-1}C_{i-1} - C_{2i-1}$ ways. The $(2n-2i+1)$-gon with vertices $i,i+1,\ldots,2n+2-i$ can be triangulated in $C_{2n-2i-1}$ ways. Therefore

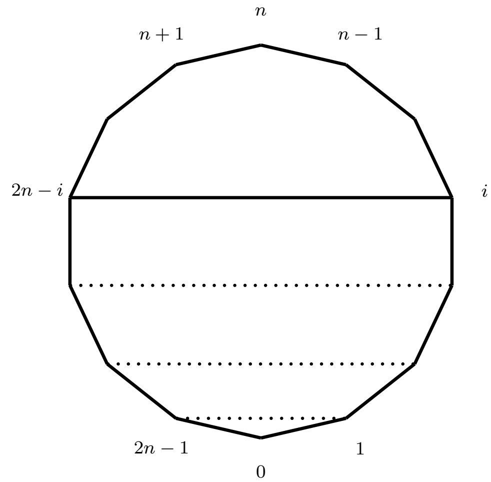

Figure 5: Illustration of the proof of (6).

$$
\begin{aligned}
f_{02}(2n) &= C_{2n-2} - \sum_{i=1}^{n-1} (2^{2i-1}C_{i-1} - C_{2i-1})C_{2n-2i-1} \\
&= C_{2n-2} - \sum_{i=1}^{n-1} 2^{2i-1}C_{i-1}C_{2n-1-2i} + \sum_{i=1}^{n-1} C_{2i-1}C_{2n-2i-1} \\
&= C_{2n-2} - (4^{n-1}C_{n-1} - C_{2n-2}) + (C_{2n-1} - 4^{n-1}C_{n-1}) \\
&= 2C_{2n-2} + C_{2n-1} - 2^{2n-1}C_{n-1},
\end{aligned}
$$

where in the penultimate equality we have used (4) and (8).

3 Including a number of diagonals of a fixed direction

The next theorem enumerates the triangulations with a fixed positive number of diagonals parallel to a fixed edge.

Theorem 4. *Let $n \geq 2$ and $k \geq 1$. Then*

$$f_{01}(2n,k) = \sum_{i_1+\ldots+i_{k+1}=n-1} 2^{k+1} C_{2i_1-1}C_{2i_2-1}\cdots C_{2i_{k+1}-1}, \tag{10}$$

and

$$f_{02}(2n,k) = \sum_{i_1+\ldots+i_{k+1}=n-1} 2^{k-1}(2^{i_1-1}C_{i_1-1} - C_{2i_1-1})(2^{i_2-1}C_{i_2-1} - C_{2i_2-1})C_{2i_3-1}C_{2i_4-1}\cdots C_{2i_{k+1}-1}. \tag{11}$$

If $n, k \geq 1$ then

$$f_{01}(2n+1,k) = \sum_{i_1+\ldots+i_{k+1}=n} (2^{i_1-1}C_{i_1-1} - C_{2i_1-1})C_{2i_2-1}C_{2i_3-1}\cdots C_{2i_{k+1}-1}. \tag{12}$$

Proof. Consider a triangulation of a $2n$-gon which includes exactly k diagonals parallel to 01. These k diagonals partition the $2n$-gon into $k+1$ triangulated polygons, and partition the $n-1$ edges $12, 23, \ldots, (n-1)n$ into $k+1$ corresponding parts consisting of $i_1, \ldots, i_{k+1} \geq 1$ edges. The number of vertices in each resulting polygon is $2i_j + 2$ for all j, and each such polygon is triangulated with diagonals which are not parallel to one of its sides. Thus

$$f_{01}(2n,k) = \sum_{\substack{i_1+\ldots+i_{k+1}=n-1\\ i_j\geq 1}} f_{01}(2i_1+2)f_{01}(2i_2+2)\cdots f_{01}(2i_{k+1}+2), \tag{13}$$

which together with (5) proves equation (10). By similar considerations,

$$f_{02}(2n,k) = \sum_{\substack{i_1+\ldots+i_{k+1}=n-1\\ i_j\geq 1}} f_{01}(2i_1+1)f_{01}(2i_2+1)f_{01}(2i_3+2)f_{01}(2i_4+2)\cdots f_{01}(2i_{k+1}+2),$$

and

$$f_{01}(2n+1,k) = \sum_{\substack{i_1+\ldots+i_{k+1}=n\\ i_j\geq 1}} f_{01}(2i_1+1)f_{01}(2i_2+2)f_{01}(2i_3+2)\cdots f_{01}(2i_{k+1}+2).$$

The differences between these equations and (13) result from considering the regions of the polygon which contain the vertices 0 and n. The proofs of (11) and (12) now follow from (6) and (7), respectively. $\square$

Note that a consequence of (1), (5) and (10) is the Catalan number identity

$$\sum_{\substack{k\geq 0\\ i_1+\ldots+i_{k+1}=n}} 2^{k+1}C_{2i_1-1}C_{2i_2-1}\cdots C_{2i_{k+1}-1} = C_{2n}. \tag{14}$$

Another Catalan number identity can be obtained by considering the set of marked triangulations of a $2n$-gon described in the proof of Lemma 3. If k is the number of diagonals parallel to 01 in a triangulation of a $2n$-gon, this triangulation corresponds to $k+2$ such marked triangulations. Thus by (10),

$$\sum_{\substack{0\leq k\leq n-2\\ i_1+\ldots+i_{k+1}=n-1}} (k+2)\,2^{k+1}C_{2i_1-1}C_{2i_2-1}\cdots C_{2i_{k+1}-1} = 4^{n-1}C_{n-1}. \tag{15}$$

Combining (14) and (15) results in the identity

$$\sum_{\substack{1\leq k\leq n-1\\ i_1+\ldots+i_{k+1}=n}} k\,2^{k}C_{2i_1-1}C_{2i_2-1}\cdots C_{2i_{k+1}-1} = 2^{2n-1}C_n - C_{2n}.$$

4 Remarks

The next theorem was proposed as a problem in the *American Mathematical Monthly* by Callan in 2003, and a solution appeared in 2005.

Theorem 5. *[2] The number of Dyck $2n$-paths that avoid the points $(4k, 0)$, $k = 1, 2, \ldots, n-1$ is twice the number of Dyck $(2n-1)$-paths.*

Callan proved Theorem 5 using a bijection on Dyck paths. The result is equivalent to (5), since the Dyck paths in question are equinumerous with the triangulations of a $(2n-2)$-gon which avoid all diagonals parallel to 01. To see this, compare the initial conditions for both sequences, and observe that the Dyck paths in question satisfy analogous recursive relations to the ones given by equations (1) and (10). Similarly, it can be shown that $f_{02}(2n)$ is equal to the number of Dyck $2n$-paths avoiding all points $(4k + 2, 0)$ with $k = 0, 1, \ldots, n-1$, and that $f_{01}(2n + 1)$ is equal to the number of Dyck $(2n + 1)$-paths avoiding all points $(4k, 0)$ with $k = 1, 2, \ldots, n-1$.

The relation with Dyck paths also gives another interpretation of these results in terms of triangulations. Consider the bijection between Dyck paths and triangulations implicit in the paper of Forder [4]. Using this bijection, the points $(4k, 0)$ and $(4k+2, 0)$ of a Dyck $2n$-path correspond to the diagonals of the form $0(2k + 1)$ and $0(2k)$, respectively, of a triangulation of an $(n + 2)$-gon. This gives analogous results to those of the present note, concerning the number of diagonals of this form instead of the number of diagonals parallel to a fixed edge.

The sequences $f_{02}(2n)$ and $f_{01}(2n + 1)$ appear in [5, A066357] and [5, A079489], respectively. The interpretation in terms Dyck paths is given there, along with other interpretations and several interesting properties. Callan ([3] and [5, A066357]) proved the analog of (6) using generating functions. Barry [5, A066357] gave an alternative formula for this sequence:

$$f_{02}(2n + 2) = \frac{1}{n} \sum_{k=0}^{n} \binom{4n}{k} \binom{3n - k - 2}{n - k - 1}.$$

Callan used Dyck paths to prove that

$$f_{02}(2n + 2) = \sum_{k=1}^{n} f_{01}(2k + 1) f_{01}(2(n - k) + 1).$$

Another relation between these sequences is evident from (6) and (7):

$$f_{01}(2n + 1) + f_{02}(2n) = 2C_{2n-2}. \tag{16}$$

A direct proof of (16) may also be of interest.

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