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Matrix Decomposition of the Unified Generalized Stirling Numbers and Inversion of the Generalized Factorial Matrices

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Abstract

In this paper, we give a matrix decomposition method used to calculate unified generalized Stirling numbers in an explicit, non-recursive mode, and some of its applications. Then, we define generalized factorial matrices which may be regarded as a generalization in the form of the Vandermonde matrices, and presents some of their properties — in particular, triangular matrix factors of the inverse matrices of the generalized factorial matrices.

1 Introduction

The unified generalized Stirling numbers, defined by Hsu and Shuie [1], are the connection coefficients of linear relations between generalized factorial functions. The generalized factorial functions of a real or complex number x with real increment α, denoted by $(x|\alpha)_n$, are special polynomials in x of degree n, as

$$(x|\alpha)_0 = 1, \quad \text{and} \quad (x|\alpha)_n = x(x-\alpha)\cdots(x-n\alpha+\alpha), \quad n = 1, 2, \ldots. \tag{1}$$

Thus, the unified generalized Stirling numbers with real parameters α, β, γ, denoted by $S(n, k; \alpha, \beta, \gamma)$, $n, k = 0, 1, 2, \ldots$, may be defined as (see [1])

$$(x|\alpha)_n = \sum_{k=0}^{\infty} S(n, k; \alpha, \beta, \gamma)(x-\gamma|\beta)_k. \tag{2}$$

We see from (2) that for any α, β, γ, $S(n, n; \alpha, \beta, \gamma) = 1$; and $S(n, k; \alpha, \beta, \gamma) = 0$ if $k > n$. Therefore, the upper limit ∞ of the summation in the right side of equality (2) may be replaced by n.

As Hsu and Shuie pointed, the definition (2) gives a more transparent unification of various Stirling-type numbers studied previously by other authors (see [1]). In particularly, we see from (2) that $S(n, k; 0, 0, 0) = \delta_{n,k}$ (the Kronecker symbols), $S(n, k; 1, 0, 0) = s(n, k)$, $S(n, k; 0, 1, 0) = S(n, k)$, and $S(n, k; 0, 0, 1) = \binom{n}{k}$, where $s(n, k)$ and $S(n, k)$ are the Stirling numbers of the first and second kind respectively.

Hsu, Shuie and other authors then investigated basic properties of the unified generalized Stirling numbers, such as recurrence relations, generating functions, convolution formulas, congruence properties, asymptotic expansion, q-analoque formulas and corresponding combinatorial interpretation (see [1, 2, 3]).

2 Matrix Decomposition of the Unified Generalized Stirling Numbers

Now let's have the following definition.

Definition 1. *the unified generalized Stirling (transform) matrix with real parameters* α, β, γ, denoted by $\mathbf{S}_{\alpha,\beta,\gamma}$, is defined to be an infinite-dimensional lower triangular matrix, the (n, k)th entries of which are $S(n, k; \alpha, \beta, \gamma)$ $(n, k = 0, 1, 2, \ldots)$, such that

$$\mathbf{v}_\alpha(x) = \mathbf{S}_{\alpha,\beta,\gamma}\mathbf{v}_\beta(x - \gamma) \tag{3}$$

where $\mathbf{v}_\alpha(x)$ is *the vector of the generalized factorials with real increment* α, as follows

$$\mathbf{v}_\alpha(x) = (1, x, (x|\alpha)_2, \ldots, (x|\alpha)_n, \ldots)^T. \tag{4}$$

Remark 2. It can easily be shown that equations (2) and (3) are equivalent. We may find immediately that matrices $\mathbf{S}_{0,0,0} = \mathbf{E}$, $\mathbf{S}_{0,0,1} = \mathbf{B}$, $\mathbf{S}_{1,0,0} = \mathbf{S}_1$, $\mathbf{S}_{0,1,0} = \mathbf{S}_2$ are, respectively, the infinite-dimensional unit matrix and the matrices of the binomial transform, the Stirling transforms of the first and second kind for integer sequences.

Remark 3. We see from the matrix definition of the unified generalized Stirling numbers, (3), that

$$\mathbf{v}_\beta(x - \gamma) = \mathbf{S}_{\beta,\alpha,-\gamma}\mathbf{v}_\alpha(x).$$

This leads to that

$$\mathbf{v}_\alpha(x) = \mathbf{S}_{\alpha,\beta,\gamma}\mathbf{S}_{\beta,\alpha,-\gamma}\mathbf{v}_\alpha(x),$$

namely $(\mathbf{S}_{\alpha,\beta,\gamma}\mathbf{S}_{\beta,\alpha,-\gamma} - \mathbf{E})\mathbf{v}_\alpha(x) = \mathbf{0}$ (the infinite-dimensional zero matrix). Because x is an arbitrary real or complex number, we have $\mathbf{S}_{\alpha,\beta,\gamma}\mathbf{S}_{\beta,\alpha,-\gamma} = \mathbf{E}$, namely matrix $\mathbf{S}_{\beta,\alpha,-\gamma}$ is the inverse of $\mathbf{S}_{\alpha,\beta,\gamma}$.

We first investigate three basic types of the generalized Stirling matrices: $\mathbf{S}_{0,0,\gamma}$, $\mathbf{S}_{\alpha,0,0}$, and $\mathbf{S}_{0,\beta,0}$.

According to the binomial theorem, we have that

$$(x|0)_n = x^n = \sum_{k=0}^{n} \binom{n}{k} \gamma^{n-k}(x-\gamma)^k = \sum_{k=0}^{n} \binom{n}{k} \gamma^{n-k}(x-\gamma|0)_k.$$

Hence, the (n,k)th entry of $\mathbf{S}_{0,0,\gamma}$ is that

$$S(n,k;0,0,\gamma) = \gamma^{n-k}\binom{n}{k}, \qquad (n,k=0,1,2,\ldots). \tag{5}$$

Remark 4. In case γ is an integer, $\mathbf{S}_{0,0,\gamma}$ is just the γ-fold binomial transform matrix for an integer sequence, namely γ successive binomial transform for an integer sequence (see [4]).

Next, we look at the generalized Stirling numbers $S(n,k;\alpha,0,0)$ ($\alpha \neq 0$) defined in expression $(x|\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,0,0)x^k$, $(n,k=0,1,2,\ldots)$.

Because $(x|\alpha)_n = \alpha^n(\frac{x}{\alpha}|1)_n$ and $x^k = \alpha^k(\frac{x}{\alpha})^k$,

$$\alpha^n(\frac{x}{\alpha}|1)_n = \sum_{k=0}^{n} S(n,k;\alpha,0,0)\alpha^k(\frac{x}{\alpha})^k.$$

On the other hand, $(\frac{x}{\alpha}|1)_n = \sum_{k=0}^{n} S(n,k;1,0,0)(\frac{x}{\alpha})^k$. Hence, we have $S(n,k;\alpha,0,0)\alpha^k = \alpha^n S(n,k;1,0,0)$, which implies that

$$S(n,k;\alpha,0,0) = \alpha^{n-k}S(n,k;1,0,0) = \alpha^{n-k}s(n,k), \qquad (n,k=0,1,2,\ldots). \tag{6}$$

Namely, the (n,k)th entry of matrix $\mathbf{S}_{\alpha,0,0}$ is the Stirling number $s(n,k)$ of the first kind, multiplied by α^{n-k}.

Similarly, we may obtain that when $\beta \neq 0$,

$$S(n,k;0,\beta,0) = \beta^{n-k}S(n,k;0,1,0) = \beta^{n-k}S(n,k), \qquad (n,k=0,1,2,\ldots). \tag{7}$$

Namely, the (n,k)th entry of matrix $\mathbf{S}_{0,\beta,0}$ is the Stirling number $S(n,k)$ of the second kind, multiplied by β^{n-k}.

Remark 5. We see from (5), (6), and (7) that, if θ is an integer, then each one of the generalized Stirling numbers $S(n,k;\theta,0,0)$, $S(n,k;0,\theta,0)$ and $S(n,k;0,0,\theta))$ is an integer number.

Remark 6. In case α (or β) is an integer, matrix $\mathbf{S}_{\alpha,0,0}$ (or $\mathbf{S}_{0,\beta,0}$) defines a *generalized Stirling transform matrix of the first (or second) kind for an integer sequence.* They both are a pair of transform and inverse transform of integer sequences, if and only if $\alpha = \beta$.

Now we may give a general decomposition formula of the unified generalized Stirling matrices, $\mathbf{S}_{\alpha,\beta,\gamma}$.

Theorem 7. *Let $\mathbf{S}_{\alpha,\beta,\gamma}$ be a generalized Stirling matrix with real parameters α,β,γ. Then*

$$\mathbf{S}_{\alpha,\beta,\gamma} = \mathbf{S}_{\alpha,0,0}\mathbf{S}_{0,0,\gamma}\mathbf{S}_{0,\beta,0}. \tag{8}$$

Proof. We see from (3) that

$$\mathbf{v}_\alpha(x) = \mathbf{S}_{\alpha,0,0}\mathbf{v}_0(x), \quad \mathbf{v}_0(x) = \mathbf{S}_{0,0,\gamma}\mathbf{v}_0(x-\gamma), \quad \mathbf{v}_0(x-\gamma) = \mathbf{S}_{0,\beta,0}\mathbf{v}_\beta(x-\gamma).$$

Hence we have

$$\mathbf{v}_\alpha(x) = \mathbf{S}_{\alpha,0,0}\mathbf{S}_{0,0,\gamma}\mathbf{S}_{0,\beta,0}\mathbf{v}_\beta(x-\gamma).$$

On the other hand, we also have $\mathbf{v}_\alpha(x) = \mathbf{S}_{\alpha,\beta,\gamma}\mathbf{v}_\beta(x-\gamma)$. Thus,

$$(\mathbf{S}_{\alpha,\beta,\gamma} - \mathbf{S}_{\alpha,0,0}\mathbf{S}_{0,0,\gamma}\mathbf{S}_{0,\beta,0})\mathbf{v}_\beta(x-\gamma) = \mathbf{0}.$$

Because x is an arbitrary real or complex number, the equality (8) holds. □

Remark 8. In case all of α, β, γ are integer numbers, we may regard $\mathbf{S}_{\alpha,\beta,\gamma}$ as a transform matrix of integer sequences. However, we see from (8) that, it is better to regard $\mathbf{S}_{\alpha,\beta,\gamma}$ as a composition of three successive transforms: $\mathbf{S}_{0,\beta,0}$, $\mathbf{S}_{0,0,\gamma}$ and $\mathbf{S}_{\alpha,0,0}$.

Remark 9. As we know that, a linear homogeneous recurrent integer sequence $a(n)$ of order q has the general-term with the following form: $a(n) = \sum_{i=1}^{q} c_i\lambda_i^n$, $(n = 0, 1, 2, \ldots)$, where λ_i, $(i = 1, \ldots, q)$ are its q real or complex characteristic values. We see from [5] that if γ is an integer, the q characteristic values of the γ-fold binomial transform $b(n)$ of integer sequence $a(n)$ are $\lambda_i + \gamma$, and the general-term of integer sequence $b(n)$ is $b(n) = \sum_{i=1}^{q} c_i(\lambda_i + \gamma)^n$. Now, we may also obtain this conclusion by using the basic relation (3). Denoting the vectors corresponding to integer sequences $a(n)$ and $b(n)$ by $\mathbf{a} = (a(0), a(1), \cdots, a(n), \cdots)^T$ and $\mathbf{b} = (b(0), b(1), \cdots, b(n), \cdots)^T$, we have $\mathbf{a} = \sum_{i=1}^{q} c_i\mathbf{v}_0(\lambda_i)$, and

$$\mathbf{b} = \mathbf{S}_{0,0,\gamma}\mathbf{a} = \mathbf{S}_{0,0,\gamma}\sum_{i=1}^{q} c_i\mathbf{v}_0(\lambda_i) = \sum_{i=1}^{q} c_i\mathbf{S}_{0,0,\gamma}\mathbf{v}_0(\lambda_i) = \sum_{i=1}^{q} c_i\mathbf{v}_0(\lambda_i + \gamma).$$

Before we mention the next theorem, let us consider first the following definition.

Definition 10. Let α be an integer, and $a(n)$ $(n = 0, 1, 2, \ldots)$ be a linear homogeneous recurrent integer sequence of order q. *The α-generalized Stirling transform of the first kind* of $a(n)$ is defined by

$$b(n) = \sum_{k=0}^{n} S(n, k; \alpha, 0, 0)a(k).$$

Now, let us give another property related to the recurrent integer sequences, as follows.

Theorem 11. *The α-generalized Stirling transform of the first kind of a linear homogeneous recurrent integer sequence*

$$a(n) = \sum_{i=1}^{q} c_i\lambda_i^n, \quad n = 0, 1, 2, \ldots,$$

is an integer sequence with the general-term

$$b(n) = \sum_{i=1}^{q} c_i(\lambda_i|\alpha)_n, \quad n = 0, 1, 2, \ldots,$$

where λ_i, $(i = 1, \ldots, q)$ are q real or complex characteristic values of $a(n)$.

Proof. Denoting the vectors corresponding to integer sequences $a(n)$ and $b(n)$ by $\mathbf{a} = (a(0), a(1), \cdots, a(n), \cdots)^T$ and $\mathbf{b} = (b(0), b(1), \cdots, b(n), \cdots)^T$, we have $\mathbf{a} = \sum_{i=1}^q c_i \mathbf{v}_0(\lambda_i)$, and

$$\mathbf{b} = \mathbf{S}_{\alpha,0,0}\mathbf{a} = \mathbf{S}_{\alpha,0,0}\sum_{i=1}^q c_i \mathbf{v}_0(\lambda_i) = \sum_{i=1}^q c_i \mathbf{S}_{\alpha,0,0}\mathbf{v}_0(\lambda_i) = \sum_{i=1}^q c_i \mathbf{v}_\alpha(\lambda_i),$$

that is, $b(n) = \sum_{i=1}^q c_i(\lambda_i|\alpha)_n$, $(n = 0, 1, 2, \ldots)$. At the same time, we see from (6) that if α is integer, each of entries of $\mathbf{S}_{\alpha,0,0}$ is also an integer, which implies that $b(n)$ is an integer sequences. □

Example 12. For example, the general-term of the recurrent integer sequence of order 2 of Lucas numbers $L(n) = 2, 1, 3, 4, 7, 11, 18, \ldots$ (A000032[6]) is $L(n) = c_1\lambda_1^n + c_2\lambda_2^n = (\frac{1}{2} + \frac{\sqrt{5}}{2})^n + (\frac{1}{2} - \frac{\sqrt{5}}{2})^n$. Hence, we find from Theorem 11 that the general-term of its Stirling transform of the first kind is

$$L_1(n) = (\frac{1}{2} + \frac{\sqrt{5}}{2}|1)_n + (\frac{1}{2} - \frac{\sqrt{5}}{2}|1)_n, \qquad n = 0, 1, 2, \ldots,$$

that is, $L_1(n) = 2, 1, 2, -3, 10, -45, 250, -1645, \ldots$ (A213593[6]).

3 Generalized Factorial Matrices

We may see that if taking $\mathbf{S}_{\alpha,\beta,\gamma}$ to be a p-dimensional $(p = 1, 2, 3, \ldots)$ matrix, which in fact is the $p \times p$ upper-left sub-matrix of the (*original*) infinite dimensional generalized Stirling matrix, then all of the conclusions presented in the preceding section still hold. We need to remember always this point of view while reading this section.

Definition 13. Let $x_1, x_2, \ldots, x_p$ be p distinct real or complex numbers, and α be a given real parameter. The *Generalized Factorial Matrices* of order p is a $p \times p$ matrix, denoted by $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p)$, whose ith column entries are the entries of the p-dimensional column vector $\mathbf{v}_\alpha(x_i)$ $(i = 1, \ldots, p)$. That is

$$\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p) = [\mathbf{v}_\alpha(x_1), \mathbf{v}_\alpha(x_2), \cdots, \mathbf{v}_\alpha(x_p)]. \tag{9}$$

In particular, $\mathbf{V}_0(x_1, x_2, \ldots, x_p)$ is just the Vandermonde matrix of p distinct parameters $x_1, x_2, \ldots, x_p$. Hence we may regard the generalized factorial matrices as a generalization in form of the Vandermonde matrices.

Example 14. For example, the 6×6 matrices $\mathbf{V}_\alpha(0, 1, 2, 3, 4, 5)$, $(\alpha = 0, 1, 2, 3)$ are, respectively,

$$\mathbf{V}_0(0, 1, 2, 3, 4, 5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & 1 & 32 & 243 & 1024 & 3125 \end{pmatrix},$$

$$\mathbf{V}_1(0,1,2,3,4,5)=\begin{pmatrix}1&1&1&1&1&1\\0&1&2&3&4&5\\0&0&2&6&12&20\\0&0&0&6&24&60\\0&0&0&0&24&120\\0&0&0&0&0&120\end{pmatrix},$$

$$\mathbf{V}_2(0,1,2,3,4,5)=\begin{pmatrix}1&1&1&1&1&1\\0&1&2&3&4&5\\0&-1&0&3&8&5\\0&3&0&-3&0&15\\0&-15&0&9&0&-15\\0&105&0&-45&0&45\end{pmatrix},$$

and

$$\mathbf{V}_3(0,1,2,3,4,5)=\begin{pmatrix}1&1&1&1&1&1\\0&1&2&3&4&5\\0&-2&-2&0&4&10\\0&10&8&0&-8&-10\\0&-80&-56&0&40&40\\0&880&560&0&-320&-280\end{pmatrix}.$$

The next theorem gives basic properties of the generalized factorial matrices.

Theorem 15. *Let $x_1, x_2, \ldots, x_p$ be p distinct real or complex numbers, and α,β,γ be three real parameters. Then the generalized factorial matrices have the following basic properties.*

(i) Two generalized factorial matrices can be connected with a suitable generalized Stirling matrice, generally like

$$\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p) = \mathbf{S}_{\alpha,\beta,\gamma}\mathbf{V}_\beta(x_1-\gamma, x_2-\gamma, \ldots, x_p-\gamma). \tag{10}$$

(ii) Determinant of the generalized factorial matrix $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p)$ is

$$\det \mathbf{V}_\alpha(x_1, x_2, \ldots, x_p) = \det \mathbf{V}_0(x_1, x_2, \ldots, x_p) = \prod_{1\le i<j\le p}(x_j - x_i). \tag{11}$$

This implies that matrix $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p)$ is invertible.

(iii) The inverse of the generalized factorial matrix $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p)$ is

$$\mathbf{V}_\alpha^{-1}(x_1, x_2, \ldots, x_p) = \mathbf{V}_0^{-1}(x_1, x_2, \ldots, x_p)\mathbf{S}_{0,\alpha,0}, \tag{12}$$

namely, the inverse of the Vandermonde matrix with the same parameters, right-multiplied by the α-generalized Stirling matrix of the second kind.

Proof. The matrix equality (10) is the matrix form of p equalities $\mathbf{v}_\alpha(x_i) = \mathbf{S}_{\alpha,\beta,\gamma}\mathbf{v}_\beta(x_i-\gamma)$, $(i = 1, 2, \ldots, p)$. From (10), we have $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p) = \mathbf{S}_{\alpha,0,0}\mathbf{V}_0(x_1, x_2, \ldots, x_p)$. Then, we obtain

$$\det \mathbf{V}_\alpha(x_1, x_2, \ldots, x_p) = \det \mathbf{S}_{\alpha,0,0} \det \mathbf{V}_0(x_1, x_2, \ldots, x_p).$$

Because $\det \mathbf{S}_{\alpha,0,0} = 1$ and $\det \mathbf{V}_0(x_1, x_2, \ldots, x_p) = \prod_{1\le i<j\le p}(x_j - x_i)$, the property (11) holds. Besides, we also see that

$$\mathbf{V}_\alpha^{-1}(x_1, \ldots, x_p) = \mathbf{V}_0^{-1}(x_1, \ldots, x_p)\mathbf{S}_{\alpha,0,0}^{-1} = \mathbf{V}_0^{-1}(x_1, \ldots, x_p)\mathbf{S}_{0,\alpha,0},$$

that is, equality (12) holds. □

Corollary 16. *Let $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p)$ be a generalized factorial matrices of p distinct real or complex numbers $x_1, x_2, \ldots, x_p$, with real parameter α. Then, we may express the inverse of matrix $\mathbf{V}_\alpha(x_1, x_2, \ldots, x_p)$ as*

$$\mathbf{V}_\alpha^{-1}(x_1, x_2, \ldots, x_p) = \mathbf{H}_p\mathbf{L}_p\mathbf{S}_{0,\alpha,0}. \tag{13}$$

where factor matrix $\mathbf{L}_p$ is a lower triangular matrix, the arbitrary entry $L_p(n,k)$ $(n, k = 0, 1, \ldots, p-1)$ of which is determined by the following linear relations

$$L_p(0,0) = 1, \quad \textit{and} \quad \prod_{i=1}^{n}(x - x_i) = \sum_{k=0}^{n} L_p(n,k)x^k, \tag{14}$$

and the matrix factor $\mathbf{H}_p$ is a upper triangular matrix, the arbitrary entry $h_p(n,k)$ $(n, k = 0, 1, \ldots, p-1)$ of which is given by

$$h_p(0,0) = 1, \quad h_p(n,k) = \begin{cases} 0, & \textit{if} \quad k < n, \\ \left(\prod_{i=1, i\ne n+1}^{k+1}(x_{n+1} - x_i)\right)^{-1}, & \textit{if} \quad k \ge n. \end{cases} \tag{15}$$

Proof. By using a triangular-matrix decomposition method given by Hou and Hou [7], we may obtain

$$\mathbf{V}_0^{-1}(x_1, x_2, \ldots, x_p) = \mathbf{H}_p\mathbf{L}_p,$$

where the lower triangular matrix $\mathbf{L}_p$ and the upper triangular matrix $\mathbf{H}_p$ are calculated by using (14) and (15). Hence, we see from (12) that the Corollary holds. □

Example 17. In case $x_1 = 0$, $x_2 = 1$, $x_3 = 2, \ldots,$ $x_p = p - 1$, we have $\mathbf{L}_p = \mathbf{S}_{1,0,0}$, and $\mathbf{H}_p = \mathbf{S}_{0,0,-1}^T\mathbf{D}_\mathbf{p}$, where matrix $\mathbf{D}_\mathbf{p}$ is a diagonal matrix, the kth diagonal entry of which is $\frac{1}{k!}$, $(k = 0, 1, \ldots, p-1)$. Hence,

$$\mathbf{V}_\alpha^{-1}(0, 1, \ldots, p-1) = \mathbf{H}_p\mathbf{S}_{1,0,0}\mathbf{S}_{0,\alpha,0} = \mathbf{S}_{0,0,-1}^T\mathbf{D}_\mathbf{p}\mathbf{S}_{1,0,0}\mathbf{S}_{0,\alpha,0}. \tag{16}$$

namely, $\mathbf{V}_\alpha^{-1}(0, 1, \ldots, p-1)$ is a product of several triangular matrices. For example, we may obtain the inverse matrices of the 6×6 generalized factorial matrix $\mathbf{V}_\alpha(0, 1, 2, 3, 4, 5)$ $(\alpha = 0, 1, 2, 3)$ (see Example 14), as follows

$$\mathbf{V}_0^{-1}(0,1,2,3,4,5) = \mathbf{H}_6\mathbf{S}_{1,0,0}\mathbf{S}_{0,0,0} = \mathbf{H}_6\mathbf{S}_{1,0,0} = \begin{pmatrix} 5 & -\frac{77}{12} & \frac{71}{24} & -\frac{7}{12} & \frac{1}{24} \\ -10 & \frac{107}{6} & -\frac{59}{6} & \frac{13}{6} & -\frac{1}{6} \\ 10 & -\frac{39}{2} & \frac{49}{4} & -3 & \frac{1}{4} \\ -5 & \frac{61}{6} & -\frac{41}{6} & \frac{11}{6} & -\frac{1}{6} \\ 1 & -\frac{25}{12} & \frac{35}{24} & -\frac{5}{12} & \frac{1}{24} \\ & & & & \end{pmatrix},$$

$$\mathbf{V}_1^{-1}(0,1,2,3,4,5) = \mathbf{H_6}\mathbf{S}_{1,0,0}\mathbf{S}_{0,1,0} = \mathbf{H_6} = \begin{pmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & -\frac{1}{120} \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{24} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{12} \\ 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{1}{24} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{120} \end{pmatrix},$$

$$\mathbf{V}_2^{-1}(0,1,2,3,4,5) = \mathbf{H_6}\mathbf{S}_{1,0,0}\mathbf{S}_{0,2,0} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{24} & -\frac{1}{24} & -\frac{1}{120} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{24} \\ 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{7}{12} & -\frac{1}{12} \\ 0 & 0 & 0 & \frac{5}{12} & \frac{2}{3} & \frac{1}{12} \\ 0 & 0 & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{12} & \frac{1}{120} \end{pmatrix},$$

and

$$\mathbf{V}_3^{-1}(0,1,2,3,4,5) = \mathbf{H_6}\mathbf{S}_{1,0,0}\mathbf{S}_{0,3,0} = \begin{pmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{8} & -\frac{1}{120} \\ 0 & 0 & \frac{1}{3} & \frac{11}{6} & \frac{2}{3} & \frac{1}{24} \\ 0 & 0 & -\frac{5}{6} & -\frac{25}{6} & -\frac{17}{12} & -\frac{1}{12} \\ 0 & \frac{1}{3} & 1 & \frac{29}{6} & \frac{3}{2} & \frac{1}{12} \\ 0 & 0 & -\frac{5}{6} & -\frac{17}{6} & -\frac{19}{24} & -\frac{1}{24} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{6} & \frac{1}{120} \end{pmatrix},$$

where matrices $\mathbf{S}_{0,2,0}$ and $\mathbf{S}_{0,3,0}$ are the 2- and 3-generalized Stirling matrices of the second kind, their arbitrary entries are $2^{n-k}S(n,k)$ and $3^{n-k}S(n,k)$ respectively.

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