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A Special Case of the Generalized Girard-Waring Formula

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Abstract

In this note we introduce a new method to proving and discovering some identities involving binomial coefficents and factorials.

1 Introduction.

Let n be a positive integer. Being given a set of variables $\{x_1, x_2, \ldots, x_n\}$, the kth elementary symmetric function $e_k(x_1, x_2, \ldots, x_n)$ on these variables is the sum of all possible products of k of these n variables, chosen without replacement

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k} \ ,$$

for $k = 1, 2, \ldots, n$. We set $e_0(x_1, x_2, \ldots, x_n) = 1$ by convention (a single choice of the empty product, if you like that kind of thing). For $k > n$ or $k < 0$, we set $e_k(x_1, x_2, \ldots, x_n) = 0$.

The starting point of this paper is the following result:

Theorem 1. *Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be n independent variables. Then*

$$e_k\left(x_1^2, \ldots, x_n^2\right) = \sum_{i=-k}^{k} (-1)^i e_{k+i}\left(x_1, \ldots, x_n\right) e_{k-i}\left(x_1, \ldots, x_n\right) \ . \tag{1}$$

Proof. Taking into account that

$$\prod_{i=1}^{n}(x+x_i)=\sum_{k=0}^{n}e_{n-k}\left(x_1,\ldots,x_n\right)x^k$$

and

$$e_k\left(-x_1,\ldots,-x_n\right)=(-1)^k e_k\left(x_1,\ldots,x_n\right)\ ,$$

we can write

$$\begin{aligned}\prod_{i=1}^{n}\left(x^2-x_i^2\right) &= \sum_{k=0}^{n}e_{n-k}\left(-x_1^2,\ldots,-x_n^2\right)x^{2k}\\ &= \sum_{k=0}^{n}(-1)^{n-k}e_{n-k}\left(x_1^2,\ldots,x_n^2\right)x^{2k}\ .\end{aligned}\tag{2}$$

On the other hand, we have

$$\begin{aligned}&\prod_{i=1}^{n}\left(x^2-x_i^2\right)=\\ &\quad=\left(\prod_{i=1}^{n}(x-x_i)\right)\left(\prod_{i=1}^{n}(x+x_i)\right)\\ &\quad=\left(\sum_{k=0}^{n}(-1)^{n-k}e_{n-k}\left(x_1,\ldots,x_n\right)x^k\right)\left(\sum_{k=0}^{n}e_{n-k}\left(x_1,\ldots,x_n\right)x^k\right)\\ &\quad=\sum_{k=0}^{n}\left(\sum_{i=0}^{2k}(-1)^{n-i}e_{n-i}\left(x_1,\ldots,x_n\right)e_{n-2k+i}\left(x_1,\ldots,x_n\right)\right)x^{2k}\ .\end{aligned}\tag{3}$$

By (2) and (3), we deduce the relation

$$(-1)^{n-k}e_{n-k}\left(x_1^2,\ldots,x_n^2\right)=\sum_{i=0}^{2k}(-1)^{n-i}e_{n-i}\left(x_1,\ldots,x_n\right)e_{n-2k+i}\left(x_1,\ldots,x_n\right)\ ,$$

that can be rewritten in the following way

$$\begin{aligned}(-1)^k e_k\left(x_1^2,\ldots,x_n^2\right) &= \sum_{i=0}^{2(n-k)}(-1)^{n-i}e_{n-i}\left(x_1,\ldots,x_n\right)e_{2k-n+i}\left(x_1,\ldots,x_n\right)\\ &= \sum_{i=k-n}^{n-k}(-1)^{k-i}e_{k-i}\left(x_1,\ldots,x_n\right)e_{k+i}\left(x_1,\ldots,x_n\right)\ .\end{aligned}$$

Since $e_k\left(x_1,\ldots,x_n\right)=0$ for $k<0$ or $k>n$, we have

$$\begin{aligned}&\sum_{i=k-n}^{n-k}(-1)^i e_{k-i}\left(x_1,\ldots,x_n\right)e_{k+i}\left(x_1,\ldots,x_n\right)\\ &\qquad=\sum_{i=-k}^{k}(-1)^i e_{k-i}\left(x_1,\ldots,x_n\right)e_{k+i}\left(x_1,\ldots,x_n\right)\ .\end{aligned}\tag{4}$$

The proof is finished. $\square$

It is well-known that the power sum symmetric functions can be expressed in terms of elementary symmetric functions using Girard-Waring formula [3, eq. 8]. In [4, 5, 8], the Girard-Waring formula is generalised to monomial symmetric functions with equal exponents. The relation (1) is the case $n = 2$ in the generalized Girard-Waring formula [8, Eq. (3)] and can be used to proving and discovering some identities. To illustrate this we present two applications involving binomial coefficients and Stirling numbers of the first kind.

2 Identities involving binomial coefficients

Let us consider the binomial coefficients

$$\binom{n}{k} = e_k(\underbrace{1,\ldots,1}_{n}) .$$

The following identity is a direct consequence of Theorem 1.

Corollary 1. *Let k and n be two nonnegative integers. Then*

$$\sum_{i=-k}^{k} (-1)^i \binom{n}{k+i}\binom{n}{k-i} = \binom{n}{k} .$$

Taking into account that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} ,$$

by Corollary 1, we obtain a new identity:

Corollary 2. *Let n be a positive integer. Then*

$$\sum_{0<i\leq k<n} (-1)^i \binom{n}{k+i}\binom{n}{k-i} = 2^{n-1} - \binom{2n-1}{n} .$$

This corollary is related in [7] with the sequences A108958. By Theorem 1, we obtain the following result which is a generalization of Corollary 1.

Corollary 3. *Let k and n be two positive integers, and let p be a real number. Then*

$$\sum_{i=-k}^{k} (-1)^i \left(1+\frac{(p-1)(k+i)}{n}\right)\left(1+\frac{(p-1)(k-i)}{n}\right)\binom{n}{k+i}\binom{n}{k-i}$$

$$= \left(1+\frac{(p^2-1)k}{n}\right)\binom{n}{k} .$$

Proof. Taking into account that

$$e_k(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_{n-1}) + x_n e_{k-1}(x_1, \ldots, x_{n-1}) \ ,$$

we can write

$$\begin{aligned} e_k(\underbrace{1, \ldots, 1}_{n-1}, p) &= \binom{n-1}{k} + p\binom{n-1}{k-1} \\ &= \binom{n}{k} + (p-1)\frac{k}{n}\binom{n}{k} \\ &= \left(1 + \frac{(p-1)k}{n}\right)\binom{n}{k} \ . \end{aligned}$$

According to Theorem 1, the corollary is proved. □

The following result is a consequence of Corollary 3.

Corollary 4. *Let k and n be two positive integers. Then*

$$\sum_{i=1}^{k} (-1)^{i+1} i^2 \binom{n}{k+i}\binom{n}{k-i} = \frac{k(n-k)}{2}\binom{n}{k} \ .$$

Proof. Replacing p by 2 in Corollary 3, we obtain

$$\begin{aligned} \left(1 + \frac{3k}{n}\right)\binom{n}{k} &= \sum_{i=-k}^{k} (-1)^i \left(1 + \frac{k-i}{n}\right)\left(1 + \frac{k+i}{n}\right)\binom{n}{k-i}\binom{n}{k+i} \\ &= \sum_{i=-k}^{k} (-1)^i \left(1 + \frac{2k}{n} + \frac{k^2 - i^2}{n^2}\right)\binom{n}{k-i}\binom{n}{k+i} \\ &= \left(1 + \frac{k}{n}\right)^2 \sum_{i=-k}^{k} (-1)^i \binom{n}{k-i}\binom{n}{k+i} \\ &\qquad - \left(\frac{1}{n}\right)^2 \sum_{i=-k}^{k} (-1)^i i^2 \binom{n}{k-i}\binom{n}{k+i} \end{aligned}$$

Now, we use Corollary 1 and, after some simple calculations, we obtain

$$\sum_{i=-k}^{k} (-1)^{i+1} i^2 \binom{n}{k-i}\binom{n}{k+i} = k(n-k)\binom{n}{k} \ .$$

The corollary is proved. □

Remark. To prove Corollary 4 we use Corollary 3 with $p = 2$. In fact, we could choose for p any value with the exception of 1. Corollary 4 is related in [7] with the sequence A094305.

Taking into account the identities

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \text{and} \qquad \sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2} ,$$

by Corollary 4, we get the following identity:

Corollary 5. *Let n be a nonnegative integer. Then*

$$\sum_{0<i\le k<n} (-1)^{i+1} i^2 \binom{n}{k+i}\binom{n}{k-i} = n(n-1)2^{n-3} .$$

This corollary is related in [7] with the sequence A001788.

At the end of this section we propose the following two exercises:

Exercise 1. *Let $x_1, x_2, \dots, x_n$ be the zeros of the polynomial*

$$x^n + \sum_{k=1}^{n} (-1)^k k \binom{n}{k} x^{n-k} .$$

Show that

$$e_k\left(x_1^2, x_2^2, \dots, x_n^2\right) = n^2\binom{n-1}{k-1} + (-1)^k 4k \binom{n}{2k} .$$

Exercise 2. *Let k and n be two positive integers. Prove that*

$$\sum_{i=1}^{k} (-1)^i i^4 \binom{n}{k+i}\binom{n}{k-i} = \frac{k(n-k)(k(n-k)-n)}{2}\binom{n}{k} .$$

3 Central factorial numbers of the first kind

The numbers

$$s(n+1, n+1-k) = (-1)^k e_k(1, 2, \dots, n) \tag{5}$$

are known as Stirling numbers of the first kind. They are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k ,$$

where $(x)_n$ is the falling factorial, namely

$$(x)_n = \prod_{k=0}^{n-1} (x-k)$$

(see [1, p. 278]).

Similarly, the central factorial numbers of the first kind are defined in Riordan's book [6, p. 213-217] by

$$x^{[n]} = \sum_{k=0}^{n} t(n,k) x^k \ ,$$

where

$$x^{[n]} = x \left(x + \frac{n}{2} - 1 \right)_{n-1} \ .$$

It is clearly that the $t(n,k)$ are not always integers. For $n = 2m$, we have

$$x^{[2m]} = \prod_{k=0}^{m-1} \left(x^2 - k^2 \right) = \sum_{k=0}^{m} t(2m, 2k) x^{2k} \ .$$

In [2] the central factorial numbers of the first kind with even indices are denoted by $u(n,k) = t(2n, 2k)$. Thus, we can see that

$$u(n+1, n+1-k) = (-1)^k e_k(1^2, 2^2, \ldots, n^2) \ . \tag{6}$$

Corollary 6. *Let k and n be two positive integers such that $k \leq n$. Then*

$$u(n,k) = \sum_{i=-k}^{k} (-1)^{n-k+i} s(n, k+i) s(n, k-i) \ .$$

Proof. By (1), (5) and (6), we deduce that

$$u(n, n-k) = \sum_{i=-k}^{k} (-1)^{k+i} s(n, n-k+i) s(n, n-k-i) \ .$$

According to (4), the corollary is proved. □

Corollary 6 is related in [7] to the sequences A008955, A000330, A000596, A000597, A001819, A001820, A001821 and A204579.

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