Journal of Integer Sequences, Vol. 15 (2012),
Article 12.8.8

Representation of Integers by Near Quadratic Sequences

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Abstract

Following a statement of the well-known Erdős-Turán conjecture, Erdős mentioned the following even stronger conjecture: if the n-th term a_n of a sequence A of positive integers is bounded by αn^2, for some positive real constant α, then the number of representations of n as a sum of two terms from A is an unbounded function of n. Here we show that if a_n differs from αn^2 (or from a quadratic polynomial with rational coefficients $q(n)$) by at most $o(\sqrt{\log n})$, then the number of representations function is indeed unbounded.

1 Introduction

In 1941, Erdős and Turán [5] conjectured that if a sequence $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ of positive integers is an asymptotic basis of the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, i.e., if all large enough integers n are sums of two terms from A, then the number of representations $r_A(n) = |\{(a_i, a_j) \in A \times A : a_i + a_j = n\}|$ of n, as a sum of two terms from A, is unbounded. This is the well-known "Erdős-Turán conjecture". A few years later (the earliest we are aware of), in 1955 and 1956, Erdős [6], and Erdős and Fuchs [7] asserted that an even stronger conjecture would be that if $a_n \leq \alpha n^2$, for all n, with a real constant $\alpha > 0$, then $\limsup r_A(n) = \infty$. This came to be known as the "generalized Erdős-Turán conjecture". It is indeed stronger than the former one, since if A is an asymptotic basis of $\mathbb{N}$, then $a_n \ll n^2$ [13, p. 105].

Much work has been done concerning the "Erdős-Turán conjecture", e.g., [3, 7, 8, 16, 1, 21, 19], including disproofs of analogues of this conjecture in many semigroups other than $\mathbb{N}$, e.g., [20, 16, 17, 11, 12, 2, 14]. In contrast, much less has been done about the "generalized Erdős-Turán conjecture". In a previous, co-authored, paper [9], we studied the class of sequences that can replace $\{\alpha n^2\}$ in the condition $a_n \leq \alpha n^2$ for all n, to imply that $r_A(n)$ is unbounded, and we gave several statements equivalent to the "generalized Erdős-Turán conjecture". In particular, we showed that if the conjecture holds with $\alpha = 1$, then it holds with any $\alpha > 0$. Moreover, it is not difficult to see that if $a_n = o(n^2)$, then the conjecture holds [9, 10]. So we can essentially focus on the case where a_n is not too small compared to n^2, while bounded by a constant multiple of n^2. In particular, we can consider the case where a_n is, in a sense, "close" to a constant multiple of n^2, or to a quadratic polynomial in n. This is basically the goal of the present paper. We thus show that if $|a_n - \alpha n^2| = o\left(\sqrt{\log n}\right)$, with a real constant $\alpha > 0$, or if $|a_n - q(n)| = o(\sqrt{\log n})$, where $q(n)$ is a quadratic polynomial with rational coefficients, then the representation function $r_A(n)$ of A is unbounded.

2 Technical tools

Let $C = \{c_1 < c_2 < \cdots < c_n < \cdots\} \subset \mathbb{R}^+$ be a strictly increasing sequence, in the set $\mathbb{R}^+$ of real numbers ≥ 0. For any $x \in \mathbb{R}^+$, let $C[x] = C \cap [0, x] = \{c \in C : c \leq x\}$, and $C(x) = |C[x]|$ the cardinality of $C[x]$. Note that $C(x)$ is finite for every $x \geq 0$ if and only if the sequence C is unbounded. This is in particular true when $c_{n+1} - c_n \geq 1$ for large enough n, and more particularly if C is a subset of the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers.

The sumset $C + C$ is defined by $C + C = \{c + d : (c, d) \in C \times C\}$.

Now let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}$ be a strictly increasing sequence of natural numbers. In addition to the above notions, valid for A as for C, the representation function r_A of A is defined by $r_A(n) = |\{(a, b) \in A \times A : a + b = n\}|$, for $n \in \mathbb{N}$, and we set $s(A) = \sup_{n \in \mathbb{N}} r_A(n)$, in $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

In the sequel, i, j, k, l, m, n generally denote positive integers, unless it is specified that they lie in $\mathbb{N}$, i.e., that they are integers ≥ 0, while x, y denote real numbers ≥ 0, i.e., they lie in $\mathbb{R}^+$.

Note that if $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}^*$, where $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ is the set of positive integers, then $a_n \geq n$ for all $n \in \mathbb{N}^*$.

For any $x \in \mathbb{R}^+$, let

$$U_A(x) = |\{(a, b) \in A \times A : a + b \leq x\}| = \sum_{0 \leq n \leq x} r_A(n). \tag{1}$$

Then

$$U_A(x) = \sum_{n \in (A+A)[x]} r_A(n) \leq \sum_{n \in (A+A)[x]} s(A) = (A+A)(x) \cdot s(A) \tag{2}$$

and

$$\begin{aligned} A(x)^2 &= |\{(a,b) \in A \times A : a, b \leq x\}| \leq |\{(a,b) \in A \times A : a+b \leq 2x\}| = U_A(2x) \leq \\ &\leq (A+A)(2x) \cdot s(A), \end{aligned} \tag{3}$$

so that, for all $x \in \mathbb{R}^+$,

$$\frac{(A+A)(2x)}{A(x)^2} s(A) \geq 1. \tag{4}$$

Define

$$h(A) = \liminf_{x \to \infty} \frac{(A+A)(2x)}{A(x)^2}. \tag{5}$$

Lemma 1. *If $h(A) = 0$, then $s(A) = \infty$.*

Proof. This follows immediately from (4). □

Corollary 2. *If $\liminf_{n \to \infty} \frac{A(x)}{\sqrt{x}} > 0$ and $\liminf_{n \to \infty} \frac{(A+A)(x)}{x} = 0$, then $h(A) = 0$, and therefore $s(A) = \infty$.*

Proof. By assumption, $\limsup_{n \to \infty} \frac{\sqrt{x}}{A(x)} = \frac{1}{\liminf_{n \to \infty} \frac{A(x)}{\sqrt{x}}}$ is finite, while $\liminf_{n \to \infty} \frac{(A+A)(2x)}{2x} = 0$. So, using properties of the lower and upper limits, we get

$$\begin{aligned} h(A) &= \liminf_{x \to \infty} \frac{(A+A)(2x)}{A(x)^2} = 2 \liminf_{x \to \infty} \frac{(A+A)(2x)}{2x} \left(\frac{\sqrt{x}}{A(x)} \right)^2 \leq \\ &\leq 2 \left(\liminf_{x \to \infty} \frac{(A+A)(2x)}{2x} \right) \cdot \left(\limsup_{x \to \infty} \frac{\sqrt{x}}{A(x)} \right)^2 = 0. \end{aligned}$$

The conclusion follows from Lemma 2.1. □

Lemma 3. *Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}^*$ be a strictly increasing sequence of positive integers, and $C = \{c_1 < c_2 < \cdots < c_n < \cdots\} \subset \mathbb{R}^+$. For $x \in \mathbb{R}^+$, set $e(x) = \sup_{n \leq x} |a_n - c_n|$. We then have, for all $x \in \mathbb{R}^+$,*

$$(A+A)(x) \leq (4e(x) + 1) \cdot (C+C)(x + 2e(x)). \tag{6}$$

If we further assume that $c_1 \geq 1$ and $c_{n+1} - c_n \geq 1$ for all $n \geq 1$, we then also have, for all $x \in \mathbb{R}^+$,

$$A(x) \geq C(x - e(x)). \tag{7}$$

Proof. Note first that the function $e(x)$ is increasing, in the sense that $x \le y$ implies $e(x) \le e(y)$.

Note also that, since $A \subset \mathbb{N}^*$, we have $i \le a_i$ for all i. So, for $n \le x$, if $n = a_i + a_j$, then $i \le a_i \le n \le x$ and similarly $j \le x$, and therefore $|n - c_i - c_j| = |a_i + a_j - c_i - c_j| \le |a_i - c_i| + |a_j - c_j| \le 2e(x)$. Hence

$$(A + A)\,[x] = \{n \le x : \exists i, j,\ n = a_i + a_j\} \subset \{n \le x : \exists i, j,\ |n - c_i - c_j| \le 2e(x)\},$$

and setting $s = c_i + c_j$, we get $s \in C + C$ and $|n - s| \le 2e(x)$, so that $s \le n + 2e(x) \le x + 2e(x)$, and therefore

$$\{n \le x : \exists i, j, |n - c_i - c_j| \le 2e(x)\} \subset \{n : \exists s \in (C + C)[x + 2e(x)]), |n - s| \le 2e(x)\}.$$

Thus

$$(A + A)\,[x] \subset \bigcup_{s \in (C+C)[x+2e(x)]} \left([s - 2e(x),\ s + 2e(x)] \cap \mathbb{N}\right),$$

and therefore

$$(A + A)\,(x) \le \sum_{n \in (C+C)[x+2e(x)]} (4e(x) + 1) = (C + C)\,(x + 2e(x)) \cdot (4e(x) + 1)\,.$$

This proves (6).

Now, if $c_1 \ge 1$ and $c_{n+1} - c_n \ge 1$ for all n, then $c_n \ge n$ for all n. So if $c_n \le x - e(x)$, then $n \le c_n \le x$, so that $|a_n - c_n| \le e(x)$, and therefore $a_n \le c_n + e(x) \le x$.

Hence $\{n : c_n \le x - e(x)\} \subset \{n : a_n \le x\}$, and thus

$$C\,(x - e(x)) = |\{n : c_n \le x - e(x)\}| \le |\{n : a_n \le x\}| = A(x),$$

which proves (7). $\square$

Lemma 4. *Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}^*$ and $C = \{c_1 < c_2 < \cdots < c_n < \cdots\} \subset \mathbb{R}^+$ be two strictly increasing sequences in $\mathbb{N}^*$ and in $\mathbb{R}^+$, respectively. For $x \in \mathbb{R}^+$, set $e(x) = \sup_{n \le x} |a_n - c_n|$. Assume that $e(x)$ is not identically zero, and that $c_1 \ge 1$ and $c_{n+1} - c_n \ge 1$ for all $n \ge 1$. Then the condition*

$$\liminf_{x \to \infty} \frac{e\,(2x) \cdot (C + C)\,(2x + 2e\,(2x))}{C(x - e(x))^2} = 0 \tag{H}$$

implies that $h\,(A) = 0$, and therefore $s\,(A) = \infty$.

Proof. Since $e(x)$ is increasing and not identically zero, there exists a real constant $t > 0$ such that $e(x) \ge \dfrac{1}{t}$ for large enough x. In view of the inequalities (6) and (7) in Lemma 2.3, we have

$$\frac{(A + A)\,(2x)}{A\,(x)^2} \le \frac{(4e\,(2x) + 1) \cdot (C + C)\,(2x + 2e\,(2x))}{C\,(x - e(x))^2}.$$

Moreover, for large enough x, we have $t \cdot e(2x) \geq 1$, and therefore $4e\,(2x)+1 \leq (4+t)\cdot e\,(2x)$. Thus

$$\frac{(A+A)\,(2x)}{A\,(x)^2} \leq (4+t)\,\frac{e\,(2x)\cdot(C+C)\,(2x+2e\,(2x))}{C(x-e(x))^2},$$

for large enough x, so that the condition (H) implies that $\liminf_{x\to\infty}\dfrac{(A+A)\,(2x)}{A\,(x)^2} = 0$, i.e., $h\,(A)=0$, and therfore, by Lemma 2.1, $s\,(A)=\infty$. ☐

Remark 5. The scope of Lemma 2.4 is broader than it seems to be. Indeed, for a subset A of $\mathbb{N}$, modifying, removing or adding finitely many elements does not modify the fact that $s(A)$ is infinite or finite. Thus Lemma 2.4 can be used in more general situations than specified by its assumptions, as shown by the next result.

Fundamental Lemma 6. *Let $B=\{b_1<b_2<\cdots<b_n<\cdots\}\subset\mathbb{N}$ and $D=\{d_1<d_2<\cdots<d_n<\cdots\}\subset\mathbb{R}^+$ be two strictly increasing sequences in $\mathbb{N}$ and in $\mathbb{R}^+$ respectively. Assume that there exists an increasing function $f:\mathbb{R}^+\to\mathbb{R}^+$ and a positive integer m such that $d_m\geq 1$, $d_{n+1}-d_n\geq 1$ for $n\geq m$, and $\sup_{m\leq n\leq x}|b_n-d_n|\leq f(x)$ for $x\geq m$. Then the condition*

$$\liminf_{x\to\infty}\frac{f\,(2x)\cdot(D+D)\,(2x+2f\,(2x))}{D(x-f(x))^2}=0 \tag{K}$$

implies that $s\,(B)=\infty$.

Proof. For $n\in\mathbb{N}^*$, set $a_n=b_{n+m}$ and $c_n=d_{n+m}$, and let $A=\{a_1<a_2<\cdots<a_n<\cdots\}\subset\mathbb{N}^*$ and $C=\{c_1<c_2<\cdots<c_n<\cdots\}\subset\mathbb{R}^+$ be the strictly increasing sequences, in $\mathbb{N}^*$ and $\mathbb{R}^+$, obtained by deleting the first m terms of B and D respectively. Then $c_1=d_{m+1}\geq 2$ and $c_{n+1}-c_n=d_{n+m+1}-d_{n+m}\geq 1$ for $n\geq 1$. Moreover, setting $e\,(x)=\sup_{n\leq x}|a_n-c_n|$, for $x\in\mathbb{R}^+$, and using the assumptions on B and D, we have

$$e(x)=\sup_{n\leq x}|a_n-c_n|=\sup_{n\leq x}|b_{n+m}-d_{n+m}|=\sup_{m<i\leq x+m}|b_i-d_i|\leq f(x+m).$$

Thus, setting $y=x+m$, we have $e(x)\leq f(y)$, and since the functions e and f are increasing,

$$e\,(2x)\leq f\,(2x+m)\leq f\,(2y)\,.$$

Also, taking into account that $C\subset D$ and $C+C\subset D+D$, so that $(C+C)\,(t)\leq(D+D)\,(t)$ for all $t\in\mathbb{R}^+$, and that the function $t\mapsto(C+C)(t)$ is increasing, we get

$$(C+C)\,(2x+2e\,(2x))\leq(C+C)\,(2y+2f\,(2y))\leq(D+D)\,(2y+2f\,(2y))\,.$$

Thus

$$e\,(2x)\cdot(C+C)\,(2x+2e\,(2x))\leq f\,(2y)\cdot(D+D)\,(2y+2f\,(2y))\,, \tag{8}$$

for $x\in\mathbb{R}^+$, and $y=x+m$.

Moreover, for $t \geq m$, we have

$$D(t) - C(t) = |\{d_n \in D : d_n \leq t\}| - |\{c_n \in C : c_n = d_{n+m} \leq t\}| = m$$

and

$$C(t) - C(t-m) = |\{c_n \in C : t - m < c_n \leq t\}| \leq m,$$

since $c_{n+1} - c_n \geq 1$ for all $n \in \mathbb{N}^*$, so that $C(t) \leq C(t-m) + m$ and $D(t) = C(t) + m \leq C(t-m) + 2m$. Therefore $C(t-m) \geq D(t) - 2m$ for $t \geq m$. Hence, taking into account that the function $t \mapsto C(t)$ is increasing and that $e(x) \leq f(y)$ we get, for large enough x,

$$C(x - e(x)) \geq C(x - f(y)) = C(y - m - f(y)) \geq D(y - f(y)) - 2m. \tag{9}$$

It follows from (8) and (9) that, for large enough x and for $y = x + m$,

$$\frac{e(2x) \cdot (C + C)(2x + 2e(2x))}{C(x - e(x))^2} \leq \frac{f(2y) \cdot (D + D)(2y + 2f(2y))}{(D(y - f(y)) - 2m)^2}. \tag{10}$$

Set $P(x) = f(2x) \cdot (D + D)(2x + 2f(2x))$ and $Q(x) = D(x - f(x))$, and suppose that the condition (K) is satisfied, i.e., that $\liminf_{x \to \infty} \frac{P(x)}{Q(x)^2} = 0$. Then there exists a strictly increasing sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^+$, tending to infinity, such that $\lim_{n \to \infty} \frac{P(x_n)}{Q(x_n)^2} = 0$. Since $P(x)$ is an increasing unbounded function, $\lim_{n \to \infty} P(x_n) = \infty$, and therefore the sequence $(Q(x_n))_{n \geq 1}$ is unbounded. So there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}^*}$ of $(x_n)_{n \geq 1}$ such that $\lim_{k \to \infty} Q(x_{n_k}) = \infty$, while $\lim_{k \to \infty} \frac{P(x_{n_k})}{Q(x_{n_k})^2} = 0$. Hence $\lim_{k \to \infty} \frac{P(x_{n_k})}{(Q(x_{n_k}) - 2m)^2} = 0$, and therefore

$$\liminf_{y \to \infty} \frac{f(2y) \cdot (D + D)(2y + 2f(2y))}{(D(y - f(y)) - 2m)^2} = \liminf_{x \to \infty} \frac{P(x)}{(Q(x) - 2m)^2} = 0.$$

It then follows from (10) that $\liminf_{x \to \infty} \frac{e(2x) \cdot (C + C)(2x + 2e(2x))}{C(x - e(x))^2} = 0$. Thus the condition (H) of Lemma 2.4 holds, and therefore, in view of this Lemma, $s(A) = \infty$. As $A \subset B$, it follows that $s(B) = \infty$ too. □

Remark 7. In the statement of Lemma 2.6, we may replace D by $D' = D + \gamma$, i.e., d_n by $d'_n = d_n + \gamma$ ($n \in \mathbb{N}^*$), where γ is any fixed real number, since a translation of the general term of D does not affect the condition (K).

3 Main results

Theorem 8. *Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}$ be a strictly increasing sequence of natural numbers, and $q(x) = \alpha x^2$ with a real number $\alpha > 0$. If the function $e(x) = \sup_{n \leq x} |a_n - q(n)|$ ($x \in \mathbb{R}^+$) satisfies $e(x) = o\left(\sqrt{\log x}\right)$ as $x \to \infty$, then $s(A) = \infty$.*

Proof. We apply Lemma 2.6 to $B = A$ and $D = \{q(1) < q(2) < \cdots < q(n) < \cdots\}$. Indeed, the sequence $(q(n))_{n\geq 1}$ is strictly increasing and unbounded, with $q(n+1) - q(n) = \alpha\,(2n+1)$ unbounded too, so that $q(n) \geq 1$ and $q(n+1) - q(n) \geq 1$ for large enough n. There remains to show that the condition (K) holds for $f(x) = e(x)$.

Let $S = \{n^2 : n \in \mathbb{N}^*\}$. By a classical result of Landau [15], there exists a constant $c > 0$ such that $(S+S)(x) \sim c\dfrac{x}{\sqrt{\log x}}$ as $x \to \infty$.

For $m, n \in \mathbb{N}^*$ and $x \in \mathbb{R}^+$, as $q(m) + q(n) \leq x$ is equivalent to $m^2 + n^2 \leq \dfrac{x}{\alpha}$, we have $(D+D)(x) = (S+S)\left(\dfrac{x}{\alpha}\right) \sim \dfrac{c}{\alpha}\dfrac{x}{\sqrt{\log x}}$, so that

$$(D+D)(x) \leq c_1 \frac{x}{\sqrt{\log x}},$$

for large enough x, with a constant $c_1 > \dfrac{c}{\alpha}$.

Moreover, as $q(n) \leq x$ if and only if $n \leq \sqrt{\dfrac{x}{\alpha}}$, we also have $D(x) = \left[\sqrt{\dfrac{x}{\alpha}}\right] > \sqrt{\dfrac{x}{\alpha}} - 1$. It follows that, for large enough x,

$$\begin{aligned}\frac{e\,(2x)\cdot(D+D)\,(2x+2e\,(2x))}{D(x-e(x))^2} &\leq \frac{c_1\cdot e\,(2x)\cdot(2x+2e\,(2x))}{\sqrt{\log\,(2x+2e\,(2x))}\left(\sqrt{\frac{x-e(x)}{\alpha}}-1\right)^2} = \\ &= \frac{c_1\alpha\cdot e\,(2x)\cdot(2x+2e\,(2x))}{\sqrt{\log\,(2x+2e\,(2x))}\left(\sqrt{x-e(x)}-\sqrt{\alpha}\right)^2}.\end{aligned}$$

As $e(x) = o\left(\sqrt{\log x}\right)$,

$$\frac{e\,(2x)\cdot(2x+2e\,(2x))}{\sqrt{\log\,(2x+2e\,(2x))}\left(\sqrt{x-e(x)}-\sqrt{\alpha}\right)^2} \sim \frac{2x\cdot e\,(2x)}{\sqrt{\log\,(2x)}\cdot x} \sim \frac{2e(2x)}{\sqrt{\log\,(2x)}},$$

and, since $e\,(x) = o\left(\sqrt{\log x}\right)$, we have $\lim_{x\to\infty}\dfrac{2e(2x)}{\sqrt{\log\,(2x)}} = 0$. Therefore

$$\lim_{x\to\infty}\frac{e\,(2x)\cdot(D+D)\,(2x+2e\,(2x))}{D(x-e(x))^2} = 0,$$

and the condition (K) holds. Thus, by Lemma 2.6, $s\,(B) = \infty$, i.e., $s\,(A) = \infty$. $\square$

Remark 9. In the statement of Theorem 3.1, we may replace $q(x) = \alpha x^2$ by $q(x) = \alpha x^2 + \gamma$, where γ is any real constant, in view of Remark 2.7.

Also, if $A = \{a_n = [\alpha n^2 + \gamma] : n \in \mathbb{N}\}$ is the set of the integral parts $[\alpha n^2 + \gamma] = [q(n)]$, then $s(A) = \infty$, since $e(x) = \sup\limits_{n\leq x}|a_n - q(n)| \leq 1$ trivially satisfies the condition in Theorem 3.1.

Theorem 10. *Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}$ and $q(x)$ be a quadratic polynomial with rational coefficients and positive leading coefficient. If the function $e(x) = \sup_{n \le x} |a_n - q(n)|$ ($x \in \mathbb{R}^+$) satisfies $e(x) = o\left(\sqrt{\log x}\right)$ as $x \to \infty$, then $s(A) = \infty$.*

Proof. As $q(x)$ has rational coefficients, there exist integers a, b, c, d, with $a, d > 0$, such that $dq(x) = (ax + b)^2 + c$.

Let $b_n = da_n - c$ and $d_n = (an + b)^2$, for $n \in \mathbb{N}^*$. Clearly, there exists $m \in \mathbb{N}^*$ such that $b_m \ge 1$, $d_m \ge 1$ and $d_{n+1} - d_n \ge 1$ for $n \ge m$. Set $B = \{b_n : n \ge m\}$ and $D = \{d_n : n \ge m\}$. Then B and D are strictly increasing sequences in $\mathbb{N}$, and, for all $n \ge m$,

$$|d_n - b_n| = |(an + b)^2 - da_n + c| = d|q(n) - a_n|.$$

For $x > m$, Let $f(x) = \sup_{m \le n \le x} |d_n - b_n|$, for $x \in \mathbb{R}^+$. Then $f(x)$ is an increasing nonnegative function satisfying $f(x) \le d \cdot e(x)$, so that $f(x) = o\left(\sqrt{\log x}\right)$ (like $e(x)$). Thus, we may apply Lemma 2.6, provided we show that the condition (K) is satisfied.

Let $S = \{n^2 : n \in \mathbb{N}\}$. Then $D \subset S$, and therefore $D + D \subset S + S$, so that $(D + D)(x) \le (S + S)(x)$, for $x \in \mathbb{R}^+$.

By Landau's theorem [15], $(S + S)(x) \sim c_0 \dfrac{x}{\sqrt{\log x}}$, with a constant $c_0 > 0$. So there exists a constant $c_1 > 0$ such that $(D + D)(x) \le (S + S)(x) \le c_1 \dfrac{x}{\sqrt{\log x}}$, and therefore

$$(D + D)(2x + 2f(2x)) \le c_1 \frac{2x + 2f(2x)}{\sqrt{\log (2x + 2f(2x))}}. \tag{11}$$

Moreover, for $x > \max(m, b^2)$, if $n \le \frac{\sqrt{x} - |b|}{a}$, then $d_n = (an + b)^2 \le x$. Hence, for large enough x,

$$\begin{aligned} D(x) = |\{n \ge m : d_n \le x\}| &\ge \left| \left\{ n \ge m : n \le \frac{\sqrt{x} - |b|}{a} \right\} \right| \\ &\ge \frac{\sqrt{x} - |b|}{a} - m \ge c_2\sqrt{x} - c_3, \end{aligned}$$

with constants $c_2, c_3 > 0$, and therefore

$$D(x - f(x)) \ge c_2\sqrt{x - f(x)} - c_3. \tag{12}$$

It follows from (11) and (12) that, for large enough x,

$$\frac{f(2x) \cdot (D + D)(2x + 2f(2x))}{D(x - f(x))^2} \le c_1 \frac{f(2x) \cdot (2x + 2f(2x))}{\sqrt{\log (2x + 2f(2x))} \left(c_2\sqrt{x - f(x)} - c_3\right)^2},$$

and, since $f(x) = o\left(\sqrt{\log x}\right)$, we have

$$\frac{f(2x) \cdot (2x + 2f(2x))}{\sqrt{\log (2x + 2f(2x))} \left(c_2\sqrt{x - f(x)} - c_3\right)^2} \sim \frac{2f(2x)}{c_2^2\sqrt{\log x}} = o(1).$$

Therefore

$$\liminf_{x\to\infty}\frac{f\left(2x\right)\cdot\left(D+D\right)\left(2x+2f\left(2x\right)\right)}{D\left(x-f(x)\right)^{2}}=0.$$

Thus the condition (K) is satisfied, and by Lemma 2.6, $s(B)=\infty$. As B is a translate of a homothetic of a subsequence $A_m=\{a_n : n\geq m\}$ of A, namely $B=d\cdot A_m+|c|$, we conclude, e.g., see [9], that $s(A_m)=s(B)=\infty$, and therefore $s\left(A\right)=\infty$. □

4 Acknowledgment

We thank an anonymous reader who suggested the use of Landau's theorem to improve a previous result.

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2010 *Mathematics Subject Classification*: Primary 11B34; Secondary 11B83.
Keywords: sequences, representation functions, quadratic, Erdős-Turán conjecture.

Received July 19 2012; revised version received October 14 2012. Published in *Journal of Integer Sequences*, October 23 2012.

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