Journal of Integer Sequences, Vol. 14 (2011),
Article 11.8.4

Another Proof of Nathanson's Theorems

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Abstract

In this paper, without using generating functions, we give new combinatorial proofs of several theorems by Nathanson on the representation functions, and we also obtain generalizations of these theorems.

1 Introduction

Let $\mathcal{A}$ be a set of nonnegative integers. Let $r_h^{\mathcal{A}}(n)$ denote the number of representations of n as a sum of h elements of $\mathcal{A}$ and $r^{\mathcal{A}}(n)$ denote the number of representations of n as a sum of an arbitrary number of elements of $\mathcal{A}$, where representations differing only in the arrangement of their summands are counted separately. We notice that if $0 \notin \mathcal{A}$, then $r^{\mathcal{A}}(n) = \sum_{h=1}^{\infty} r_h^{\mathcal{A}}(n)$ is finite for all n. Representation functions have been extensively studied by many authors [1, 2, 3, 5, 6] and are still a fruitful area of research in additive number theory. Using generating functions, Nathanson [4] proved the following results.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets of nonnegative integers, and let $r_h^{\mathcal{A}}(n)$ and $r_h^{\mathcal{B}}(n)$ denote the number of representations of n as a sum of h elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. If $r_h^{\mathcal{A}}(n) = r_h^{\mathcal{B}}(n)$ for all $n \geqslant 0$, then $\mathcal{A} = \mathcal{B}$.*

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets of positive integers, and let $r^{\mathcal{A}}(n)$ and $r^{\mathcal{B}}(n)$ denote the number of representations of n as a sum of an arbitrary number of elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. If $r^{\mathcal{A}}(n) = r^{\mathcal{B}}(n)$ for all $n \geqslant 1$, then $\mathcal{A} = \mathcal{B}$.*

Theorem 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be sets of positive integers, and let $p^{\mathcal{A}}(n)$ and $p^{\mathcal{B}}(n)$ denote the number of representations of n as a sum of an arbitrary number of elements of $\mathcal{A}$ and $\mathcal{B}$, respectively, where representations differing only in the arrangement of their summands are not counted separately. If $p^{\mathcal{A}}(n) = p^{\mathcal{B}}(n)$ for all $n \geqslant 1$, then $\mathcal{A} = \mathcal{B}$.*

In this paper, we give new proofs of theorems above. Indeed, we shall prove slightly more. We first introduce some notation. If $\mathcal{A}$ is a strictly increasing sequence of integers, then a_n denotes the nth element of $\mathcal{A}$. Let $\mathcal{A}$ be a set of nonnegative integers and $\mathcal{H}$ be a set of positive integers. If $|\mathcal{H}|$ is finite, then $r_{\mathcal{H}}^{\mathcal{A}}(n)$ denotes the number of representations of n as a sum of h_1 or h_2 or $\ldots$ elements of $\mathcal{A}$; if $|\mathcal{H}|$ is infinite, then $r_{\mathcal{H}}^{\mathcal{A}}(n)$ denotes the number of representations of n as a sum of h_1 or h_2 or $\ldots$ elements of $\mathcal{A}\backslash\{0\}$.

Theorem 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty sets of nonnegative integers. Let $\mathcal{H}$ be a nonempty set of positive integers, and let $S = \{\min\{a_i, b_i\} : i = 1, 2, \ldots\}$. Write $t = (\min(\mathcal{H}) - 1)\min\{a_1, b_1\}$. If $r_{\mathcal{H}}^{\mathcal{A}}(n) = r_{\mathcal{H}}^{\mathcal{B}}(n)$ for all $n \in t + S$, then $\mathcal{A} = \mathcal{B}$.*

Let $\mathcal{H} = \{h\}$. Since $t + S \subseteq \{0, 1, 2, \ldots\}$, Theorem 4 is a generalization of Theorem 1. Let $\mathcal{H} = \{1, 2, 3, \ldots\}$. If $\mathcal{A}$ and $\mathcal{B}$ are sets of positive integers, then $t+S \subseteq \{1, 2, \ldots\}$. Hence, Theorem 4 is also a generalization of Theorem 2.

Theorem 5. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{H}$ be nonempty sets of positive integers. Let $S = \{\min\{a_i, b_i\} : i = 1, 2, \ldots\}$, and let $p_{\mathcal{H}}^{\mathcal{A}}(n)$ denote the number of representations of n as a sum of h_1 or h_2 or $\ldots$ elements of $\mathcal{A}$, where representations differing only in the arrangement of their summands are not counted separately. Write $t = (\min(\mathcal{H}) - 1)\min\{a_1, b_1\}$. If $p_{\mathcal{H}}^{\mathcal{A}}(n) = p_{\mathcal{H}}^{\mathcal{B}}(n)$ for all $n \in t + S$, then $\mathcal{A} = \mathcal{B}$.*

Let $\mathcal{H} = \{1, 2, 3, \ldots\}$. Since $t+S \subseteq \{1, 2, \ldots\}$, Theorem 5 is a generalization of Theorem 3.

Let A, B, and T be finite sets of integers. If each residue class modulo m contains exactly the same number of elements of A as elements of B, then we write $A \equiv B \pmod{m}$. If the number of solutions of the congruence $a + t \equiv n \pmod{m}$ with $a \in A$, $t \in T$, equals the number of solutions of the congruence $b+t \equiv n \pmod{m}$ with $b \in B$, $t \in T$, for each residue class n modulo m, then we write $A + T \equiv B + T \pmod{m}$. Nathanson [4] also proved the following theorem.

Theorem 6. *Let $\mathcal{A}$ and $\mathcal{B}$ be distinct nonempty sets of nonnegative integers such that $r_2^{\mathcal{A}}(n) = r_2^{\mathcal{B}}(n)$ for all sufficiently large n. Then there exist finite sets A, B, and T with $A \cup B \subset \{0, 1, \ldots, N\}$ and $T \subset \{0, 1, \ldots, m-1\}$ such that $A + T \equiv B + T \pmod{m}$, and $\mathcal{A} = A \cup\ \mathcal{C}$ and $\mathcal{B} = B \cup\ \mathcal{C}$, where $\mathcal{C} = \{c > N |\ c \equiv t \pmod{m}$ for some $t \in T\}$.*

In this paper, we prove theorems above without using generating functions. We notice that for a prime number p, if $\mathcal{A}$ and $\mathcal{B}$ are sets of nonnegative integers such that $r_p^{\mathcal{A}}(n) = r_p^{\mathcal{B}}(n)$ for all sufficiently large n, then $\mathcal{A}$ and $\mathcal{B}$ eventually coincide. Now, I pose the following problem.

Problem 7. *Let $p \geqslant 3$ be a prime number and $\mathcal{A}$ be a set of nonnegative integers. Does there exist a set of nonnegative integers $\mathcal{B}$ with $\mathcal{B} \neq \mathcal{A}$ such that $r_p^{\mathcal{A}}(n) = r_p^{\mathcal{B}}(n)$ for all sufficiently large n?*

2 Proof of Theorems 4 and 5

Suppose that $\mathcal{A} \neq \mathcal{B}$. Let $h = \min(\mathcal{H})$ and j_0 be the smallest index such that $a_{j_0} \neq b_{j_0}$. Without loss of generality, we can assume that $a_{j_0} < b_{j_0}$. Let $C = \{a_j : j < j_0\}$. Since $a_j = b_j$ for all $j < j_0$ and $t = (h-1)a_1$, we have $(h-1)a_1 + a_{j_0} \in t + S$ and

$$\begin{aligned} r_{\mathcal{H}}^{\mathcal{A}}((h-1)a_1 + a_{j_0}) &= r_{\mathcal{H}}^{C}((h-1)a_1 + a_{j_0}) + 1 \\ &= r_{\mathcal{H}}^{\mathcal{B}}((h-1)a_1 + a_{j_0}) + 1, \end{aligned}$$

which is a contradiction. Hence, we have $\mathcal{A} = \mathcal{B}$. This completes the proof of Theorem 4.

The proof of Theorem 5 is very similar to the proof of Theorem 4, and we omit it here.

3 Proof of Theorem 6

Clearly, $r_2^{\mathcal{A}}(2n)$ is odd if and only if $n \in \mathcal{A}$. Similarly, $n \in \mathcal{B}$ if and only if $r_2^{\mathcal{B}}(2n)$ is odd. If $r_2^{\mathcal{A}}(n) = r_2^{\mathcal{B}}(n)$ for all $n > N_0$, then for all $n > N_0$ we have $n \in \mathcal{A}$ if and only if $n \in \mathcal{B}$. Let

$$\mathcal{D} = \mathcal{A} \cap [N_0 + 1, \infty) = \mathcal{B} \cap [N_0 + 1, \infty)$$

and write

$$\eta(n) = \begin{cases} 1, & \text{if } \ n \in \mathcal{D}; \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n > 2N_0$, we have

$$\begin{aligned} r_2^{\mathcal{A}}(n) &= 2\sharp\{(a,d) : a \in \mathcal{A}\backslash\mathcal{D}, d \in \mathcal{D}, a + d = n\} \\ &\quad + \sharp\{(d', d'') : d', d'' \in \mathcal{D}, d' + d'' = n\} \\ &= 2\sum_{a \in \mathcal{A}\backslash\mathcal{D}} \eta(n-a) + \sharp\{(d', d'') : d', d'' \in \mathcal{D}, d' + d'' = n\}. \end{aligned} \tag{1}$$

Similarly, we have

$$r_2^{\mathcal{B}}(n) = 2\sum_{b \in \mathcal{B}\backslash\mathcal{D}} \eta(n-b) + \sharp\{(d', d'') : d', d'' \in \mathcal{D}, d' + d'' = n\}. \tag{2}$$

Since $r_2^{\mathcal{A}}(n) = r_2^{\mathcal{B}}(n)$ for all $n > N_0$, by (1) and (2), we have

$$\sum_{a \in \mathcal{A}\backslash\mathcal{D}} \eta(n-a) = \sum_{b \in \mathcal{B}\backslash\mathcal{D}} \eta(n-b) \tag{3}$$

for all $n > 2N_0$. Let i_0 be the smallest index such that $a_{i_0} \neq b_{i_0}$. Without loss of generality, we may assume that $a_{i_0} < b_{i_0}$.

Let

$$t = n - a_{i_0}$$

and

$$\mathcal{D}' = \{a : a < a_{i_0}, a \in \mathcal{A}\}.$$

Then by (3), we have

$$\eta(t) = \sum_{b \in \mathcal{B} \setminus (\mathcal{D} \cup \mathcal{D}')} \eta(t + a_{i_0} - b) - \sum_{a \in \mathcal{A} \setminus (\mathcal{D} \cup \mathcal{D}' \cup a_{i_0})} \eta(t + a_{i_0} - a).$$

Since $\eta(t)$ defined by a linear recurrence on a finite set $\{0, 1\}$, we have that it must be eventually periodic. Hence, for some $N > N_0$, $\mathcal{D} \cap [N+1, \infty)$ is periodic. We denote such a period by m. Let $T = \{t : t \equiv d \pmod{m}$ for some $d \in \mathcal{D} \cap [N+1, \infty)$ and $0 \leqslant t < m\}$. Then we have $n \in \mathcal{A} \cap \mathcal{B} \cap [N+1, \infty)$ if and only if $n \equiv t \pmod{m}$ for some $t \in T$.

The remainder of the proof is the same as that of the proof by Nathanson. To make this paper self-contained, we formulate it here. Let

$$A = \{a \leqslant N : a \in \mathcal{A}\}, \quad B = \{b \leqslant N : b \in \mathcal{B}\},$$

and

$$\mathcal{C} = \{c > N : c \in \mathcal{A} \cap \mathcal{B}\} = \{c > N : c \equiv t \pmod{m} \text{ for some } \ t \in T\}.$$

Then $\mathcal{A} = A \cup \mathcal{C}$ and $\mathcal{B} = B \cup \mathcal{C}$. Next we prove that $A + T \equiv B + T \pmod{m}$.

For $n > 2N$, we have

$$\begin{aligned} r_2^{\mathcal{A}}(n) &= r_2^{\mathcal{C}}(n) + 2\sharp\{(a, c) : a \in A, c \in \mathcal{C}, a + c = n\} \\ &= r_2^{\mathcal{C}}(n) + 2\sharp\{(a, t) : a \in A, t \in T, a + t \equiv n \pmod{m}\}. \end{aligned} \tag{4}$$

Similarly,

$$r_2^{\mathcal{B}}(n) = r_2^{\mathcal{C}}(n) + 2\sharp\{(b, t) : b \in B, t \in T, b + t \equiv n \pmod{m}\}. \tag{5}$$

Since $r_2^{\mathcal{A}}(n) = r_2^{\mathcal{B}}(n)$ for $n > 2N$, by (4) and (5), we have that $A + T \equiv B + T \pmod{m}$. This completes the proof of Theorem 6.

4 Acknowledgement

I sincerely thank my supervisor Professor Yong-Gao Chen for his valuable suggestions and useful discussions. I am also grateful to the referee for his/her valuable comments.

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2010 *Mathematics Subject Classification*: Primary 11B34; Secondary 11D85.
Keywords: Nathanson's theorems; representation functions; generating functions.

Received March 13 2011; revised version received August 13 2011. Published in *Journal of Integer Sequences*, September 25 2011.

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