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On the Solutions of $\sigma(n) = \sigma(n+k)$

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Abstract

Given a fixed even integer k, we show that Schinzel's hypothesis H implies that $\sigma(n) = \sigma(n+k)$ infinitely often. We also discuss the case of odd k and the more general equation $\sigma_\alpha(n) = \sigma_\alpha(n+k)$.

1 Introduction

Let $\sigma(n)$ be the sum of the positive divisors of n. Sierpiński [9, p. 166] asked whether

$$\sigma(n) = \sigma(n+1) \tag{1}$$

infinitely often. The sequence of solutions (OEIS A002961) begins with

$$14,\ 206,\ 957,\ 1334,\ 1364,\ 1634,\ 2685,\ 2974,\ 4364,\ 14841, \ldots$$

Although Sierpiński's question has not been answered yet, computational results [5, 6, 7] seem to suggest that the sequence may have infinitely many terms. Guy and Shanks [3] constructed a large solution of (1) based on a pattern observed in smaller ones, but they point out that their construction is unlikely to yield an infinite number of solutions.

More generally, Mientka and Vogt [7] asked for which values of $k \geq 1$,

$$\sigma(n) = \sigma(n+k) \tag{2}$$

has infinitely many solutions. Hunsucker, Nebb and Stearns [6] found at least two solutions of (2) for each $k \leq 5000$.

Erdős [2] made the much stronger conjecture that for every integer $k \geq 1$ there is an n such that

$$\sigma(n) = \sigma(n+1) = \cdots = \sigma(n+k),$$

which would clearly imply that (2) has infinitely many solutions for each k. However, a computer search showed that there is no solution to $\sigma(n) = \sigma(n+1) = \sigma(n+2)$ for $n \leq 10^{10}$.

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be irreducible polynomials with integer coefficients and positive leading coefficients. Schinzel's hypothesis H [8] says that, if there is no fixed integer greater than one which divides

$$f(x) = f_1(x)f_2(x)\cdots f_n(x)$$

for all integers x, then $f_1(x), \ldots, f_n(x)$ are simultaneously prime for infinitely many values of x.

In Section 2 we show that if k is even, the existence of an infinite number of solutions of (2) follows from a particular case of Schinzel's hypothesis H. For $k = 1$ we show that if odd multiperfect numbers do not exist, then there are only finitely many solutions n to (1) with $\Omega(n) + \Omega(n+1)$ bounded by an absolute constant, where $\Omega(n)$ denotes the number of prime divisors of n, counted with multiplicity.

In Section 3 we take a look at the corresponding questions for $\sigma_\alpha(n)$, the sum of the α-th powers of the positive divisors of n.

2 The case $\alpha = 1$.

The first ten solutions to (2) for $k = 2$ are (OEIS A007373)

$$33,\ 54,\ 284,\ 366,\ 834,\ 848,\ 918,\ 1240,\ 1504,\ 2910, \ldots.$$

The following result shows that this sequence, and all other sequences corresponding to even values of k (A015863 ($k = 4$), A015866 ($k = 6$), A015876 ($k = 8$), A015880 ($k = 10$), A015882 ($k = 12$), A181647 ($k = 20$)), are infinite sequences, provided Schinzel's hypothesis H holds.

Proposition 1. *Let k be a fixed even integer. Assuming Schinzel's hypothesis H, the equation $\sigma(n) = \sigma(n+k)$ has infinitely many solutions.*

Proof. We begin with the case $k = 2$. Define

$$p = 2x+1, \quad q = 3x+8, \quad r = 2x+5, \quad s = 3x+2, \tag{3}$$

$$n = pq, \quad m = rs,$$

and

$$f(x) = pqrs = (2x+1)(3x+8)(2x+5)(3x+2).$$

The function $f(x)$ does not have a fixed divisor > 1, since, for example, $f(0) = 2^4 \cdot 5$ and $f(3) = 7 \cdot 11^2 \cdot 17$ have no common factor. Schinzel's hypothesis H says that there are infinitely many values of x such that each one of the four numbers p, q, r, and s is prime. For these values of x we have $\sigma(n) = \sigma(m)$, since

$$\sigma(n) = (p+1)(q+1) = 6(x+1)(x+3) = (r+1)(s+1) = \sigma(m).$$

Furthermore, $m = n + 2$ for all x. This completes the case $k = 2$.

If $k = 2l$ with $l > 1$, let m and n be as above, and write

$$n' = ln, \quad m' = lm.$$

We have $m' = n' + 2l$ and

$$\sigma(m') = \sigma(l)\sigma(m) = \sigma(l)\sigma(n) = \sigma(n'),$$

if $\gcd(l, mn) = 1$, which is the case as soon as $p = 2x + 1 > l$. $\square$

Without Schinzel's hypothesis H, we have the following unconditional result.

Proposition 2. *The equation $\sigma(n) = \sigma(n + k)$ has at least one solution for every even k with $2 \leq k \leq 10^{10^7}$.*

Proof. Using a computer, we generate a sequence $33 = x_1 < x_2 < \cdots < x_N = 146344933173$, with $N = 913685$ terms, such that $p_i = 2x_i + 1$, $q_i = 3x_i + 8$, $r_i = 2x_i + 5$, and $s_i = 3x_i + 2$ are distinct primes for $1 \leq i \leq N$. It is easy to see that $x_i \equiv 3 \pmod{30}$ if p_i, q_i, r_i and s_i are not divisible by 2, 3, or 5. To ensure that we obtain $4N$ distinct primes in all, we further restrict the search to $x_i \equiv 33 \pmod{60}$. As in the proof of Proposition 1, we have $\sigma(p_iq_i) = \sigma(r_is_i) = \sigma(p_iq_i + 2)$, and therefore $\sigma(p_iq_il) = \sigma(p_iq_il + 2l)$ if $\gcd(l, p_iq_ir_is_i) = 1$, where $k = 2l$. If this fails for all $i \leq N$, then l must be divisible by at least one of the four primes p_i, q_i, r_i, or s_i, for each $i \leq N$. Thus $k > l \geq \prod_{i=1}^{N}(2x_i + 1) > 10^{10^7}$. $\square$

For odd values of k, the situation is quite different. Assume that there is a sequence n_j of solutions to (2) of the form

$$n_j = a \prod_{i=1}^{N} p_{i,j}^{\alpha_{i,j}}, \quad n_j + k = b \prod_{i=1}^{M} q_{i,j}^{\beta_{i,j}}, \tag{4}$$

where a, b, N, M are fixed and the primes $p_{i,j}, q_{i,j}$ satisfy $\lim_{j\to\infty} p_{i,j} = \infty$ for $1 \leq i \leq N$ and $\lim_{j\to\infty} q_{i,j} = \infty$ for $1 \leq i \leq M$. Then, as n_j grows,

$$\frac{\sigma(n_j)}{n_j} = \frac{\sigma(a)}{a} \prod_{i=1}^{N} \frac{p_{i,j}^{\alpha_{i,j}+1} - 1}{(p_{i,j} - 1)p_{i,j}^{\alpha_{i,j}}} = \frac{\sigma(a)}{a}(1 + o(1)),$$

$$\frac{\sigma(n_j + k)}{n_j + k} = \frac{\sigma(b)}{b} \prod_{i=1}^{M} \frac{q_{i,j}^{\beta_{i,j}+1} - 1}{(q_{i,j} - 1)q_{i,j}^{\beta_{i,j}}} = \frac{\sigma(b)}{b}(1 + o(1)).$$

Since $\sigma(n_j) = \sigma(n_j + k)$, we also have

$$\frac{\sigma(n_j)}{n_j} = \frac{\sigma(n_j + k)}{n_j + k} \frac{n_j + k}{n_j} = \frac{\sigma(n_j + k)}{n_j + k}(1 + o(1)).$$

It follows that

$$\frac{\sigma(a)}{a} = \frac{\sigma(b)}{b}, \tag{5}$$

where one of a and b is even, and the other one is odd, since k is odd. When $k = 1$ we also have $\gcd(a, b) = 1$, which means that the fractions in (5) must reduce to integers ≥ 2. Thus a and b are both multiperfect numbers, with one of them being odd. Since no odd multiperfect numbers are known, we can not use this approach to show that there are infinitely many solutions to (1). We summarize this in the following statement.

Proposition 3. *Let a, b, N, M be fixed positive integers, such that neither a nor b is an odd multiperfect number, and let n_j be a sequence of the form (4) with $k = 1$, where the primes $p_{i,j}, q_{i,j}$ satisfy $\lim_{j\to\infty} p_{i,j} = \infty$ for $1 \leq i \leq N$ and $\lim_{j\to\infty} q_{i,j} = \infty$ for $1 \leq i \leq M$. Then the sequence n_j contains at most a finite number of solutions to (1).*

The following result is a consequence of Proposition 3.

Proposition 4. *Let $C \geq 1$ be fixed. If there are no odd multiperfect numbers, then there are only finitely many solutions to $\sigma(n) = \sigma(n + 1)$ with $\Omega(n) + \Omega(n + 1) \leq C$.*

Proof. Assume that n_j is an infinite increasing sequence with $\sigma(n_j) = \sigma(n_j + 1)$ and $\Omega(n_j) + \Omega(n_j + 1) \leq C$. We show that n_j has an infinite subsequence of the form (4) with $k = 1$. By restricting n_j to a subsequence if necessary, we may assume $\Omega(n_j) = \mu$ and $\Omega(n_j + 1) = \nu$, for some constants $\mu, \nu \in \mathbb{N}$. Write $n_j = p_{1,j} \cdots p_{\mu,j}$ and $n_j + 1 = q_{1,j} \cdots q_{\nu,j}$. For each $1 \leq i \leq \mu$, either $\liminf_{j\to\infty} p_{i,j} = p_i$ for some fixed prime p_i, or $\lim_{j\to\infty} p_{i,j} = \infty$. In the first case we restrict the sequence n_j to an infinite subsequence such that $p_{i,j} = p_i$ for all $j \geq 1$. Let a denote the product of all the fixed primes p_i. After applying the same process to the prime divisors of $n_j + 1$, we arrive at an infinite sequence of the form (4) with $k = 1$. Thus the result follows from Proposition 3. $\square$

For odd values of $k \geq 3$, it may still be possible to construct an infinite sequence of solutions of the form (4), because there are solutions to (5) with a even and b odd, such as

$$a = 3472 = 2^4 \cdot 7 \cdot 31, \quad b = 544635 = 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19.$$

In this case $\gcd(a, b) = 7$, so it may be possible to use this as a starting point to show that $\sigma(n) = \sigma(n + 7)$ has infinitely many solutions, assuming Schinzel's hypothesis H, although we were not able to do so.

While we can't prove or disprove that there is at least one solution of (2) for every odd value of k, in some cases a solution is easy to find. For example, $n = 14k$ is a solution to (2) if $\gcd(k, 14 \cdot 15) = 1$, since $n = 14$ is a solution to (1). The same idea can be applied to other solutions of (1).

3 The case $\alpha \geq 2$.

The case $\alpha = 2$ was treated by De Koninck [1] with the following result.

Proposition 5 (De Koninck)**.** *Let k be a fixed positive integer.*

(i) If k is odd, then $\sigma_2(n) = \sigma_2(n+k)$ has at most a finite number of solutions.

(ii) If k is even, then Schinzel's hypothesis H implies that $\sigma_2(n) = \sigma_2(n+k)$ has an infinite number of solutions.

The only solution to $\sigma_2(n) = \sigma_2(n+1)$ is $n = 6$. According to [1], $\sigma_2(n) = \sigma_2(n+k)$ has no solution when $k = 3, 9, 15, 27, 33, 35, 39, 45, 51, 57, 69, 75, 81, 87, 93$ or 99. Assuming Schinzel's hypothesis H, a sequence of solutions to $\sigma_2(n) = \sigma_2(n+2)$ given in [1] can be written as

$$n = x^3 - 1 = (x-1)(x^2+x+1), \quad n+2 = x^3+1 = (x+1)(x^2-x+1).$$

It is easy to verify that $\sigma_2(n) = \sigma_2(n+2)$ as long as the four quantities $x-1, x^2+x+1, x+1$ and x^2-x+1 are all prime.

We can extend the first half of Proposition 5 to $\alpha \geq 3$.

Proposition 6. *For every $\alpha \geq 2$ there is a constant $c_\alpha > 0$ such that if $\sigma_\alpha(n) = \sigma_\alpha(n+k)$ for an odd value of k, then $k > c_\alpha n$.*

Proof. We first show that

$$1 + \frac{1}{2^\alpha} > \sum_{j\geq 0} \frac{1}{(2j+1)^\alpha} \qquad (\alpha \geq 2). \tag{6}$$

For $\alpha = 2$, (6) simplifies to $\frac{5}{4} > (1 - 1/2^2)\zeta(2) = \frac{\pi^2}{8} = 1.2337...$ If we subtract 1 from both sides of (6) and then multiply by 2^α, we see that (6) is equivalent to

$$1 > \sum_{j\geq 1} \frac{1}{(j+1/2)^\alpha} \qquad (\alpha \geq 2).$$

Note that the right-hand side is decreasing in α. Since the inequality is valid for $\alpha = 2$, it is valid for all $\alpha \geq 2$.

Now let $m = n+k$. Since k is odd, one of m, n is even and the other one is odd. Without loss of generality we assume $2|n$ and $2 \nmid m$, allowing $k < 0$ if necessary. Then

$$\frac{\sigma_\alpha(n)}{n^\alpha} \geq \frac{\sigma_\alpha(2)}{2^\alpha} = 1 + \frac{1}{2^\alpha},$$

and

$$\frac{\sigma_\alpha(m)}{m^\alpha} < \sum_{j\geq 0} \frac{1}{(2j+1)^\alpha}.$$

If $\sigma_\alpha(n) = \sigma_\alpha(m)$,

$$\left(1 + \frac{1}{2^\alpha}\right)\left(\frac{n}{n+k}\right)^\alpha \leq \frac{\sigma_\alpha(n)}{n^\alpha}\frac{n^\alpha}{m^\alpha} = \frac{\sigma_\alpha(m)}{m^\alpha} < \sum_{j\geq 0} \frac{1}{(2j+1)^\alpha},$$

or

$$\left(\frac{1}{1+k/n}\right)^\alpha < \frac{1 - \frac{1}{2^\alpha}}{1 + \frac{1}{2^\alpha}}\zeta(\alpha) =: b_\alpha,$$

say. From (6) we have $b_\alpha < 1$ for $\alpha \geq 2$, hence

$$\frac{k}{n} > b_\alpha^{-1/\alpha} - 1 =: c_\alpha > 0.$$

$\square$

The remaining case is where $\alpha \geq 3$ and k is even. Guy [4, Problem B13] writes, "Erdős thinks that $\sigma_3(n) = \sigma_3(n+2)$ has no solution at all". A computer search showed that there is in fact no solution to $\sigma_3(n) = \sigma_3(n+2)$ for $n \leq 10^{10}$. Although we are not able to prove that there are no solutions in this case, we have the following partial result. We omit the proof since the ideas are similar to those of Proposition 6.

Proposition 7. *If n is a solution of $\sigma_3(n) = \sigma_3(n+2)$, then neither n nor $n+2$ is divisible by $2, 3, 5$ or 7.*

For $\alpha \geq 3$, solutions to $\sigma_\alpha(n) = \sigma_\alpha(n+k)$ are quite rare even if k is allowed to vary. There are only three primitive solutions to $\sigma_3(n) = \sigma_3(n+k)$ with $n \leq 5 \cdot 10^6$ and $k \geq 1$, namely

$$(n, k) = (184926, 9389), (291741, 3560), (1880574, 6346),$$

while the other fourteen solutions in the same range are obtained from these by multiplying both n and k by some l with $\gcd(l, nk) = 1$.

There are no solutions to $\sigma_\alpha(n) = \sigma_\alpha(n+k)$ for $\alpha \geq 4$, $n \leq 10^6$ and $k \geq 1$.

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