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Dixon's Formula and Identities Involving Harmonic Numbers

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Abstract

Inspired by the recent work of Chu and Fu, we derive some new identities with harmonic numbers from Dixon's hypergeometric summation formula by applying the derivation operator to the summation of binomial coefficients.

1 Introduction

For an indeterminate x and a nonnegative integer n, the shifted factorial or *Pochhammer's symbol* is defined by

$$(x)_0 := 1 \text{ and } (x)_n := \Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1),\ \ n = 1, 2, \cdots$$

with the Γ-function given through Euler integral

$$\Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u}\mathrm{d}u \quad \text{with} \quad \Re(x) > 0.$$

Following Bailey [1, §2.1] and Slater [9, §2.1], the generalized hypergeometric series is defined by

$${}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!},$$

where we suppose that none of the denominator parameters is a nonpositive integer, so that the series is well defined.

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The generalized harmonic numbers are defined by

$$H_n^{\langle \ell \rangle}(x) := \sum_{k=1}^{n} \frac{1}{(k+x)^\ell} \quad \text{and} \quad H_n^{\langle \ell \rangle} := H_n^{\langle \ell \rangle}(0) = \sum_{k=1}^{n} \frac{1}{k^\ell}, \qquad H_0^{\langle \ell \rangle} := 0,$$

with indeterminate x and natural numbers n and ℓ. When $\ell = 1$, they will be abbreviated as $H_n(x)$ and H_n respectively. These numbers come naturally from the derivatives of binomial coefficients

$$D_x \binom{n+x}{n} = H_n(x) \binom{n+x}{n} \quad \text{and} \quad D_x \binom{n+x}{n}^{-1} = -H_n(x) \binom{n+x}{n}^{-1},$$

where the differential operator is defined as

$$D_x f(x) = \frac{d}{dx} f(x),$$

with differentiable function $f(x)$. Obviously, the generalized harmonic numbers satisfy the following recurrence relation

$$D_x H_n(x) = -H_n^{\langle 2 \rangle}(x) \quad \text{and} \quad D_x H_n^{\langle \ell \rangle}(x) = -\ell H_n^{\langle \ell+1 \rangle}(x).$$

This fact can be traced back to Issac Newton [7], and has been explored recently in several papers [2, 3, 4, 6, 8].

Chu and Fu [5] derived many identities involving harmonic numbers from Dougall–Dixon's summation formula. There exist $\binom{n}{k}^3$ or $\binom{n}{k}^4$ in the coefficients of identities. In recent work, Chen and Chu [2] established a general formula involving harmonic numbers and the Riemann zeta function. The purpose of this article is to present some new identities with harmonic numbers from Dixon's summation formula by applying the derivation operator to binomial coefficients. In this paper, there are $\binom{2n}{k}^3$, $\binom{2n}{k}$ and $\binom{2n}{k}^{-1}$ in the coefficients of the identities.

2 The identities due to Dixon's ${}_3F_2(1)$ summation formula

In this section, we will obtain several identities with harmonic numbers from Dixon's summation formula. Dixon's summation theorem is presented as follows:

Theorem 1 (Dixon [9]).

$${}_3F_2 \left[\begin{matrix} a, \quad b, \quad c \\ 1+a-b, 1+a-c \end{matrix} \,\middle|\, 1 \right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}.$$

(I) Letting $a = -2n, b = 1+\lambda x$ and $c = 1+\theta x$ in Theorem 1, we have

$${}_3F_2 \left[\begin{matrix} -2n, 1+\lambda x, 1+\theta x \\ -\lambda x - 2n, -\theta x - 2n \end{matrix} \,\middle|\, 1 \right] = \frac{(2n)!(1+\lambda x)_n (1+\theta x)_n (1+\lambda x+\theta x)_{2n+1}}{n!(1+\lambda x)_{2n}(1+\theta x)_{2n}(1+\lambda x+\theta x)_{n+1}}.$$

Multiplying both sides by the binomial coefficients $\binom{2n+\lambda x}{2n}\binom{2n+\theta x}{2n}$, we reformulate the result as the following finite summation identity.

$$\sum_{k=0}^{2n} \frac{(-1)^k \binom{\lambda x+k}{k}\binom{\theta x+k}{k}\binom{2n-k+\lambda x}{2n-k}\binom{2n-k+\theta x}{2n-k}}{\binom{2n}{k}} = \frac{(2n+1)\binom{n+\lambda x}{n}\binom{n+\theta x}{n}\binom{2n+1+\lambda x+\theta x}{2n+1}}{(n+1)\binom{n+1+\lambda x+\theta x}{n+1}}. \tag{1}$$

Computing the derivation of identity (1) with respect to x for one time, we get

Theorem 2.

$$\sum_{k=0}^{2n} \frac{(-1)^k \binom{\lambda x+k}{k}\binom{\theta x+k}{k}\binom{2n-k+\lambda x}{2n-k}\binom{2n-k+\theta x}{2n-k}\Omega(x)}{\binom{2n}{k}} = \frac{(2n+1)\binom{n+\lambda x}{n}\binom{n+\theta x}{n}\binom{2n+1+\lambda x+\theta x}{2n+1}W(x)}{(n+1)\binom{n+1+\lambda x+\theta x}{n+1}},$$

where $\Omega(x)$ and $W(x)$ are given respectively by

$$\begin{aligned}
&\Omega(x) = \lambda H_{2n-k}(\lambda x) + \theta H_{2n-k}(\theta x) + \lambda H_k(\lambda x) + \theta H_k(\theta x);\\
&W(x) = (\lambda+\theta)H_{2n+1}(\lambda x+\theta x) + \lambda H_n(\lambda x) + \theta H_n(\theta x) - (\lambda+\theta)H_{n+1}(\lambda x+\theta x).
\end{aligned}$$

Proof. The derivatives of binomial coefficients are as follows

$$D_x\binom{n+\lambda x}{n} = \lambda H_n(\lambda x)\binom{n+\lambda x}{n} \quad \text{and} \quad D_x\binom{n+\lambda x}{n}^{-1} = -\lambda H_n(\lambda x)\binom{n+\lambda x}{n}^{-1}.$$

Applying the derivation operator on the left side of the identity (1) to x, changing the calculation of the orders of summation and derivation and simplifying the result, we have

$$\begin{aligned}
&\sum_{k=0}^{2n} \frac{(-1)^k \binom{\lambda x+k}{k}\binom{\theta x+k}{k}\binom{2n-k+\lambda x}{2n-k}\binom{2n-k+\theta x}{2n-k}}{\binom{2n}{k}}\\
&\times\Big\{\lambda H_k(\lambda x) + \theta H_k(\theta x) + \lambda H_{2n-k}(\lambda x) + \theta H_{2n-k}(\theta x)\Big\}.
\end{aligned}$$

By the same method, we evaluate the derivation of the right side of identity (1) to x as follows

$$\begin{aligned}
&\frac{(2n+1)\binom{n+\lambda x}{n}\binom{n+\theta x}{n}\binom{2n+1+\lambda x+\theta x}{2n+1}}{(n+1)\binom{n+1+\lambda x+\theta x}{n+1}}\\
&\times\Big\{\lambda H_n(\lambda x) + \theta H_n(\theta x) + (\lambda+\theta)H_{2n+1}(\lambda x+\theta x) + (\lambda+\theta)H_{n+1}(\lambda x+\theta x)\Big\}.
\end{aligned}$$

Comparing both results, we get Theorem 2. ☐

Setting $x = 0$ in Theorem 2 and noting that

$$\Omega(0) = (\lambda+\theta)(H_{2n-k}+H_k); \quad W(0) = (\lambda+\theta)(H_{2n+1}+H_n-H_{n+1}),$$

we have the new identity with the harmonic numbers and $\binom{2n}{k}^{-1}$ as follows:

Corollary 3.

$$\sum_{k=0}^{2n} \frac{(-1)^k}{\binom{2n}{k}} H_k = \frac{2n+1}{2(n+1)} \Big\{ H_{2n+1} + H_n - H_{n+1} \Big\}.$$

Furthermore, computing the derivation of the identity (1) with respect to x for two times, we have another relation about the harmonic numbers.

Theorem 4.

$$\sum_{k=0}^{2n} \frac{(-1)^k \binom{\lambda x+k}{k}\binom{\theta x+k}{k}\binom{2n-k+\lambda x}{2n-k}\binom{2n-k+\theta x}{2n-k}\{\Omega^2(x)+\Omega'(x)\}}{\binom{2n}{k}}$$
$$= \frac{(2n+1)\binom{n+\lambda x}{n}\binom{n+\theta x}{n}\binom{2n+1+\lambda x+\theta x}{2n+1}\{W^2(x)+W'(x)\}}{(n+1)\binom{n+1+\lambda x+\theta x}{n+1}},$$

where $\Omega'(x)$ *and* $W'(x)$ *are given respectively by*

$$\begin{aligned}
&\Omega'(x) = -\lambda^2 H^{\langle 2\rangle}_{2n-k}(\lambda x) - \theta^2 H^{\langle 2\rangle}_{2n-k}(\theta x) - \lambda^2 H^{\langle 2\rangle}_{k}(\lambda x) - \theta^2 H^{\langle 2\rangle}_{k}(\theta x);\\
&W'(x) = -(\lambda+\theta)^2 H^{\langle 2\rangle}_{2n+1}(\lambda x+\theta x) + (\lambda+\theta)^2 H^{\langle 2\rangle}_{n+1}(\lambda x+\theta x) - \lambda^2 H^{\langle 2\rangle}_{n}(\lambda x) - \theta^2 H^{\langle 2\rangle}_{n}(\theta x).
\end{aligned}$$

Noting further that

$$\begin{aligned}
&\Omega'(0) = -(\lambda^2+\theta^2)(H^{\langle 2\rangle}_{2n-k} + H^{\langle 2\rangle}_{k});\\
&W'(0) = (\lambda+\theta)^2(H^{\langle 2\rangle}_{n+1} - H^{\langle 2\rangle}_{2n+1}) - (\lambda^2+\theta^2)H^{\langle 2\rangle}_{n},
\end{aligned}$$

we have the following new identity with harmonic numbers when $x = 0$ in Theorem 4.

Corollary 5.

$$\sum_{k=0}^{2n} \frac{(-1)^k}{\binom{2n}{k}} \Big\{ (\lambda+\theta)^2(H_{2n-k}+H_k)^2 - (\lambda^2+\theta^2)(H^{\langle 2\rangle}_{2n-k} + H^{\langle 2\rangle}_{k}) \Big\}$$
$$= \frac{2n+1}{n+1} \Big\{ (\lambda+\theta)^2(H_{2n+1}+H_n-H_{n+1})^2 + (\lambda+\theta)^2(H^{\langle 2\rangle}_{n+1} - H^{\langle 2\rangle}_{2n+1}) - (\lambda^2+\theta^2)H^{\langle 2\rangle}_{n} \Big\}.$$

Now, we present some examples with the harmonic numbers from Corollary 5.

Example 6. [$\lambda = 0, \theta \neq 0$ or $\lambda \neq 0, \theta = 0$ in Corollary 5]

$$\sum_{k=0}^{2n} \frac{(-1)^k}{\binom{2n}{k}} \{ (H_{2n-k}+H_k)^2 - (H^{\langle 2\rangle}_{2n-k} + H^{\langle 2\rangle}_{k}) \}$$
$$= \frac{2n+1}{n+1} \{ (H_{2n+1}+H_n-H_{n+1})^2 + H^{\langle 2\rangle}_{n+1} - H^{\langle 2\rangle}_{2n+1} - H^{\langle 2\rangle}_{n} \}.$$

Example 7. [$\lambda = -\theta \neq 0$ in Corollary 5]

$$\sum_{k=0}^{2n} \frac{(-1)^k}{\binom{2n}{k}} H^{\langle 2\rangle}_{k} = \frac{2n+1}{2(n+1)} H^{\langle 2\rangle}_{n}.$$

(II) Letting $a = -2n$, $b = \lambda x - 2n$ and $c = \theta x - 2n$ in Dixon's Theorem 1, we have the following identity

$$ {}_3F_2\left[\begin{matrix} -2n, \lambda x - 2n, \theta x - 2n \\ 1-\lambda x, \quad 1-\theta x \end{matrix} \,\middle|\, 1\right] = \frac{(2n)!(1-\theta x-\lambda x)_{3n}}{n!(1-\lambda x)_n(1-\theta x)_n(1-\theta x-\lambda x)_{2n}}. $$

Dividing both sides of the above identity by the binomial coefficients $\binom{2n-\lambda x}{2n}\binom{2n-\theta x}{2n}$, we reformulate the result as the following finite summation identity.

$$ \sum_{k=0}^{2n} \frac{(-1)^k\binom{2n}{k}^3}{\binom{k-\lambda x}{k}\binom{k-\theta x}{k}\binom{2n-k-\lambda x}{2n-k}\binom{2n-k-\theta x}{2n-k}} = \frac{(-1)^n\binom{2n}{n}\binom{3n}{n}\binom{3n-\lambda x-\theta x}{3n}}{\binom{n-\lambda x}{n}\binom{2n-\lambda x}{2n}\binom{n-\theta x}{n}\binom{2n-\theta x}{2n}\binom{2n-\theta x-\lambda x}{2n}}. \tag{2} $$

When $x = 0$ in the above identity, we have the well known identity with binomial coefficients.

Corollary 8.

$$ \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = (-1)^n\binom{2n}{n}\binom{3n}{n}. $$

Evaluating the derivation of identity (2) with respect to x for one time, we get the following result.

Theorem 9.

$$ \sum_{k=0}^{2n} \frac{(-1)^k\binom{2n}{k}^3\Omega(x)}{\binom{k-\lambda x}{k}\binom{k-\theta x}{k}\binom{2n-k-\lambda x}{2n-k}\binom{2n-k-\theta x}{2n-k}} = \frac{(-1)^n\binom{2n}{n}\binom{3n}{n}\binom{3n-\lambda x-\theta x}{3n}W(x)}{\binom{n-\lambda x}{n}\binom{2n-\lambda x}{2n}\binom{n-\theta x}{n}\binom{2n-\theta x}{2n}\binom{2n-\theta x-\lambda x}{2n}}. $$

where $\Omega(x)$ and $W(x)$ are given respectively by

$$ \begin{aligned} \Omega(x) &= \lambda H_k(-\lambda x) + \theta H_k(-\theta x) + \lambda H_{2n-k}(-\lambda x) + \theta H_{2n-k}(-\theta x); \\ W(x) &= \lambda\big\{H_n(-\lambda x) + H_{2n}(-\lambda x)\big\} + \theta\big\{H_n(-\theta x) + H_{2n}(-\theta x)\big\} \\ &\quad +(\theta+\lambda)\big\{H_{2n}(-\theta x-\lambda x) - H_{3n}(-\theta x-\lambda x)\big\}. \end{aligned} $$

Letting $x = 0$ in Theorem 9 and noting that

$$ \begin{aligned} \Omega(0) &= (\lambda+\theta)(H_k + H_{2n-k}); \\ W(0) &= (\lambda+\theta)(H_n + 2H_{2n} - H_{3n}), \end{aligned} $$

we have the following identity.

Corollary 10.

$$ \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\{H_k + H_{2n-k}\} = (-1)^n\binom{2n}{n}\binom{3n}{n}\{H_n + 2H_{2n} - H_{3n}\}, $$

which is a special case of Example 1 in [5]. Evaluating the derivation of identity (2) with respect to x for two times, we obtain the identity as follows.

Theorem 11.

$$\sum_{k=0}^{2n} \frac{(-1)^k \binom{2n}{k}^3 \big\{\Omega(x)^2 + \Omega'(x)\big\}}{\binom{k-\lambda x}{k}\binom{k-\theta x}{k}\binom{2n-k-\lambda x}{2n-k}\binom{2n-k-\theta x}{2n-k}} = \frac{(-1)^n \binom{2n}{n}\binom{3n}{n}\binom{3n-\lambda x-\theta x}{3n}\big\{W(x)^2 + W'(x)\big\}}{\binom{n-\lambda x}{n}\binom{2n-\lambda x}{2n}\binom{n-\theta x}{n}\binom{2n-\theta x}{2n}\binom{2n-\theta x-\lambda x}{2n}}.$$

where $\Omega'(x)$ and $W'(x)$ are given respectively by

$$\begin{aligned}
\Omega'(x) &= \lambda^2 H_k^{\langle 2\rangle}(-\lambda x) + \lambda^2 H_{2n-k}^{\langle 2\rangle}(-\lambda x) + \theta^2 H_k^{\langle 2\rangle}(-\theta x) + \theta^2 H_{2n-k}^{\langle 2\rangle}(-\theta x);\\
W'(x) &= \lambda^2\big\{H_n^{\langle 2\rangle}(-\lambda x) + H_{2n}^{\langle 2\rangle}(-\lambda x)\big\} + \theta^2\big\{H_n^{\langle 2\rangle}(-\theta x) + H_{2n}^{\langle 2\rangle}(-\theta x)\big\}\\
&\quad + (\theta+\lambda)^2\big\{H_{2n}^{\langle 2\rangle}(-\theta x-\lambda x) - H_{3n}^{\langle 2\rangle}(-\theta x-\lambda x)\big\}.
\end{aligned}$$

Letting $x = 0$ in Theorem 11 and noting that

$$\begin{aligned}
\Omega'(0) &= (\lambda^2+\theta^2)(H_k^{\langle 2\rangle} + H_{2n-k}^{\langle 2\rangle});\\
W'(0) &= (\lambda+\theta)^2(H_{2n}^{\langle 2\rangle} - H_{3n}^{\langle 2\rangle}) + (\lambda^2+\theta^2)(H_n^{\langle 2\rangle} + H_{2n}^{\langle 2\rangle}),
\end{aligned}$$

we derive the identity as follows.

Corollary 12.

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\Big\{(\lambda+\theta)^2(H_k+H_{2n-k})^2 + (\lambda^2+\theta^2)(H_k^{\langle 2\rangle} + H_{2n-k}^{\langle 2\rangle})\Big\}\\
&= (-1)^n\binom{2n}{n}\binom{3n}{n}\Big\{(\lambda+\theta)^2\big[(H_n+2H_{2n}-H_{3n})^2 + H_{2n}^{\langle 2\rangle} - H_{3n}^{\langle 2\rangle}\big] + (\lambda^2+\theta^2)(H_n^{\langle 2\rangle} + H_{2n}^{\langle 2\rangle})\Big\}.
\end{aligned}$$

This identity is the same as Theorem 2 in [5]. Now we present some examples from Corollary 12.

Example 13. [$\lambda = 0,\ \theta \neq 0$ or $\lambda \neq 0,\ \theta = 0$ in Corollary 12]

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\Big\{(H_k+H_{2n-k})^2 + H_k^{\langle 2\rangle} + H_{2n-k}^{\langle 2\rangle}\Big\}\\
&= (-1)^n\binom{2n}{n}\binom{3n}{n}\Big\{(H_n+2H_{2n}-H_{3n})^2 + H_n^{\langle 2\rangle} + 2H_{2n}^{\langle 2\rangle} - H_{3n}^{\langle 2\rangle}\Big\},
\end{aligned}$$

which is equal to Example 3 in [5].

Example 14. [$\lambda = -\theta \neq 0$ in Corollary 12]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 H_k^{\langle 2\rangle} = \frac{(-1)^n}{2}\binom{2n}{n}\binom{3n}{n}\Big\{H_n^{\langle 2\rangle} + H_{2n}^{\langle 2\rangle}\Big\}.$$

Example 15. [$\lambda = 1$ and $\theta = 1$ in Corollary 12]

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\Big\{(H_k+H_{2n-k})^2 + H_k^{\langle 2\rangle}\Big\}\\
&= (-1)^n\binom{2n}{n}\binom{3n}{n}\Big\{(H_n+2H_{2n}-H_{3n})^2 + \frac{1}{2}H_n^{\langle 2\rangle} + \frac{3}{2}H_{2n}^{\langle 2\rangle} - H_{3n}^{\langle 2\rangle}\Big\}.
\end{aligned}$$

In fact, there are many identities involving harmonic numbers which can be obtained from Corollary 12.

(III) Letting $a=-2n, b=-\lambda x-2n$ and $c=1-\theta x$ in Dixon's Theorem refdixon, we have the following identity.

$$
{}_3F_2\left[\begin{matrix}-2n,-\lambda x-2n,1-\theta x\\ 1+\lambda x,\quad \theta x-2n\end{matrix}\,\Big|\,1\right]=\frac{(2n)!(\lambda x+\theta x)_n(1-\theta x)_n}{n!(1+\lambda x)_n(1-\theta x)_{2n}}.
$$

Dividing both sides of the above identity by the binomial coefficients $\binom{2n+\lambda x}{2n}/\binom{2n-\theta x}{2n}$, we reformulate the result as the following finite summation identity.

$$
\sum_{k=0}^{2n}\frac{(-1)^k\binom{2n}{k}\binom{k-\theta x}{k}\binom{2n-k-\theta x}{2n-k}}{\binom{k+\lambda x}{k}\binom{2n-k+\lambda x}{2n-k}}=\frac{(\lambda x+\theta x)\binom{n-\theta x}{n}\binom{n-1+\lambda x+\theta x}{n-1}}{n\binom{n+\lambda x}{n}\binom{2n+\lambda x}{2n}}. \tag{3}
$$

Evaluating the derivation of identity (3) with respect to x for two times, we have the following identity with the harmonic numbers.

Theorem 16.

$$
\begin{aligned}
&\sum_{k=0}^{2n}\frac{(-1)^k\binom{2n}{k}\binom{k-\theta x}{k}\binom{2n-k-\theta x}{2n-k}}{\binom{k+\lambda x}{k}\binom{2n-k+\lambda x}{2n-k}}\Big\{\Omega^2(x)+\Omega'(x)\Big\}\\
&\qquad=\frac{(\lambda x+\theta x)\binom{n-\theta x}{n}\binom{n-1+\lambda x+\theta x}{n-1}}{n\binom{n+\lambda x}{n}\binom{2n+\lambda x}{2n}}\Big\{W(x)^2+\frac{2}{x}W(x)+W'(x)\Big\},
\end{aligned}
$$

where $\Omega(x)$, $\Omega'(x)$, $W(x)$ and $W'(x)$ are given respectively by

$$
\begin{aligned}
\Omega(x)&=-\theta H_k(-\theta x)-\theta H_{2n-k}(-\theta x)-\lambda H_k(\lambda x)-\lambda H_{2n-k}(\lambda x);\\
\Omega'(x)&=-\theta^2 H_k^{\langle 2\rangle}(-\theta x)-\theta^2 H_{2n-k}^{\langle 2\rangle}(-\theta x)+\lambda^2 H_k^{\langle 2\rangle}(\lambda x)+\lambda^2 H_{2n-k}^{\langle 2\rangle}(\lambda x);\\
W(x)&=(\lambda+\theta)H_{n-1}(\lambda x+\theta x)-\theta H_n(-\theta x)-\lambda H_n(\lambda x)-\lambda H_{2n}(\lambda x);\\
W'(x)&=\lambda^2 H_{2n}^{\langle 2\rangle}(\lambda x)+\lambda^2 H_n^{\langle 2\rangle}(\lambda x)-\theta^2 H_n^{\langle 2\rangle}(-\theta x)-(\lambda+\theta)^2 H_{n-1}^{\langle 2\rangle}(\lambda x+\theta x).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\Omega(0)=-(\lambda+\theta)(H_k+H_{2n-k}); &&W(0)=(\lambda+\theta)(H_{n-1}-H_n)-\lambda H_{2n};\\
&\Omega'(0)=(\lambda^2-\theta^2)(H_k^{\langle 2\rangle}+H_{2n-k}^{\langle 2\rangle}); &&W'(0)=\lambda^2 H_{2n}^{\langle 2\rangle}+(\lambda^2-\theta^2)H_n^{\langle 2\rangle}-(\lambda+\theta)^2 H_{n-1}^{\langle 2\rangle}.
\end{aligned}
$$

the case $x\to 0$ of Theorem 16 reads as the following new general formula.

Corollary 17.

$$
\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\Big\{(\lambda+\theta)(H_k+H_{2n-k})^2+2(\lambda-\theta)H_k^{\langle 2\rangle}\Big\}=\frac{2}{n}\big\{(\lambda+\theta)(H_{n-1}-H_n)-\lambda H_{2n}\big\}.
$$

Now we present some examples with harmonic numbers from Corollary 17.

Example 18. [$\lambda = 0$ and $\theta = 1$ in Corollary 17]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\Big\{(H_k + H_{2n-k})^2 - 2H_k^{\langle 2\rangle}\Big\} = -\frac{2}{n^2}.$$

Example 19. [$\lambda = 1$ and $\theta = 0$ in Corollary 17]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\Big\{(H_k + H_{2n-k})^2 + 2H_k^{\langle 2\rangle}\Big\} = \frac{2}{n}\Big\{H_{n-1} - H_n - H_{2n}\Big\}.$$

Example 20. [$\lambda = 1$ and $\theta = 1$ in Corollary 17]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(H_k + H_{2n-k})^2 = \frac{1}{n}\{2H_{n-1} - 2H_n - H_{2n}\}.$$

Example 21. [$\lambda = 1$ and $\theta = -1$ in Corollary 17]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}H_k^{\langle 2\rangle} = -\frac{H_{2n}}{2n}.$$

Also there are many other identities can be obtained from Corollary 17. Here, we have just presented several of them as examples.

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