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The 4-Nicol Numbers Having Five Different Prime Divisors

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Abstract

A positive integer n is called a Nicol number if $n \mid \varphi(n)+\sigma(n)$, and a t-Nicol number if $\varphi(n)+\sigma(n)=tn$. In this paper, we show that if n is a 4-Nicol number that has five different prime divisors, then $n = 2^{\alpha_1}\cdot 3\cdot 5^{\alpha_3}\cdot p^{\alpha_4}\cdot q^{\alpha_5}$, or $n = 2^{\alpha_1}\cdot 3\cdot 7^{\alpha_3}\cdot p^{\alpha_4}\cdot q^{\alpha_5}$ with $p \le 29$.

1 Introduction

For any positive integer n, let $\phi(n)$, $\omega(n)$ and $\sigma(n)$ be the Euler function of n, the number of prime divisors of n and the sum of divisors of n, respectively. We call n is a Nicol number if $n \mid \varphi(n)+\sigma(n)$, and a t-Nicol number if $\varphi(n)+\sigma(n)=tn$. It is well-known that $t \ge 2$, and n is prime if and only if $\varphi(n)+\sigma(n)=2n$. In 1966, Nicol [4] conjectured that Nicol numbers are all even, and proved that if α is such that $p = 2^{\alpha-2}\cdot 7-1$ is prime, then $n = 2^{\alpha}\cdot 3\cdot p$ is 3-Nicol number. In 1995, Ming-Zhi Zhang [6] showed that if $n = p^{\alpha}q$ then n cannot be a Nicol number, where p and q are distinct primes and α is a positive integer. In 1997, Lin and Zhang [2] showed that if $\omega(n)=2$, then n cannot be a Nicol number. In 2008, Luca and Sandor [3] showed that if n is a Nicol number and $\omega(n)=3$, then either $n \in \{560, 588, 1400\}$ or $n = 2^{\alpha}\cdot 3\cdot p$ with $p = 2^{\alpha-2}\cdot 7-1$ prime. In 2008, Wang [5] studied the Nicol numbers that have four different prime divisors. In 2009, Harris [1] showed that the Nicol numbers that have four different prime divisors must be one of the following forms:

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1. $n = 2^3 \cdot 3^3 \cdot 5^2 \cdot 11$, $2^4 \cdot 3^3 \cdot 5 \cdot 11$, $2^7 \cdot 5 \cdot 11 \cdot 79$, $2^3 \cdot 3^3 \cdot 5^3 \cdot 13^2$, $2^2 \cdot 3^2 \cdot 17 \cdot 241$, $2^2 \cdot 3^2 \cdot 17^2 \cdot 2243$;

2. $n \in \left\{2^a \cdot 3 \cdot p_3 \cdot p_4 \middle| p_4 = \dfrac{(7 \cdot 2^{a-2} - 1)p_3 + 9 \cdot 2^{a-2} - 1}{p_3 - (7 \cdot 2^{a-2} - 1)}\right.$, where p_3, p_4 are distinct primes.$\left.\right\}$

Moreover, Harris [1] proved that all but finitely many Nicol numbers that have 5 different prime divisors are divisible by 6 and not 9.

In this paper, we study the 4-Nicol numbers that have five different prime divisors and obtain the following result:

Theorem 1. *If n is a 4-Nicol number with $\omega(n) = 5$, then either $n = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}p^{\alpha_4}q^{\alpha_5}$, or $n = 2^{\alpha_1}3^{\alpha_2}7^{\alpha_3}p^{\alpha_4}q^{\alpha_5}$ with $p \leq 29$, where p, q are distinct primes, and $\alpha_i (i = 1, 2, \cdots, 5)$ are positive integers.*

By the Harris result and Theorem 1, we have the following result:

Corollary 2. *All but finitely many 4-Nicol numbers with 5 different prime divisors have the form $n = 2^{\alpha_1} \cdot 3 \cdot 5^{\alpha_3} \cdot p^{\alpha_4} \cdot q^{\alpha_5}$, or $n = 2^{\alpha_1} \cdot 3 \cdot 7^{\alpha_3} \cdot p^{\alpha_4} \cdot q^{\alpha_5}$ with $p \leq 29$.*

Throughout this paper, let a and m be relatively prime positive integers, the least positive integer x such that $a^x \equiv 1 \pmod m$ is called the order of a modulo m. We denote the order of a modulo m by $\mathrm{ord}_m(a)$. Let $V_p(m)$ be the exponent of the highest power of p that divides m.

2 Lemmas

The following three lemmas are motivated by the work of Luca and Sándor [3]. Here we make some minor revisions.

Lemma 3. *Let a, b be two natural numbers and p be an odd prime. If $V_p(a-1) \geq 1$, then*

$$V_p(a^b - 1) = V_p(b) + V_p(a-1).$$

Proof. Let $V_p(b) = m$ and $V_p(a-1) = n$. We may assume that $b = p^m t$ with $p \nmid t$ and $a = 1 + p^n a_0$ with $p \nmid a_0$.

Since $n \geq 1$, we have

$$a^t = (1 + p^n a_0)^t = 1 + C_t^1 p^n a_0 + \cdots + C_t^t (p^n a_0)^t = 1 + p^n c, \quad p \nmid c.$$

Thus

$$a^{tp} = (1 + p^n c)^p = 1 + C_p^1 p^n c + \cdots + C_p^p (p^n c)^p = 1 + p^{n+1} a_1, \quad p \nmid a_1.$$

By induction on m, for all $m \geq 0$ we have $a^b = a^{tp^m} = 1 + p^{m+n} a_m$ with $p \nmid a_m$. Hence

$$V_p(a^b - 1) = m + n = V_p(b) + V_p(a-1).$$

This completes the proof of Lemma 3. $\square$

Lemma 4. *Let t be a natural number and p, q be two primes. We have*

$$V_p(q^t - 1) \leq V_p(q^f - 1) + V_p(t),$$

where $f = \mathrm{ord}_p(q)$, if $p \neq 2$; and $f = 2$, if $p = 2$.

Proof. (i) $p = 2$. By [3, Lemma 1], we have

$$V_2(q^t - 1) \leq V_2(q^2 - 1) + V_2(t).$$

(ii) $p > 2$. Now consider the following two cases:

Case 1. $q^t \not\equiv 1 \pmod{p}$. The above inequality is obvious.

Case 2. $q^t \equiv 1 \pmod{p}$. Then $\mathrm{ord}_p(q) \mid t$ and $\mathrm{ord}_p(q) \mid p - 1$. Let $V_p(t) = m$. We may assume that $t = \mathrm{ord}_p(q) \cdot p^m \cdot k$ with $p \nmid k$. Thus

$$\begin{aligned} V_p(q^t - 1) &= V_p\big((q^{\mathrm{ord}_p(q)})^{p^m \cdot k} - 1\big) \\ &= V_p(q^{\mathrm{ord}_p(q)} - 1) + V_p(p^m \cdot k) \\ &= V_p(q^{\mathrm{ord}_p(q)} - 1) + m \\ &= V_p(q^{\mathrm{ord}_p(q)} - 1) + V_p(t). \end{aligned}$$

This completes the proof of Lemma 4. $\square$

Lemma 5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the standard factorization of n and $X = \max\{\alpha_j \mid j = 1, 2, \cdots, k\}$. We fix $i \in \{1, \cdots, k\}$ such that $X = \alpha_i$. If n is a Nicol number, then we have*

$$X - 1 \leq \sum_{j=1, j\neq i}^{k} V_{p_i}(p_j^{f_j} - 1) + \frac{k-1}{\log p_i} \log(X + 1),$$

where $f_j = ord_{p_i}(p_j)$, if $p_i \neq 2$; and $f_j = 2$, if $p_i = 2$.

Proof. Since $n \mid \phi(n) + \sigma(n)$ and $p_i^{X-1} \mid \phi(n)$, we have

$$p_i^{X-1} \mid \sigma(n) = \prod_{j=1}^{k} \Big(\frac{p_j^{\alpha_j+1} - 1}{p_j - 1}\Big).$$

Hence

$$p_i^{X-1} \Big| \prod_{j=1}^{k} (p_j^{\alpha_j+1} - 1).$$

The above relation implies that

$$X - 1 \leq \sum_{j=1}^{k} V_{p_i}(p_j^{\alpha_j+1} - 1) = \sum_{j=1, j\neq i}^{k} V_{p_i}(p_j^{\alpha_j+1} - 1).$$

By Lemma 4

$$
\begin{aligned}
X-1 &\leq \sum_{j=1, j\neq i}^{k} V_{p_i}(p_j^{f_j}-1) + \sum_{j=1, j\neq i}^{k} V_{p_i}(\alpha_j+1) \\
&\leq \sum_{j=1, j\neq i}^{k} V_{p_i}(p_j^{f_j}-1) + \sum_{j=1, j\neq i}^{k} \frac{\log(\alpha_j+1)}{\log p_i} \\
&\leq \sum_{j=1, j\neq i}^{k} V_{p_i}(p_j^{f_j}-1) + \frac{k-1}{\log p_i}\log(X+1).
\end{aligned}
$$

This completes the proof of Lemma 5. $\square$

Lemma 6. *If n is a 4-Nicol number and $\omega(n)=5$, then n must be one of the following three forms:*

1. $n = 2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\cdot p^{\alpha_4}\cdot q^{\alpha_5}$, *$p, q$ are distinct primes and $7 \leq p < q$.*
2. $n = 2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 7^{\alpha_3}\cdot p^{\alpha_4}\cdot q^{\alpha_5}$, *$p < q$ are distinct primes and $p \leq 29$.*
3. $n = 2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 11^{\alpha_3}\cdot 13^{\alpha_4}\cdot p^{\alpha_5}$, *$p \leq 23$ is prime.*

Proof. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}p_5^{\alpha_5}$ be the standard factorization of n. Put $l = \dfrac{\sigma(n)}{n}$.

Noting that n is a 4-Nicol number and $\varphi(n)\sigma(n) < n^2$, we have $4 = \dfrac{\varphi(n)}{n} + \dfrac{\sigma(n)}{n} < l + l^{-1}$, hence $l > 2+\sqrt{3}$. By $\dfrac{n}{\varphi(n)} > \dfrac{\sigma(n)}{n} = l$, we have

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\frac{p_3}{p_3-1}\frac{p_4}{p_4-1}\frac{p_5}{p_5-1} > l > 2+\sqrt{3}.$$

If $p_2 \geq 5$ then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\frac{p_3}{p_3-1}\frac{p_4}{p_4-1}\frac{p_5}{p_5-1} \leq 2\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12} < 2+\sqrt{3},$$

a contradiction. Thus $p_2 = 3$ and $p_1 = 2$.

If $p_3 \geq 13$ then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\frac{p_3}{p_3-1}\frac{p_4}{p_4-1}\frac{p_5}{p_5-1} \leq 2\cdot\frac{3}{2}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18} < 2+\sqrt{3},$$

a contradiction, thus $p_3 \leq 11$.

Case 1. $p_3 = 7$. Then $p_4 \leq 29$. In fact, if $p_4 \geq 31$ then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\frac{p_3}{p_3-1}\frac{p_4}{p_4-1}\frac{p_5}{p_5-1} \leq 2\cdot\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{31}{30}\cdot\frac{37}{36} < 2+\sqrt{3},$$

a contradiction.

Case 2. $p_3 = 11$. Then $p_4 = 13$ and $p_5 \leq 23$. In fact, if $p_4 \geq 17$ then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\frac{p_3}{p_3-1}\frac{p_4}{p_4-1}\frac{p_5}{p_5-1} \leq 2\cdot\frac{3}{2}\cdot\frac{11}{10}\cdot\frac{17}{16}\cdot\frac{19}{18} < 2+\sqrt{3},$$

a contradiction.

If $p_5 \geq 29$ then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\frac{p_3}{p_3-1}\frac{p_4}{p_4-1}\frac{p_5}{p_5-1} \leq 2 \cdot \frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{29}{28} < 2+\sqrt{3},$$

a contradiction.

This completes the proof of Lemma 6. $\square$

3 Proof of Theorem 1

By Lemma 6, it is enough to show that there is no 4-Nicol numbers $n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 11^{\alpha_3} \cdot 13^{\alpha_4} \cdot p^{\alpha_5}$ with $p \leq 23$.

Assume that $n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 11^{\alpha_3} \cdot 13^{\alpha_4} \cdot p^{\alpha_5}$ with $p \leq 23$ be a 4-Nicol number, then by $\dfrac{\varphi(n)}{n} + \dfrac{\sigma(n)}{n} = 4$ we have:

$$\begin{aligned}&2^{\alpha_1+6} \cdot 3^{\alpha_2+1} \cdot 5 \cdot 11^{\alpha_3-1} \cdot 13^{\alpha_4-1} \cdot p^{\alpha_5-1} \cdot (133p+10) \cdot (p-1)\\ &= (2^{\alpha_1+1}-1) \cdot (3^{\alpha_2+1}-1) \cdot (11^{\alpha_3+1}-1) \cdot (13^{\alpha_4+1}-1) \cdot (p^{\alpha_5+1}-1).\end{aligned}$$

Case 1. $p = 17$, $n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 11^{\alpha_3} \cdot 13^{\alpha_4} \cdot 17^{\alpha_5}$. Then

$$\begin{aligned}&2^{\alpha_1+10} \cdot 3^{\alpha_2+2} \cdot 5 \cdot 11^{\alpha_3-1} \cdot 13^{\alpha_4-1} \cdot 17^{\alpha_5-1} \cdot 757\\ &= (2^{\alpha_1+1}-1)(3^{\alpha_2+1}-1)(11^{\alpha_3+1}-1)(13^{\alpha_4+1}-1)(17^{\alpha_5+1}-1).\end{aligned} \tag{1}$$

By Lemma 5 we have $X \leq 35$. Noting that

$$\mathrm{ord}_{757}(2) = 756, \mathrm{ord}_{757}(3) = 9, \mathrm{ord}_{757}(11) = \mathrm{ord}_{757}(13) = \mathrm{ord}_{757}(17) = 189,$$

thus

$$757 \nmid 2^{\alpha_1+1}-1, 757 \nmid 11^{\alpha_3+1}-1, 757 \nmid 13^{\alpha_4+1}-1, 757 \nmid 17^{\alpha_5+1}-1.$$

By (1) we have $757 \mid 3^{\alpha_2+1}-1$, thus $\alpha_2+1 = 9k, k \in \mathbb{Z}$. By $X \leq 35$, we have $k = 1, 2, 3$.

Subcase 1: $k = 1$, $\alpha_2 + 1 = 9$. Then

$$\begin{aligned}&2^{\alpha_1+9} \cdot 3^{10} \cdot 5 \cdot 11^{\alpha_3-1} \cdot 13^{\alpha_4-2} \cdot 17^{\alpha_5-1}\\ &= (2^{\alpha_1+1}-1)(11^{\alpha_3+1}-1)(13^{\alpha_4+1}-1)(17^{\alpha_5+1}-1).\end{aligned}$$

(i) $\alpha_4 = 2$. By $\mathrm{ord}_{61}(13) = 3$ we have $61 \mid 13^3 - 1$, this is impossible.

(ii) $\alpha_4 > 2$. Then $13 \mid (2^{\alpha_1+1}-1)(11^{\alpha_3+1}-1)(17^{\alpha_5+1}-1)$. On the other hand, we have the following facts: If $13 \mid 2^{\alpha_1+1}-1$, by $\mathrm{ord}_{13}(2) = 12$, thus $12 \mid \alpha_1+1$, and noting that $\mathrm{ord}_7(2) = 3$ we have $7 \mid 2^{\alpha_1+1}-1$, which is impossible. If $13 \mid 11^{\alpha_3+1}-1$, by $\mathrm{ord}_{13}(11) = 12$, thus $12 \mid \alpha_3+1$, and noting that $\mathrm{ord}_7(11) = 3$ we have $7 \mid 11^{\alpha_3+1}-1$, which is impossible. If $13 \mid 17^{\alpha_5+1}-1$, by $\mathrm{ord}_{13}(17) = 6$, thus $6 \mid \alpha_5+1$, and noting that $\mathrm{ord}_7(17) = 6$ we have $7 \mid 17^{\alpha_5+1}-1$, which is impossible. Thus $13 \nmid (2^{\alpha_1+1}-1)(11^{\alpha_3+1}-1)(17^{\alpha_5+1}-1)$, a contradiction.

Subcase 2: $k=2$, $\alpha_2+1=18$. By $\mathrm{ord}_7(3)=6$, we have $7 \mid 3^{18}-1$, thus $7 \mid 3^{\alpha_2+1}-1$, which contradicts (1).

Subcase 3: $k=3$, $\alpha_2+1=27$. By $\mathrm{ord}_{757}(3)=9$, we have $757 \mid 3^{27}-1$, thus $757 \mid 3^{\alpha_2+1}-1$, which contradicts (1).

Case 2. $p=19$, $n=2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 11^{\alpha_3}\cdot 13^{\alpha_4}\cdot 19^{\alpha_5}$. Then

$$\begin{aligned} &2^{\alpha_1+7}\cdot 3^{\alpha_2+3}\cdot 5\cdot 11^{\alpha_3-1}\cdot 13^{\alpha_4-1}\cdot 19^{\alpha_5-1}\cdot 43\cdot 59\\ &=(2^{\alpha_1+1}-1)(3^{\alpha_2+1}-1)(11^{\alpha_3+1}-1)(13^{\alpha_4+1}-1)(19^{\alpha_5+1}-1). \end{aligned} \tag{2}$$

By Lemma 5 we have $X\le 33$. Noting that

$$\mathrm{ord}_{59}(2)=\mathrm{ord}_{59}(11)=\mathrm{ord}_{59}(13)=58, \mathrm{ord}_{59}(3)=\mathrm{ord}_{59}(19)=29,$$

we have

$$59\nmid 2^{\alpha_1+1}-1, 59\nmid 11^{\alpha_3+1}-1, 59\nmid 13^{\alpha_4+1}-1.$$

By (2) we have $59 \mid 3^{\alpha_2+1}-1$ or $59 \mid 19^{\alpha_5+1}-1$. If $59 \mid 3^{\alpha_2+1}-1$, then $29 \mid \alpha_2+1$. Since $\mathrm{ord}_{28537}(3)=29$, we have $28537 \mid 3^{\alpha_2+1}-1$, which contradicts with (2), thus $59\nmid 3^{\alpha_2+1}-1$. If $59 \mid 19^{\alpha_5+1}-1$, then $29 \mid \alpha_5+1$. Since $\mathrm{ord}_{233}(19)=29$, we have $233 \mid 19^{\alpha_5+1}-1$, which contradicts with (2), thus $59\nmid 19^{\alpha_5+1}-1$.

Case 3. $p=23$, $n=2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 11^{\alpha_3}\cdot 13^{\alpha_4}\cdot 23^{\alpha_5}$. Then

$$\begin{aligned} &2^{\alpha_1+7}\cdot 3^{\alpha_2+3}\cdot 5\cdot 11^{\alpha_3+1}\cdot 13^{\alpha_4-1}\cdot 23^{\alpha_5-1}\cdot 31\\ &=(2^{\alpha_1+1}-1)(3^{\alpha_2+1}-1)(11^{\alpha_3+1}-1)(13^{\alpha_4+1}-1)(23^{\alpha_5+1}-1). \end{aligned} \tag{3}$$

By Lemma 5 we have $X\le 34$. Noting that

$$\mathrm{ord}_{31}(2)=5, \mathrm{ord}_{31}(3)=\mathrm{ord}_{31}(11)=\mathrm{ord}_{31}(13)=30, \mathrm{ord}_{31}(23)=10,$$

by $\mathrm{ord}_{31}(3)=\mathrm{ord}_{31}(11)=\mathrm{ord}_{31}(13)=30$, we know that $30 \mid \alpha_i+1, i=2,3,4$. Noting that $\mathrm{ord}_{61}(3)=10, \mathrm{ord}_{19}(11)=\mathrm{ord}_{61}(13)=3$, we have $61 \mid 3^{\alpha_2+1}-1, 19 \mid 11^{\alpha_3+1}-1, 61 \mid 13^{\alpha_4+1}-1$. Which contradicts with (3), then we have

$$31\nmid 3^{\alpha_2+1}-1, 31\nmid 11^{\alpha_3+1}-1, 31\nmid 13^{\alpha_4+1}-1.$$

By (3) we know that $31 \mid 23^{\alpha_5+1}-1$ or $31 \mid 2^{\alpha_1+1}-1$.

If $31 \mid 23^{\alpha_5+1}-1$, then by $\mathrm{ord}_{31}(23)=10$, we know that $10 \mid \alpha_5+1$. Noting that $\mathrm{ord}_{41}(23)=5$ we have $41 \mid 23^{\alpha_5+1}-1$, which contradicts (3).

If $31 \mid 2^{\alpha_1+1}-1$, then $\alpha_1+1=5k, k\in\mathbb{Z}$. By $X\le 34$, we have $k=1,2,3,4,5,6$.

Subcase 1: $k=1$, $\alpha_1+1=5$, $n=2^4\cdot 3^{\alpha_2}\cdot 11^{\alpha_3}\cdot 13^{\alpha_4}\cdot 23^{\alpha_5}$. Put $m=3^{\alpha_2}\cdot 11^{\alpha_3}\cdot 13^{\alpha_4}\cdot 23^{\alpha_5}$. Then

$$\frac{\sigma(m)}{m}<\frac{m}{\varphi(m)}=\frac{3}{2}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{23}{22}=\frac{299}{160}=1.86875.$$

On the other hand, noting that $\varphi(n)+\sigma(n)=4n$, then $8\varphi(m)+31\sigma(m)=64m$, thus

$$\frac{\sigma(m)}{m}>1.9264>1.86875,$$

a contradiction.

Subcase 2: $k = 2$, $\alpha_1 + 1 = 10$. Thus

$$\begin{aligned}&2^{16} \cdot 3^{\alpha_2+2} \cdot 5 \cdot 11^{\alpha_3} \cdot 13^{\alpha_4-1} \cdot 23^{\alpha_5-1}\\ &= (3^{\alpha_2+1} - 1)(11^{\alpha_3+1} - 1)(13^{\alpha_4+1} - 1)(23^{\alpha_5+1} - 1).\end{aligned}$$

Noting that the following facts:

(i) If $2^\alpha \mid 3^{\alpha_2+1} - 1$, then $\alpha \leq 4$. In fact, if $\alpha \geq 5$, then by $\mathrm{ord}_{32}(3) = 8$, we have $\alpha_2 + 1 = 8s, s \in \mathbb{Z}$. Noting that $\mathrm{ord}_{41}(3) = 8$, thus $41 \mid 3^{\alpha_2+1} - 1$, this is impossible.

(ii) If $2^\alpha \mid 11^{\alpha_3+1} - 1$, then $\alpha \leq 3$. In fact, if $\alpha \geq 4$, then by $\mathrm{ord}_{16}(11) = 4$, we have $\alpha_3 + 1 = 4s, s \in \mathbb{Z}$. Noting that $\mathrm{ord}_{61}(11) = 4$, thus $61 \mid 11^{\alpha_3+1} - 1$, this is impossible.

(iii) If $2^\alpha \mid 13^{\alpha_4+1} - 1$, then $\alpha \leq 3$. In fact, if $\alpha \geq 4$, then by $\mathrm{ord}_{16}(13) = 4$, thus $\alpha_4 + 1 = 4s, s \in \mathbb{Z}$. Noting that $\mathrm{ord}_7(13) = 2$, thus $7 \mid 13^{\alpha_4+1} - 1$, this is impossible.

(iv) If $2^\alpha \mid 23^{\alpha_5+1} - 1$, then $\alpha \leq 4$. In fact, if $\alpha \geq 5$, then by $\mathrm{ord}_{32}(23) = 4$, we have $\alpha_5 + 1 = 4s, s \in \mathbb{Z}$. Noting that $\mathrm{ord}_{53}(23) = 4$, thus $53 \mid 23^{\alpha_5+1} - 1$, this is impossible.

Let

$$\begin{aligned}A &= (3^{\alpha_2+1} - 1)(11^{\alpha_3+1} - 1)(13^{\alpha_4+1} - 1)(23^{\alpha_5+1} - 1),\\ B &= 2^{16} \cdot 3^{\alpha_2+2} \cdot 5 \cdot 11^{\alpha_3} \cdot 13^{\alpha_4-1} \cdot 23^{\alpha_5-1}.\end{aligned}$$

We have $V_2(A) \leq 14$ and $V_2(B) = 16$, this is impossible.

Subcase 3: $k = 3$, $\alpha_1 + 1 = 15$. By $\mathrm{ord}_7(2) = 3$, we have $7 \mid 2^{\alpha_1+1} - 1$, which contradicts (3).

Subcase 4: $k = 4$, $\alpha_1 + 1 = 20$. By $\mathrm{ord}_{41}(2) = 20$, we have $41 \mid 2^{\alpha_1+1} - 1$, which contradicts (3).

Subcase 5: $k = 5, \alpha_1 + 1 = 25$. By $\mathrm{ord}_{601}(2) = 25$, we have $601 \mid 2^{\alpha_1+1} - 1$, which contradicts (3).

Subcase 6: $k = 6, \alpha_1 + 1 = 30$. By $\mathrm{ord}_{151}(2) = 15$, we have $151 \mid 2^{\alpha_1+1} - 1$, which contradicts (3).

This completes the proof of Theorem 1.

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