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Some Properties of the Multiple Binomial Transform and the Hankel Transform of Shifted Sequences

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Abstract

In this paper, the author studies the multiple binomial transform and the Hankel transform of shifted sequences of an integer sequence, particularly a linear homogeneous recurrence sequence, and some of their properties.

1 Notation

In this paper, we generally use function symbols, like $a(t)$, $b(t)$, etc., to express integer sequences, where $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. However sometimes, to employ matrix tools in deduction process, we also denote the integer sequences by using (infinite-dimensional) vector symbols, like $a = (a(0), a(1), a(2), a(3), \cdots, \cdots)^T$, $b = (b(0), b(1), b(2), b(3), \cdots, \cdots)^T$, etc.

2 Multiple binomial transforms of shifted sequences

Definition 1 (Shifting integer sequences)**.** Let $a(t)$ be an integer sequence and σ be the shift operator. Then we define *the pth-order shifted sequence* $a_{(p)}(t))$, $(p = 0, 1, 2, \ldots)$, of $a(t)$, as follows:

$$a_{(p)}(t) = \sigma^p(a) = a(t + p), \qquad t = 0, 1, 2, \ldots, \tag{1}$$

Note that in the case $p = 0$, $a_{(0)}(t) = \sigma^0(a) = a(t)$.

Definition 2 (Multiple binomial transforms)**.** Let $a(t)$ be an integer sequence. Then according to Pan [1], we define *the n-fold binomial transform* of $a(t)$, and denote its image sequence by $\mathcal{B}_n(a)$ or $a^{(n)}(t)$, as follows:

$$a^{(1)}(t) = \mathcal{B}_1(a) = \sum_{k=0}^{t} \binom{t}{k} a(k), \qquad a^{(n)}(t) = \mathcal{B}_n(a) = \overbrace{\mathcal{B}_1(\mathcal{B}_1(\cdots(\mathcal{B}_1}^{n-fold}(a)))), \tag{2}$$

where $n = 0, 1, 2, \ldots$. Note that in the case $n = 0$, $\mathcal{B}_0(a) = a^{(0)}(t) = a(t)$, that is, the transform $\mathcal{B}_0$ just is the identity transform.

Definition 3 (Inverse multiple binomial transform)**.** Let $a(t)$ be an integer sequence. Then according to Pan [1], we define *the m-fold inverse binomial transform* of $a(t)$, and denote its image sequence by $\mathcal{B}_{-m}(a)$ or $a^{(-m)}(t)$, as follows:

$$a^{(-1)}(t) = \mathcal{B}_{-1}(a) = \sum_{k=0}^{t} (-1)^{t-k} \binom{t}{k} a(k), \quad a^{(-m)}(t) = \mathcal{B}_{-m}(a) = \overbrace{\mathcal{B}_{-1}(\mathcal{B}_{-1}(\cdots(\mathcal{B}_{-1}}^{m-fold}(a)))), \tag{3}$$

where $m = 1, 2, \ldots$.

Remark 4. We can express (2) in the matrix form: $a^{(1)} = B_1 a$, where the transform matrix B_1 is an infinite-order lower-triangular matrix, as follows:

$$B_1 = \begin{pmatrix} \binom{0}{0} & & & & \\ \binom{1}{0} & \binom{1}{1} & & & \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4}$$

and

$$a^{(n)} = (a^{(n)}(0), a^{(n)}(1), a^{(n)}(2), \cdots, \cdots)^T = B_n a = B_1^n a, \tag{5}$$

where $n = 0, 1, 2, 3, \ldots$. The transform matrix of the n-fold binomial transform B_n $(= B_1^n)$ is always a lower-triangular transform matrix with each of the diagonal elements being one.

Remark 5. We can also express (3) in matrix form, as $a^{(-1)} = B_{-1} a$, where the transform matrix B_{-1} is an infinite-order lower-triangular matrix, as

$$B^{-1} = \begin{pmatrix} \binom{0}{0} & & & & \\ -\binom{1}{0} & \binom{1}{1} & & & \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & & \\ -\binom{3}{0} & \binom{3}{1} & -\binom{3}{2} & \binom{3}{3} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6}$$

and

$$a^{(-m)} = (a^{(-m)}(0), a^{(-m)}(1), a^{(-m)}(2), \cdots, \cdots)^T = B_{-m} a = B_{-1}^m a, \tag{7}$$

where $m = 1, 2, 3, \ldots$. The transform matrix B_{-m} $(= B_{-1}^m)$ is also always a lower-triangular transform matrix with each of the diagonal elements being one. We see that $B_1 B_{-1} = B_{-1} B_1 = E$, where E is the infinite-order unit matrix. It is the matrix form of well-known inversion relation: $\sum_{k=i}^{t} (-1)^{t-k} \binom{t}{k} \binom{k}{i} = \sum_{k=i}^{t} (-1)^{k-i} \binom{t}{k} \binom{k}{i} = \delta_{ti}$, where $t, i = 0, 1, 2, \ldots$.

Remark 6. We view the n-fold binomial or inverse binomial transform $\mathcal{B}_n$, $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$, to be one simple transform of integer sequences, because such inversion relations as $B_2 B_{-2} = B_{-2} B_2 = E$, $B_3 B_{-3} = B_{-3} B_3 = E$ hold, and so forth. For example, for 2-fold binomial and inverse binomial transforms, the transform matrices are respectively

$$B_2 = \begin{pmatrix} 1 & & & & \\ 2 & 1 & & & \\ 4 & 4 & 1 & & \\ 8 & 12 & 6 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad B_{-2} = \begin{pmatrix} 1 & & & & \\ -2 & 1 & & & \\ 4 & -4 & 1 & & \\ -8 & 12 & -6 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{8}$$

Now, let us give the multiple binomial transforms of the shifting sequences $a_{(p)}(t)$, $(p = 0, 1, 2, \ldots)$, of an integer sequence $a(t)$.

Theorem 7. *Let $a(t)$ be an integer sequence. Then*

$$\mathcal{B}_n(a_{(p)}) = (\sigma - n)^p (\mathcal{B}_n(a)) = (\sigma - n)^p (a^{(n)}) = \sum_{k=0}^{p} (-n)^{p-k} \binom{p}{k} \sigma^k (a^{(n)}), \tag{9}$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Proof. Use the mathematical induction. When $n = \pm 1$ and $p = 1$,

$$\mathcal{B}_1(\sigma(a)) = \sum_{k=0}^{t} \binom{t}{k} a(k+1) = \sum_{k=1}^{t+1} \binom{t}{k-1} a(k) = \sum_{k=1}^{t+1} \binom{t+1}{k} a(k) - \sum_{k=1}^{t+1} \binom{t}{k} a(k)$$
$$= \sum_{k=0}^{t+1} \binom{t+1}{k} a(k) - a(0) - [\sum_{k=0}^{t} \binom{t}{k} a(k) - a(0)] = \sigma(\mathcal{B}_1(a)) - \mathcal{B}_1(a) = (\sigma - 1)(\mathcal{B}_1(a)),$$

and

$$\mathcal{B}_{-1}(\sigma(a)) = \sum_{k=0}^{t} (-1)^{t-k} \binom{t}{k} a(k+1) = \sum_{k=1}^{t+1} (-1)^{t+1-k} \binom{t}{k-1} a(k)$$
$$= \sum_{k=0}^{t+1} (-1)^{t+1-k} \left[\binom{t+1}{k} - \binom{t}{k} \right] a(k) = \sum_{k=0}^{t+1} (-1)^{t+1-k} \binom{t+1}{k} a(k) + \sum_{k=0}^{t} (-1)^{t-k} \binom{t}{k} a(k)$$
$$= \sigma(\mathcal{B}_{-1}(a)) + \mathcal{B}_{-1}(a) = (\sigma + 1)(\mathcal{B}_{-1}(a)).$$

If for $n = \pm k$(k is some positive integer), $\mathcal{B}_{\pm k}(\sigma(a)) = (\sigma \mp k)(\mathcal{B}_{\pm k}(a))$ holds, then for $n = \pm(k+1)$, $\mathcal{B}_{\pm(k+1)}(\sigma(a)) = \mathcal{B}_{\pm 1}(\sigma(\mathcal{B}_{\pm k}(a))) \mp k\mathcal{B}_{\pm 1}(\mathcal{B}_{\pm k}(a)) = (\sigma \mp 1)(\mathcal{B}_{\pm(k+1)}(a)) \mp k\mathcal{B}_{\pm(k+1)}(a) = (\sigma \mp (k+1))(\mathcal{B}_{\pm(k+1)}(a))$ also holds. Hence, for any integer n, $\mathcal{B}_n(\sigma(a)) =$

$(\sigma - n)(\mathcal{B}_n(a))$ holds. On the other hand, if for $p = m$(m is some positive integer) that $\mathcal{B}_n(\sigma^m(a)) = (\sigma - n)^m(\mathcal{B}_n(a))$ holds, then when $p = m+1$, we get that $\mathcal{B}_n(\sigma^{m+1}(a)) = (\sigma - n)^m(\mathcal{B}_n(\sigma(a))) = (\sigma - n)^m((\sigma - n)(\mathcal{B}_n(a))) = (\sigma - n)^{m+1}(\mathcal{B}_n(a))$. Hence, for any positive integer n and p, $\mathcal{B}_n(\sigma^p(a)) = (\sigma - n)^p(\mathcal{B}_n(a))$. Special cases that $n = 0$ and/or $p = 0$ are trivial. □

Corollary 8. *Let $a(t)$ be an integer sequence, and $P(\sigma)$ be an integer-coefficient polynomial in σ. Then*

$$\mathcal{B}_n(P(\sigma)(a)) = P(\sigma - n)(\mathcal{B}_n(a)) = P(\sigma - n)(a^{(n)}), \tag{10}$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Proof. Let $P(\sigma)$ be a integer-coefficient polynomial of degree p ($p = 0, 1, 2, \ldots$) in σ: $P(\sigma) = \sum_{k=0}^{p} c_k \sigma^k$, where c_ks are $(p+1)$ integers. From Theorem 7, we have that $\mathcal{B}_n(P(\sigma)(a)) = \mathcal{B}_n(\sum_{k=0}^{p} c_k \sigma^k(a)) = \sum_{k=0}^{p} c_k \mathcal{B}_n(\sigma^k(a)) = \sum_{k=0}^{p} c_k (\sigma - n)^k(\mathcal{B}_n(a)) = P(\sigma - n)(\mathcal{B}_n(a)) = P(\sigma - n)(a^{(n)})$. □

Remark 9. By using Corollary 8, we can more succinctly prove the following known property of recurrence sequences (see [1, Thm. 17]). Let $a(t)$ be a linear homogeneous recurrence sequence of order q with the recurrence equation

$$P(\sigma)(a) = \sum_{k=0}^{q} b_k \sigma^{q-k}(a) = 0, \tag{11}$$

where $b_0 = 1$, $b_1, b_2, \ldots, b_q$ are q given integers. Then its q complex characteristic values λ_k, $k = 1, 2, \ldots, q$, are the roots of polynomial (algebraic) equation:

$$P(\lambda) = \sum_{k=0}^{q} b_k \lambda^{q-k} = 0. \tag{12}$$

On the other hand, by taking transformation $\mathcal{B}_n$ of the two sides of (11), and then employing Corollary 8, we find that sequences $a^{(n)}(t)$, $(n = 0, \pm 1, \pm 2, \ldots)$, satisfy recurrence equation:

$$P(\sigma - n)(a^{(n)}) = 0. \tag{13}$$

This implies that q complex characteristic values $\lambda_k^{(n)}$, $(k = 1, 2, \ldots, q)$, of $a^{(n)}(t)$ are the roots of the algebraic equation:

$$P(\lambda^{(n)} - n) = \sum_{k=0}^{q} b_k (\lambda^{(n)} - n)^{q-k} = 0. \tag{14}$$

Comparing (12) with (14), we find that $\lambda_k^{(n)} - n = \lambda_k$, namely

$$\lambda_k^{(n)} = \lambda_k + n, \quad (k = 1, 2, \ldots, q). \tag{15}$$

3 Shifted sequences and the Hankel transform

Layman proved the invariance of the Hankel transform under applications of the binomial transform or its inverse transform (see [2]). For an integer sequence, the n-fold binomial (or inverse binomial) transform is the same as the n times successive binomial (or inverse binomial) transform operation, Pan [1] pointed out that the invariance of the Hankel transform holds under applications of the n-fold binomial (or n-fold invert binomial) transform. Now by using Theorem 7, we give a more direct and succinct proof of the invariance, as follows.

Remark 10. By using Definition 1, we express the Hankel matrix H_a of sequence $a(t)$ as

$$H_a = \begin{pmatrix} a & \sigma(a) & \sigma^2(a) & \sigma^3(a) & \cdots \end{pmatrix} = \begin{pmatrix} a & a_{(1)} & a_{(2)} & a_{(3)} & \cdots \end{pmatrix}, \tag{16}$$

and Hankel matrix $H_{a^{(n)}}$ of integer sequence $a^{(n)}(t)$ as

$$H_{a^{(n)}} = \begin{pmatrix} a^{(n)} & \sigma(a^{(n)}) & \sigma^2(a^{(n)}) & \sigma^3(a^{(n)}) & \cdots \end{pmatrix}, \tag{17}$$

According to Theorem 7, we have that

$$\begin{aligned} B_nH_a &= \begin{pmatrix} B_na & B_na_{(1)} & B_na_{(2)} & B_na_{(3)} & \cdots \end{pmatrix} \\ &= \begin{pmatrix} a^{(n)} & (\sigma-n)(a^{(n)}) & (\sigma-n)^2(a^{(n)}) & (\sigma-n)^3(a^{(n)}) & \cdots \end{pmatrix}. \end{aligned} \tag{18}$$

Comparing (18) with (17), we see that the upper-left $(t+1)\times(t+1)$ $(t=0,1,2,\ldots)$ sub-matrix of B_nH_a has the same determinant to the upper-left sub-matrix of the Hankel matrix $H_{a^{(n)}}$ of sequence $a^{(n)}(t)$. On the other hand, the determinant of the upper-left $(t+1)\times(t+1)$ $(t=0,1,2,\ldots)$ sub-matrix of matrix B_nH_a is equal to the determinant of the upper-left $(t+1)\times(t+1)$ $(t=0,1,2,\ldots)$ sub-matrix of matrix H_a, because the determinant of any upper-left sub-matrices of matrix B_n $(n=\pm1,\pm2,\pm3,\ldots)$ is always equal to one. In other words, the sequences a and $a^{(n)}$ both have the same Hankel transform, for any integer n.

Remark 11. This result gives an affirmative answer to one of Layman's two questions raised in [2]: *Are there other interesting transforms, T, of an integer sequence S, in addition to the Binomial and Invert transforms, with the property that the Hankel transform of S is the same as the Hankel transform of the T transform of S?* For example, $\mathcal{T}=\mathcal{B}_2$ or $\mathcal{B}_{-2}$, which have transform matrices listed in (8).

Next, we investigate the Hankel transform of recurrence sequences. The following theorem gives a basic property of the Hankel transform of recurrence sequences.

Theorem 12. *Let $a(t)$ be a linear homogeneous recurrence sequence of order q, with recurrence equation (11). Then the Hankel transform $h_a(t)$ of sequence $a(t)$ is a finite sequence with length q, that is, for $t \geq q$, $h_a(t)\equiv 0$.*

Proof. We see from (16) and (11) that if multiplying the first, the second, …, the q-th column vectors of the Hankel matrix H_a by b_q, b_{q-1}, …, b_1 respectively, and then adding them to the $(q+1)$th column $\sigma^q(a)$, we cause the $(q+1)$-th column to be a zero-column. This operation does not change the determinants of principal sub-matrices of H_a. On the other hand, for a infinite-order square matrix with its $(q+1)$-th column being a zero-column, determinants of the principal sub-matrices of order $q+1$, $q+2$, $q+3$, …, namely $h(q)$, $h(q+1)$, $h(q+2)$, …, are always equal to zeros. That is, the Hankel transform $h(t)$ is a finite integer sequence with the length of q. □

Corollary 13. *All of the n-fold binomial transforms $a^{(n)}(t)$ $(n = 0, \pm1, \pm2, \pm3, \ldots)$ of a q-order recurrence sequence $a(t)$ have identical Hankel transform with the length of q.*

Remark 14. For example, as recurrence sequences of order 2 and 3, the Fibonacci sequence $F(t)$ (A000045 in [3]) and its multiple binomial transforms A001906, A093131, A039834, etc. (see Pan [1]) all have the same Hankel transform with length 2: $h_F(0) = 1$, $h_F(1) = 1$, and the Tribonacci sequence $T(t)$ (A000073 in [3]) and its multiple binomial transforms A115390, etc. (see Pan [1]) all have the same Hankel transform with length 3: $h_T(0) = 3$, $h_T(1) = 8$, $h_T(2) = -44$.

Finally, we give special relations of the Hankel transforms of $a^{(n)}(t)$, $(n = 0, \pm1, \pm2, \ldots)$, and $a_{(p)}(t)$, $(p = 0, 1, 2, \ldots)$, with the general term formula of the recurrent sequences $a(t)$, respectively.

Theorem 15. *Let $a(t)$ be a linear homogeneous recurrence sequence of order q, with the general-term formula: $a(t) = \sum_{i=1}^{q} c_i\lambda_i^t$, $t \in \mathbb{N}_0$. Then the Hankel transforms $h_{a^{(n)}}(t)$, $(n = 0, \pm1, \pm2, \ldots)$, are such that*

$$h_{a^{(n)}}(t) = \sum_{(i_1,i_2,\cdots,i_{t+1})} \prod_{k=1}^{t+1}(c_{i_k}\lambda_{i_k}^{k-1}) \prod_{1\le k<m\le(t+1)} (\lambda_{i_k} - \lambda_{i_m}), \quad t = 0, 1, \ldots, q-1, \tag{19}$$

where the summation is over the $q!/(q-t-1)!$ different $(t+1)$-permutations $(i_1, i_2, \cdots, i_{t+1})$ of set $\{1, 2, \ldots, q\}$. Particularly, the first term $h_{a^{(n)}}(0) = \sum_{i=1}^{q} c_i = a(0)$, and the qth (last) term $h_{a^{(n)}}(q-1) = \prod_{i=1}^{q} c_i \prod_{1\le i<j\le q}(\lambda_i - \lambda_j)^2$.

Proof. Denoting j-order vectors $(1, \lambda_i, \lambda_i^2, \cdots, \lambda_i^{j-1})$ by $\lambda(i, j)$, and $(j \times j)$ Vandermonde square-matrices $\big(\lambda(i_1, j), \lambda(i_2, j), \ldots, \lambda(i_j, j)\big)$ by $\mathbb{V}(i_1, i_2, \cdots, i_j)$ respectively, where $i \in \{1, 2, \ldots, q\}$, and $(i_1, i_2, \cdots, i_j)$ is a j-permutation of set $\{1, 2, \ldots, q\}$, $(1 \le j \le q)$, we find that the t-th term of Hankel transform $h_a(t)$ of $a(t)$, that is, the determinant of upper-left $(t+1) \times (t+1)$ sub-matrix of Hankel matrix (16), is

$$h_a(t) = \det\left[\ \sum_{i=1}^{q} c_i\lambda(i, t+1) \quad \sum_{i=1}^{q} c_i\lambda_i\lambda(i, t+1) \quad \cdots \quad \sum_{i=1}^{q} c_i\lambda_i^t\lambda(i, t+1)\ \right]$$

$$= \sum_{(i_1,i_2,\cdots,i_{(t+1)})} \Big(\prod_{k=1}^{t+1}(c_{i_k}\lambda_{i_k}^{k-1})\Big) \det\mathbb{V}(i_1, i_2, \cdots, i_{t+1}),$$

where the summation is over $q!/(q-t-1)!$ different $(t+1)$-permutations $(i_1, i_2, \cdots, i_{t+1})$ of set $\{1, 2, \ldots, q\}$. The Vandermonde determinant $\det\mathbb{V}(i_1, i_2, \cdots, i_{t+1})$ equals $\prod_{1\le k<m\le(t+1)}(\lambda_{i_k} - \lambda_{i_m})$. Because $h_{a^{(n)}}(t) = h_a(t)$, (19) holds. In case $t = 0$, we see that $h_{a^{(n)}}(0) = h_a(0) = \sum_{i=1}^{q} c_i = a(0)$; in the case $t = q-1$, we have that

$$h_{a^{(n)}}(q-1) = h_a(q-1) = \det\left[\ \sum_{i=1}^{q} c_i\lambda(i, q) \quad \sum_{i=1}^{q} c_i\lambda_i\lambda(i, q) \quad \cdots \quad \sum_{i=1}^{q} c_i\lambda_i^{q-1}\lambda(i, q)\ \right],$$

The matrix in the right side of the above equality just equals a product of three square matrices: $\mathbb{V}(1, 2, \cdots, q) \cdot \mathrm{diag}\{c_1, c_2, \ldots, c_q\} \cdot \mathbb{V}^T(1, 2, \cdots, q)$. Hence, we have that

$$h_{a^{(n)}}(q-1) = \det\mathbb{V}(1, \cdots, q) \times \det\mathrm{diag}\{c_1, \ldots, c_q\} \times \det\mathbb{V}^T(1, \cdots, q) = \prod_{i=1}^{q} c_i \prod_{1\le i<j\le q} (\lambda_i - \lambda_j)^2$$

$\square$

Theorem 16. *Let $a(t)$ be a linear homogeneous recurrence sequence of order q, with a general-term formula: $a(t) = \sum_{i=1}^{q} c_i \lambda_i^t$, $t \in \mathbb{N}_0$. Then the Hankel transform $h_{a_{(p)}}(t)$ of the shifted sequence $a_{(p)}$, $(p = 0, 1, 2, \ldots)$, of sequence $a(t)$ are given by*

$$h_{a_{(p)}}(t) = \sum_{(i_1, i_2, \cdots, i_{t+1})} \prod_{k=1}^{t+1} (c_{i_k} \lambda_{i_k}^{k-1+p}) \prod_{1 \le k < m \le (t+1)} (\lambda_{i_k} - \lambda_{i_m}), \quad t = 0, 1, \ldots, q-1, \tag{20}$$

where summarizing is over $q!/(q-t-1)!$ different $(t+1)$-permutations $(i_1, i_2, \cdots, i_{t+1})$ of set $\{1, 2, \ldots, q\}$. Particularly, the first term $h_{a_{(p)}}(0) = \sum_{i=1}^{q} c_i \lambda_i^p$, and the q-th (last) term $h_{a_{(p)}}(q-1) = \prod_{i=1}^{q} (c_i \lambda_i^p) \prod_{1 \le i < j \le q} (\lambda_i - \lambda_j)^2$.

Proof. The general term of $a_{(p)}(t)$ is $a_{(p)}(t) = \sum_{i=1}^{q} c_i \lambda_i^{t+p} = \sum_{i=1}^{q} d_i \lambda_i^t$, $t \in \mathbb{N}_0$, where $d_i = c_i \lambda_i^p$ $(i = 1, 2, \ldots, q)$. We see from Theorem 15 that the Hankel transform $h_{a_{(p)}}(t)$ of sequence $a_{(p)}$ (note that it is also a recurrence sequence of order q) is

$$h_{a_{(p)}}(t) = \sum_{(i_1, i_2, \cdots, i_{t+1})} \prod_{k=1}^{t+1} (d_{i_k} \lambda_{i_k}^{k-1}) \prod_{1 \le k < m \le (t+1)} (\lambda_{i_k} - \lambda_{i_m}),$$

where summarizing is over $q!/(q-t-1)!$ different $(t+1)$-permutations $(i_1, i_2, \cdots, i_{t+1})$ of set $\{1, 2, \ldots, q\}$. Replacing $d_1, d_2, \ldots, d_q$ by $c_1 \lambda_1^p, c_2 \lambda_2^p, \ldots, c_1 \lambda_q^p$ respectively, we obtain (20). From Theorem 15, we obtain that $h_{a_{(p)}}(0) = \sum_{i=1}^{q} d_i = \sum_{i=1}^{q} c_i \lambda_i^p$, and

$$h_{a_{(p)}}(q-1) = \prod_{i=1}^{q} d_i \prod_{1 \le i < j \le q} (\lambda_i - \lambda_j)^2 = \prod_{i=1}^{q} (c_i \lambda_i^p) \prod_{1 \le i < j \le q} (\lambda_i - \lambda_j)^2$$

. ∎

Remark 17. We take the generalized Lucas sequence $s(t) = 3, 1, 3, 7, 11, 21, 39, \ldots$ (sequence A001644 in [3]) as an example used for verification. The third order recurrent sequence has a general term formula that $s(t) = \lambda_1^t + \lambda_2^t + \lambda_3^t$ (Note that $c_1 = c_2 = c_3 = 1$), where three characteristic values λ_i $(i = 1, 2, 3)$ are the roots of algebraic equation $\lambda^3 - \lambda^2 - \lambda - 1 = 0$. They are that

$$\lambda_1 = \frac{1}{3}(1 + \alpha + \beta), \quad \lambda_2 = \frac{1}{3}(1 + \omega_1 \alpha + \omega_2 \beta), \quad \lambda_2 = \frac{1}{3}(1 + \omega_2 \alpha + \omega_1 \beta).$$

where two real numbers $\alpha = \sqrt[3]{19 + \sqrt{297}}$, $\beta = \sqrt[3]{19 - \sqrt{297}}$; and $1, \omega_1, \omega_2$ are three complex cubic roots of 1. Hence, noting that $\omega_1 + \omega_2 = -1$ and $\omega_1 \omega_2 = 1$, we get that the Hankel transform of $s(t)$ (and any of its multiple binomial transforms) has the three terms:

$$h_s(0) = c_1 + c_2 + c_3 = 1 + 1 + 1 = 3,$$

$$\begin{aligned} h_s(1) &= c_1 c_2 \lambda_2 (\lambda_2 - \lambda_1) + c_2 c_1 \lambda_1 (\lambda_1 - \lambda_2) + c_1 c_3 \lambda_3 (\lambda_3 - \lambda_1) + c_3 c_1 \lambda_1 (\lambda_1 - \lambda_3) + c_2 c_3 \lambda_3 (\lambda_3 - \lambda_2) \\ &\quad + c_3 c_2 \lambda_2 (\lambda_2 - \lambda_3) = (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2 = 2\alpha\beta = 8, \end{aligned}$$

$$\begin{aligned} h_s(2) &= c_1 c_2 c_3 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2 = (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2 \\ &= -\frac{1}{27} (\alpha^2 + \beta^2 + \alpha\beta)^2 (\alpha - \beta)^2 = -\frac{1}{27} (\alpha^3 + \beta^3 + 16)(\alpha^3 + \beta^3 - 16) = -44. \end{aligned}$$

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