Journal of Integer Sequences, Vol. 14 (2011),
Article 11.1.1

The r-Bell Numbers

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Abstract

The notion of r-Stirling numbers implies the definition of generalized Bell (or r-Bell) numbers. The r-Bell numbers have appeared in several works, but there is no systematic treatise on this topic. In this paper we fill this gap. We discuss the most important combinatorial, algebraic and analytic properties of these numbers, which generalize similar properties of the Bell numbers. Most of these results seem to be new. It turns out that in a paper of Whitehead, these numbers appeared in a very different context. In addition, we study the so-called r-Bell polynomials.

1 Introduction

The Bell number B_n [11] counts the partitions of a set with n elements. The Stirling number with parameters n and k, denoted by $\left\{ {n \atop k} \right\}$, enumerates the number of partitions of a set with n elements consisting k disjoint, nonempty sets. We get immediately that B_n can be given by the sum

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}. \tag{1}$$

The numbers $\left\{ {n \atop k} \right\}$ are also called as Stirling partition numbers. The n-th Bell polynomial is

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k.$$

These numbers and polynomials have many interesting properties and appear in several combinatorial identities. A comprehensive paper is [11].

A more general notion can be introduced. The r-Stirling number of the second kind with parameters $n \geq k \geq r$ enumerates the partitions of a set of n elements into k nonempty, disjoint subsets such that the first r elements are in distinct subsets. It is denoted by $\left\{ {n \atop k} \right\}_r$. A systematic treatment on the r-Stirling numbers is given in [4], and a different approach is described in [6, 7]. According to (1), it seems to be natural to define the numbers

$$B_{n,r} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r. \tag{2}$$

(It is obvious that $B_n = B_{n,0}$, because $\left\{ {n \atop k} \right\} = \left\{ {n \atop k} \right\}_0$ by the definitions.)

The very first question is on the meaning of the r-Bell numbers. By (2), $B_{n,r}$ is the number of the partitions of a set with $n+r$ element such that the first r elements are in distinct subsets in each partition.

The name of r-Stirling numbers suggests the name for the numbers $B_{n,r}$: we call them as r-Bell numbers, and the name of the polynomials

$$B_{n,r}(x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k$$

will be r-Bell polynomials (see also the title of [12]). Thus $B_{n,r} = B_{n,r}(1)$ and $B_{n,0}(x) = B_n(x)$, the ordinary Bell polynomial.

1.1 Some elementary facts about the r-Bell polynomials

Actually, the coefficients of $B_{n,r}(x)$ are polynomials in r, since

$$\left\{ {n+r \atop k+r} \right\} = \sum_{i=0}^{n} \binom{n}{i} \left\{ {i \atop k} \right\} r^{n-i}. \tag{3}$$

That is,

$$B_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{i=0}^{n} \binom{n}{i} \left\{ {i \atop k} \right\} r^{n-i} \right) x^k. \tag{4}$$

The equality (3) can be proven easily: $\left\{ {n+r \atop k+r} \right\}_r$ enumerates the $(k+r)$-partitions of $n+r$ elements such that the first r elements are in distinct subsets. The number of such partitions can be enumerated in the following way. We separate $1, \ldots, r$ into singletons, and we create k additional blocks to have $k+r$ blocks. To fill the k blocks, we choose i elements from $\{r+1, \ldots, n+r\}$ into them. This can happen $\binom{n}{i}$ way. We can construct $\left\{ {i \atop k} \right\}$ different k-partitions from these elements. The remaining $n-i$ elements from $\{r+1, \ldots, n+r\}$ go beside the first r elements. We may choose these blocks independently, so we have r^{n-i} possibilities. Finally we sum on i.

A consequence is that the r-Bell polynomials can be expressed by the Bell polynomials:

$$B_{n,r}(x) = \sum_{k=0}^{n} r^k \binom{n}{k} B_{n-k}(x).$$

To see the validity of this identity, just change the order of the summations in (4).

As far as we know, this paper is the first one fully devoted to the r-Bell numbers, although Carlitz [6, 7] defined these numbers and proved some identities for them. His original notation was $B(n,r)$ such that

$$B_{n,r} = B(n,r).$$

2 Example and tables

The following example illuminates again the meaning of the r-Bell numbers. By definition,

$$B_{2,2} = \left\{ {4 \atop 2} \right\}_2 + \left\{ {4 \atop 3} \right\}_2 + \left\{ {4 \atop 4} \right\}_2.$$

$\left\{ {4 \atop 2} \right\}_2$ counts the partitions of 4 element into 2 subsets such that the first 2 element are in distinct subsets:

$$\{1,3,4\},\{2\} \quad ; \quad \{1\},\{2,3,4\} \quad ; \quad \{1,3\},\{2,4\} \quad ; \quad \{1,4\},\{2,3\}.$$

$\left\{ {4 \atop 3} \right\}_2$ belongs to the partitions

$$\{1\},\{2\},\{3,4\} \quad ; \quad \{1,3\},\{2\},\{4\} \quad ; \quad \{1,4\},\{2\},\{3\} \quad ;$$

$$\{1\},\{2,3\},\{4\} \quad ; \quad \{1\},\{2,4\},\{3\}.$$

Finally, $\left\{ {4 \atop 4} \right\}_2$ equals to the number of partitions of 4 elements into 4 subsets (and necessarily, the first two elements are in distinct subsets):

$$\{1\},\{2\},\{3\},\{4\}.$$

That is,

$$B_{2,2} = \left\{ {4 \atop 2} \right\}_2 + \left\{ {4 \atop 3} \right\}_2 + \left\{ {4 \atop 4} \right\}_2 = 4 + 5 + 1 = 10$$

is the number of partitions of the set $\{1,2,3,4\}$ such that the first two elements are in distinct subsets.

3 Generating functions

We start to derive the properties of r-Bell numbers and polynomials. First of all, the generating functions are determined.

Figure 1: The first few r-Bell numbers

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$r=0$	1	1	2	5	15	52	203
$r=1$	1	2	5	15	52	203	877
$r=2$	1	3	10	37	151	674	3263
$r=3$	1	4	17	77	372	1915	10481
$r=4$	1	5	26	141	799	4736	29371
$r=5$	1	6	37	235	1540	10427	73013
$r=6$	1	7	50	365	2727	20878	163967

Figure 2: The first few r-Bell polynomials

$$
\begin{aligned}
B_{0,r}(x) &= 1 \\
B_{1,r}(x) &= x+r \\
B_{2,r}(x) &= x^2+(2r+1)x+r^2 \\
B_{3,r}(x) &= x^3+(3r+3)x^2+(3r^2+3r+1)x+r^3 \\
B_{4,r}(x) &= x^4+(4r+6)x^3+(6r^2+12r+7)x^2+ \\
&\quad (4r^3+6r^2+4r+1)x+r^4
\end{aligned}
$$

Theorem 3.1. The exponential generating function for the r-Bell polynomials is

$$\sum_{n=0}^{\infty} B_{n,r}(x)\frac{z^n}{n!} = e^{x(e^z-1)+rz}.$$

Proof. Broder [4] gave the double generating function of r-Stirling numbers

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{ {n+r \atop k+r} \right\} x^k\right)\frac{z^n}{n!} = e^{x(e^z-1)+rz}.$$

The inner sum is exactly our polynomial $B_{n,r}(x)$. We note that this identity is remarked in [6, eq. (3.19)] □

We remark that the non-polynomial version was proven by Carlitz [6, eq. (3.18)].

In order to determine the ordinary generating function we need some other notions. The falling factorial of a given real number x is denoted and defined by

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1), \quad (n = 1, 2, \dots) \tag{5}$$

and $(x)^{\underline{0}} = 1$, while the rising factorial (a.k.a. Pochhammer symbol) is

$$(x)_n \equiv x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1) \quad (n = 1, 2, \dots) \tag{6}$$

with $(x)_0 = 1$. It is obvious that $(1)_n = n!$. Fitting our notations to the theory of hypergeometric functions defined below, we apply the notation $(x)_n$ instead of $x^{\overline{n}}$. The next transformation formula holds

$$x^{\underline{n}} = (-1)^n(-x)_n. \tag{7}$$

The hypergeometric function (or hypergeometric series) is defined by

$${}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{array} \middle| \, t \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{t^k}{k!}.$$

The ordinary generating function of $B_{n,r}(x)$ can be given by this function.

Theorem 3.2. The r-Bell polynomials have the generating function

$$\sum_{n=0}^{\infty} B_{n,r}(x)z^n = \frac{-1}{rz-1}\frac{1}{e^x}\,{}_1F_1\left(\begin{array}{c} \frac{rz-1}{z} \\ \frac{rz+z-1}{z} \end{array} \middle| \, x \right).$$

Proof. It is known [4] that for the Stirling numbers

$$\sum_{n=0}^{\infty} \left\{ {n \atop m} \right\}_r z^n = \frac{z^m}{(1-rz)(1-(r+1)z)\cdots(1-mz)} \quad (m \geq r \geq 0).$$

This can be rewritten as

$$\sum_{n=m}^{\infty} \left\{ {n+r \atop m+r} \right\}_r z^n = \frac{z^m}{(1-rz)(1-(r+1)z)\cdots(1-(m+r)z)}.$$

We transform the denominator using the falling factorial:

$$(1-rz)(1-(r+1)z)\cdots(1-(m+r)z)$$

$$=\frac{(1-z)(1-2z)\cdots(1-(m+r)z)}{(1-z)(1-2z)\cdots(1-(r-1)z)}=\frac{z^{m+1}\left(\frac{1}{z}\right)^{\underline{m+r+1}}}{\left(\frac{1}{z}\right)^{\underline{r}}}.$$

Hence

$$\sum_{k=m}^{\infty}\left\{ {k+r \atop m+r} \right\}_r z^k=\frac{1}{z}\left(\frac{1}{z}\right)^{\underline{r}}\frac{1}{\left(\frac{1}{z}\right)^{\underline{m+r+1}}}.$$

Equality (7) and definitions (5)-(6) give that

$$\left(\frac{1}{z}\right)^{\underline{m+r+1}}=(-1)^{m+r+1}\left(-\frac{1}{z}\right)_{m+r+1}$$

$$=(-1)^{m+r+1}\left(-\frac{1}{z}\right)_{r+1}\left(-\frac{1}{z}+r+1\right)_m.$$

Consequently,

$$\sum_{n=m}^{\infty}\left\{ {n+r \atop m+r} \right\}_r z^n=\frac{1}{z}\frac{\left(\frac{1}{z}\right)^{\underline{r}}}{\left(-\frac{1}{z}\right)_{r+1}}\frac{(-1)^{m+r+1}}{\left(\frac{rz+z-1}{z}\right)_m}.$$

Since

$$\frac{\left(\frac{1}{z}\right)^{\underline{r}}}{\left(-\frac{1}{z}\right)_{r+1}}=(-1)^r\frac{z}{rz-1},$$

we get that

$$\sum_{n=m}^{\infty}\left\{ {n+r \atop m+r} \right\}_r z^n=\frac{-1}{rz-1}\frac{(-1)^m}{\left(\frac{rz+z-1}{z}\right)_m}.$$

We multiply both sides by x^m and take summation over the non-negative integers:

$$\sum_{n=0}^{\infty}B_{n,r}(x)z^n=\frac{-1}{rz-1}\sum_{m=0}^{\infty}\frac{(-x)^m}{\left(\frac{rz+z-1}{z}\right)_m}=\frac{-1}{rz-1}\,{}_1F_1\left(\begin{matrix}1\\ \frac{rz+z-1}{z}\end{matrix}\,\middle|\,-x\right).$$

Finally, we apply Kummer's formula [1, p. 505]

$$e^{-x}\,{}_1F_1\left(\begin{matrix}a\\ b\end{matrix}\,\middle|\,x\right)={}_1F_1\left(\begin{matrix}b-a\\ b\end{matrix}\,\middle|\,-x\right)$$

with $b=\frac{rz+z-1}{z}$ and $a=\frac{rz-1}{z}$. $\square$

4 Basic recurrences

In an earlier paper of the author [18], the polynomials $B_{n,r}(x)$ were introduced because of a very different reason. These functions were used to study the unimodality of r-Stirling numbers and some properties of them were proven in that paper. We repeat those results without proof.

Theorem 4.1. We have the following recursive identities:

$$\begin{aligned} B_{n,r}(x) &= x\left(\frac{d}{dx}B_{n-1,r}(x) + B_{n-1,r}(x)\right) + rB_{n-1,r}(x), \\ e^x x^r B_{n,r}(x) &= x\frac{d}{dx}\left(e^x x^r B_{n-1,r}(x)\right). \end{aligned}$$

Moreover, all zeros of $B_{n,r}(x)$ are real and negative.

Straightforward corollaries are that for a fixed r the constant term of the n-th polynomial is r^n:

$$B_{n,r}(0) = r^n,$$

and that the derivative of an r-Bell polynomial is determined by the relation

$$\frac{d}{dx}B_{n,r}(x) = \frac{B_{n+1,r}(x)}{x} - \frac{rB_{n,r}(x)}{x} - B_{n,r}(x).$$

The identity

$$\left\{ {n+r \atop k+r} \right\}_r = \left\{ {n+r \atop k+r} \right\}_{r-1} - (r-1)\left\{ {n-1+r \atop k+r} \right\}_{r-1}$$

was proven in [4, p. 245] and implies the recurrence relation

$$B_{n,r}(x) = xB_{n-1,r+1}(x) + rB_{n-1,r}(x).$$

Theorem 4.2. The next polynomial identity is valid:

$$B_{n,r}(x) = rB_{n-1,r}(x) + x\sum_{k=0}^{n-1}\binom{n-1}{k}B_{k,r}(x).$$

Proof. We give a combinatorial proof for the non-polynomial version ($x = 1$). First we rearrange the sum on the right hand side:

$$\sum_{k=0}^{n-1}\binom{n-1}{k}B_{k,r} = \sum_{k=0}^{n-1}\binom{n-1}{n-1-k}B_{n-1-k,r} = \sum_{k=0}^{n-1}\binom{n-1}{k}B_{n-1-k,r}.$$

Hence we need to prove that

$$B_{n,r} = rB_{n-1,r} + \sum_{k=0}^{n-1}\binom{n-1}{k}B_{n-1-k,r}.$$

If we construct partitions on $n+r$ elements and the first r elements are in distinct blocks, then we have two possibilities: 1) the last element, $n+r$, belongs to a block containing one of the first elements. Such partition can be constructed such that we construct a partition of $\{1, 2, \ldots, n+r-1\}$ and then put the last element into the block containing 1 or 2 ... or r. We see that there are $rB_{n-1,r}$ possibilities. 2) the last element belongs to a block *not* containing 1, 2, ... and r. Now we may choose k other elements from $\{r+1, \ldots, n+r-1\}$ into the block of n. There are $\binom{n-1}{k}$ ways to do this. Then the remaining $n-1-k$ elements build up a partition (such that $1, \ldots, r$ are in different blocks). This can be done $B_{n-1-k,r}$ ways. Last, we take summation over all the possible values of k. ☐

Closing this section, we cite Carlitz's identities [6, eq. (3.22-3.23)]:

$$\begin{aligned} B_{n+m,r} &= \sum_{j=0}^{m} \left\{ {m+r \atop j+r} \right\}_r B_{n,r+j}, \qquad (8) \\ B_{n,r+m} &= \sum_{j=0}^{m} (-1)^{m-j} \left[{m+r \atop j+r} \right]_r B_{n+j,r}. \end{aligned}$$

Here $\left[{n \atop m} \right]_r$ is an r-Stirling number of the first kind (see [4, 6, 7]).

5 Dobinski's formula

The Bell numbers are involved in Dobinski's nice formula [9, 13, 14, 19]:

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Our goal is to generalize this identity to our case.

Theorem 5.1 (Dobinski's formula). The r-Bell polynomials satisfy the identity

$$B_{n,r}(x) = \frac{1}{e^x} \sum_{k=0}^{\infty} \frac{(k+r)^n}{k!} x^k.$$

Consequently, the r-Bell numbers are given by

$$B_{n,r} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{(k+r)^n}{k!}.$$

Proof. The r-Stirling numbers for a fixed n (and r) have the "horizontal" generating function [4]

$$(x+r)^n = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^{\underline{k}},$$

whence, for an arbitrary integer m,

$$\frac{(m+r)^n}{m!} = \sum_{k=0}^{m} \left\{ {n+r \atop k+r} \right\}_r \frac{1}{(m-k)!}.$$

In the next step we multiply both sides by x^m and sum from $m = 0$ to ∞. Then

$$\sum_{m=0}^{\infty} \frac{(m+r)^n}{m!} x^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left\{ {n+r \atop k+r} \right\}_r \frac{x^m}{(m-k)!} =$$

$$e^x \left(\sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k \right) = e^x B_{n,r}(x).$$

□

We can determine some interesting sums with the aid of r-Bell numbers. For example, we know from the second paragraph that $B_{2,2} = 10$, so

$$\frac{1}{e} \sum_{k=0}^{\infty} \frac{(k+2)^2}{k!} = 10.$$

6 An integral representation

In 1885, Cesàro [8] found a remarkable integral representation of the Bell numbers (see also [3, 5]):

$$B_n = \frac{2n!}{\pi e} \operatorname{Im} \int_0^{\pi} e^{e^{e^{i\theta}}} \sin(n\theta) d\theta.$$

It is not hard to deduce the "r-Bell version".

Theorem 6.1. The r-Bell numbers have the integral representation

$$B_{n,r} = \frac{2n!}{\pi e} \operatorname{Im} \int_0^{\pi} e^{e^{e^{i\theta}}} e^{re^{i\theta}} \sin(n\theta) d\theta.$$

Proof. In [6] we find that

$$k! \left\{ {n+r \atop k+r} \right\}_r = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j+r)^n. \tag{9}$$

In the next step we use the next equality [5]:

$$\operatorname{Im} \int_0^{\pi} e^{je^{i\theta}} \sin(n\theta) d\theta = \frac{\pi}{2} \frac{j^n}{n!}. \tag{10}$$

Unifying equations (9) and (10), we get that

$$\begin{aligned}
\frac{\pi}{2}\frac{1}{n!}\left\{ {n+r \atop k+r} \right\}_r &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\operatorname{Im}\int_0^{\pi} e^{(j+r)e^{i\theta}}\sin(n\theta)d\theta \\
= \quad & \frac{1}{k!}\operatorname{Im}\int_0^{\pi}\left[\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(e^{e^{i\theta}}\right)^j\right]e^{re^{i\theta}}\sin(n\theta)d\theta \\
= \quad & \operatorname{Im}\int_0^{\pi}\frac{\left(e^{e^{i\theta}}-1\right)^k}{k!}e^{re^{i\theta}}\sin(n\theta)d\theta,
\end{aligned}$$

whence

$$\sum_{k=0}^{\infty}\left\{ {n+r \atop k+r} \right\}_r = \frac{2n!}{\pi}\operatorname{Im}\int_0^{\pi}\left(\sum_{k=0}^{\infty}\frac{\left(e^{e^{i\theta}}-1\right)^k}{k!}\right)e^{re^{i\theta}}\sin(n\theta)d\theta,$$

and the result follows. $\square$

The imaginary part of the above integral can be calculated with a bit of effort:

$$B_{n,r} = \frac{2n!}{\pi e}\int_0^{\pi} e^{e^{\cos\theta}\cos\sin\theta + r\cos\theta}.$$

$$\cdot\left[\cos(e^{\cos\theta}\sin\sin\theta)\sin(r\sin\theta)+\sin(e^{\cos\theta}\sin\sin\theta)\cos(r\sin\theta)\right]\sin(n\theta)d\theta.$$

Without the r-Bell numbers in background, the evaluation of this integral seems to be impossible...

Citing the general version of Dobinski's formula we find the compelling identity

$$\sum_{k=0}^{\infty}\frac{(k+r)^n}{k!} = \frac{2n!}{\pi}\operatorname{Im}\int_0^{\pi} e^{e^{e^{i\theta}}}e^{re^{i\theta}}\sin(n\theta)d\theta.$$

7 Hankel transformation and log-convexity

Since

$$e^t\sum_{n=0}^{\infty}B_{n,r}(x)\frac{t^n}{n!} = e^{x(e^t-1)+(r+1)t},$$

Cauchy's product immediately implies the next

Theorem 7.1. The r-Bell polynomials satisfy the relations

$$\begin{aligned}
B_{n,r+1}(x) &= \sum_{k=0}^{n}\binom{n}{k}B_{k,r}(x), \\
B_{n,r}(x) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}B_{k,r+1}(x).
\end{aligned}$$

An interesting corollary is connected to the Hankel transform. The H Hankel matrix [16] of an integer sequence (a_n) is

$$H = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

while the Hankel matrix of order n, denoted by h_n, is the upper-left submatrix of H of size $n \times n$. The Hankel transform of the sequence (a_n) is again a sequence formed by the determinants of the matrices h_n.

A notable result of Aigner and Lenard [2, 17] is that the Hankel transform of the Bell numbers is $(1!, 1!2!, 1!2!3!, \dots)$, that is, for any fixed n,

$$\begin{vmatrix} B_0 & B_1 & B_2 & \cdots & B_n \\ B_1 & B_2 & B_3 & \cdots & B_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ B_n & B_{n+1} & B_{n+2} & \cdots & B_{2n} \end{vmatrix} = \prod_{i=0}^{n} i!$$

We can determine the Hankel transform of r-Bell numbers easily. To reach this aim, we recall the next notion. If (a_n) is a sequence, then its binomial transform (b_n) is defined by the relation

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k,$$

while the inverse transform is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k.$$

See the paper [20] on these transformations, for instance. A useful theorem of Layman [16] states that any integer sequence has the same Hankel transform as its binomial transform. Then Theorem 7.1 yields the next

Corollary 7.2. The r-Bell numbers have the Hankel transform

$$\begin{vmatrix} B_{0,r} & B_{1,r} & B_{2,r} & \cdots & B_{n,r} \\ B_{1,r} & B_{2,r} & B_{3,r} & \cdots & B_{n+1,r} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{n,r} & B_{n+1,r} & B_{n+2,r} & \cdots & B_{2n,r} \end{vmatrix} = \prod_{i=0}^{n} i!$$

Professor J. Cigler [10] calculated more general identities with respect to Hankel determinants involving not only r-Bell numbers but polynomials. We cite his unpublished results here.

Let $d(n,k) = \det(B_{i+j+k,r}(x))_{i,j=0}^{n-1}$. Cigler's results are the following:

$$d(n,0) = x^{\binom{n}{2}} \prod_{k=0}^{n-1} k!,$$

and

$$d(n,1) = x^{\binom{n}{2}} \prod_{k=0}^{n-1} k! \sum_{k=0}^{n} \binom{n}{k} x^k (r)_{n-k}.$$

8 Some occurrences of the r-Bell numbers

Surprisingly, the r-Bell numbers turned up in a table of Whitehead's paper [22]. In his table, the (n,i)-entry is denoted by $b_{n,i}$ and it is the sum of the coefficients of the polynomial $x^i(x)_{n-i}$ with respect to the so-called complete graph base. A more detailed description on this graph theoretical notion can be found in the paper [22] and the references therein.

Our r-Bell numbers are exactly the entries of that table, more exactly,

$$B_{n,r} = b_{n+r,n} \quad (n \geq 1). \tag{11}$$

From this observation we get straightaway the next identity.

Theorem 8.1. We have for all $n \geq 1$ that

$$B_{n,r} = rB_{n-1,r} + B_{n-1,r+1}.$$

Proof. According to [22], the entries $b_{n,i}$ satisfy the recurrence

$$(n-i)b_{n,i} + b_{n+1,i} = b_{n+1,i+1}.$$

Then (11) implies the statement. On the other hand, this theorem is a special case of (8) but it is worthwhile to give a different viewpoint. □

We note that the "row sum" in the table of Whitehead can be expressed by the r-Bell numbers, too.

$$\sum_{i=1}^{n} b_{n,i} = \sum_{i=1}^{n} B_{i,n-i}.$$

Identification (11) gives also that the r-Bell numbers have meaning in the theory of chromatic polynomials.

Another occurrence is the following. The r-Bell numbers come from a problem on the maximum of r-Stirling numbers (see [18]). The author proved there that all zeros of the polynomial $B_{n,r}(x)$ are real. This implies that

$$\left\{ {n \atop k} \right\}_r^2 \geq \left\{ {n \atop k+1} \right\}_r \left\{ {n \atop k-1} \right\}_r,$$

which is an important relation – for example – in the theory of combinatorial sequences. In addition, the maximizing index of r-Stirling numbers of the second kind can be expressed approximately by the r-Bell numbers [18]. Namely,

$$\left| K - \left(\frac{B_{n+1,r}}{B_{n,r}} - (r+1) \right) \right| < 1,$$

where K is the parameter, for which

$$\left\{ {n+r \atop K} \right\}_r \geq \left\{ {n+r \atop k} \right\}_r$$

for all $k = r, r+1, \ldots, n+r$.

We remark that (beside the papers cited above), there are other articles in which the r-Bell numbers (at least implicitly) appear. C. B. Corcino [12] deals with the asymptotic properties of these numbers. The paper of Hsu and Shiue [15] concerns the Stirling-type pairs. In that article a generalized Dobinski formula is presented.

9 Acknowledgement

I thank Professor Cigler for his suggestions and results on Hankel determinants of r-Bell polynomials.

I also appreciate that Jonathan Vos Post uploaded the table of r-Bell numbers (see A134980 in [21]).

Moreover, I would like to thank the referee for his/her useful suggestions and improvements.

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2010 *Mathematics Subject Classification*: Primary 05A18; Secondary 05A15.
Keywords: Bell numbers, r-Bell numbers, Stirling numbers, r-Stirling numbers, Hankel determinants, restricted partitions.

(Concerned with sequences A000110, A005493, A005494, A045379, and A134980.)

Received November 1 2010; revised version received December 19 2010. Published in *Journal of Integer Sequences*, December 19 2010.

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