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Generalized Stirling Numbers, Exponential Riordan Arrays, and Orthogonal Polynomials

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Abstract

We define a generalization of the Stirling numbers of the second kind, which depends on two parameters. The matrices of integers that result are exponential Riordan arrays. We explore links to orthogonal polynomials by studying the production matrices of these Riordan arrays. Generalized Bell numbers are also defined, again depending on two parameters, and we determine the Hankel transform of these numbers.

1 Introduction

The Stirling numbers of the second kind [13, 17] defined by

$$\left\{ {n \atop k} \right\} = S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n,$$

are the elements of the exponential Riordan array (see below for more details)

$$S = [1, e^x - 1].$$

This matrix, A048993, begins

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 1 & 7 & 6 & 1 & 0 & \dots \\ 0 & 1 & 15 & 25 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this matrix are the well-known Bell numbers [13, 17]

$$\mathrm{Bell}(n) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

By the theory of exponential Riordan arrays, this implies that

$$\mathrm{Bell}(n) = n![x^n]e^{e^x-1},$$

corresponding to the well-known fact that the Bell numbers have exponential generating function (e.g.f.) e^{e^x-1}. The elements of the inverse of the matrix S define the (signed) Stirling numbers of the first kind, $\left[{n \atop k} \right]$ [13]. These are thus the elements of the exponential Riordan array A048894

$$s = [1, \ln(1+x)].$$

In this note, we shall define a generalization of the matrix of Stirling numbers, and in so doing, we obtain a notion of generalized Bell numbers. The generalization depends on two parameters. We also exhibit these generalized Bell numbers as the moments of families of orthogonal polynomials (except in the case of the Bell numbers themselves). Links between orthogonal polynomials [6, 12, 27] and Riordan arrays [23, 26] have been studied in [3, 4].

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence. The n-th term a_n is thus the coefficient of x^n in this series. As is customary, we can denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. The power series $g(x) = \sum_{k=0}^{n} a_n \frac{x^n}{n!}$ is called the exponential generating function or e.g.f. of the sequence a_n. In this case we have $a_n = n![x^n]g(x)$. For instance, the e.g.f. of $n!$ is $\frac{1}{1-x}$. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \dots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [13], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $f^{<-1>}(x) = \bar{f}(x)$ such that $f(\bar{f}(x)) = x$.

Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [24, 25]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.

2 Exponential Riordan arrays

The *exponential Riordan group* [2, 10, 11] is a set of infinite lower-triangular matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ where $g_0 \neq 0$ and $f_1 \neq 0$. In what follows, we shall assume that

$$g_0 = f_1 = 1.$$

The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by $[g, f]$. The group law is given by

$$[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}(x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$ [11]. Thus the row sums of the array $[g, f]$ have exponential generating function given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, the *binomial matrix* $\mathbf{B}$ with (n, k)-th element $\binom{n}{k}$ is given by $\mathbf{B} = [e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^x e^x = e^{2x}$, as expected (since e^{2x} is the e.g.f. of 2^n). Applying the matrix $\mathbf{B}$ to a sequence a_n yields the *binomial transform* of that sequence, with general term

$$\sum_{k=0}^{n} \binom{n}{k} a_k$$

and e.g.f. $e^x g(x)$ where $g(x)$ is the e.g.f. of a_n.

Example 1. We consider the exponential Riordan array $[\frac{1}{1-x}, x]$, A094587. This array has elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 6 & 6 & 3 & 1 & 0 & 0 & \ldots \\ 24 & 24 & 12 & 4 & 1 & 0 & \ldots \\ 120 & 120 & 60 & 20 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and general term $[k \leq n]\frac{n!}{k!}$, and inverse

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & -5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is the array $[1-x, x]$. In particular, we note that the row sums of the inverse, which begin $1, 0, -1, -2, -3, \ldots$, (that is, $1-n$), have e.g.f. $(1-x)e^x$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1-x)$ (which is the sequence starting $1, -1, 0, 0, 0, \ldots$).

Example 2. We consider the exponential Riordan array $L = [1, \frac{x}{1-x}]$. The general term of this matrix may be calculated as follows:

$$\begin{aligned}
T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\
&= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\
&= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\
&= \frac{n!}{k!}\binom{n-1}{n-k}.
\end{aligned}$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1, .., n\}$ into any number of lists, where a list means an ordered subset'.

We will use the following important result [9, 10, 11] concerning matrices that are production matrices for exponential Riordan arrays. We recall that if L is an invertible matrix, then its production matrix (sometimes called its Stieltjes matrix [19]) is the matrix

$$P_L = L^{-1}\tilde{L},$$

where $\tilde{L}$ is the matrix L with its first row removed. For an exponential Riordan array L, it is easy to recapture a knowledge of L from P_L.

Proposition 3. *[10, Proposition 4.1] [11] Let $L = (l_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \ldots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \ldots \tag{1}$$

be two formal power series such that

$$\begin{aligned}
r(f(x)) &= f'(x) \qquad (2)\\
c(f(x)) &= \frac{g'(x)}{g(x)}. \qquad (3)
\end{aligned}$$

Then

$$(i) \qquad l_{n+1,0} = \sum_i i! c_i l_{n,i} \tag{4}$$

$$(ii) \qquad l_{n+1,k} = r_0 l_{n,k-1} + \frac{1}{k!} \sum_{i \geq k} i!(c_{i-k} + k r_{i-k+1}) l_{n,i} \tag{5}$$

or, assuming $c_k = 0$ *for* $k < 0$ *and* $r_k = 0$ *for* $k < 0$,

$$l_{n+1,k} = \frac{1}{k!} \sum_{i \geq k-1} i!(c_{i-k} + k r_{i-k+1}) l_{n,i}. \tag{6}$$

Conversely, starting from the sequences defined by (1), the infinite array $(l_{n,k})_{n,k \geq 0}$ *defined by (6) is an exponential Riordan array.*

A consequence of this proposition is that the production matrix $P = (p_{i,j})_{i,j \geq 0}$ for an exponential Riordan array obtained as in the proposition satisfies [10, 11]

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate generating function

$$\phi_P(x, y) = \sum_{n,k} p_{n,k} \frac{x^n}{n!} y^k$$

of the matrix P is given by

$$\phi_P(x, y) = e^{xy}(c(x) + y r(x)),$$

where we have

$$r(x) = f'(\bar{f}(x)), \tag{7}$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}. \tag{8}$$

Example 4. The production matrix of $L = \left[1, \frac{x}{1+x}\right]$ A111596 is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & -4 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 6 & -6 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 12 & -8 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 20 & -10 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of L have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884. The form of the production matrix above follows since we have $g(x) = 1$ and so $g'(x) = 0$, implying that $c(x) = 0$, and $f(x) = \frac{x}{1+x}$ which gives us $\bar{f}(x) = \frac{x}{1-x}$ and $f'(x) = \frac{1}{(1+x)^2}$. Thus $f'(\bar{f}(x)) = r(x) = (1-x)^2$. Hence the bivariate generating function of P is $e^{xy}(1-x)^2 y$, as required.

Example 5. In this example, we calculate the production matrix of the Stirling matrix of the second kind, $S = [1, e^x - 1]$. We have $f(x) = e^x - 1$ and hence $f'(x) = e^x$ and $\bar{f}(x) = \ln(1+x)$. In addition, $g(x) = 1$ implies that $g'(x) = 0$. Thus $r(x) = f'(\bar{f}(x)) = \exp(\ln(1+x)) = 1 + x$ while $c(x) = 0$. It follows that the generating function of the production matrix of S is simply $e^{xy}(1+x)y$. Thus we get the matrix

$$P_S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have the following important link between exponential Riordan arrays and orthogonal polynomials.

Theorem 6. *[3, Theorem 25] An exponential Riordan array $L = [g(x), f(x)]$ is the inverse of the coefficient array of a family of orthogonal polynomials if and only if its production matrix $P = S_L$ is tri-diagonal.*

This theorem has the following corollary.

Corollary 7. *[3, Corollary 27] Let $L = [g(x), f(x)]$ be an exponential Riordan array with tri-diagonal production matrix S_L. Then the moments μ_n of the associated family of orthogonal polynomials are given by the terms of the first column of L.*

This implies [14] that $g(x)$ has the continued fraction [28] expansion of the form

$$g(x) = \cfrac{1}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}},$$

where the associated family of orthogonal polynomials $P_n(x)$ obeys the three-term recurrence

$$P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x).$$

The Hankel transform [16] of a sequence a_n is the sequence of determinants $h_n = |a_{i+j}|_{0 \le i,j \le n-1}$. If a_n has a generating function with a continued fraction expansion as above (with $a_0 = 1$), then [14, 15] its Hankel transform is given by

$$h_n = \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-1} = \prod_{k=1}^{n} \beta_k^{n-k}. \tag{9}$$

Examples of the calculation of Hankel transforms can be found in [8, 22].

3 Generalized Stirling and Bell numbers

In this section, we shall be interested in the power series

$$g(x;\alpha,\beta) = e^{\alpha(e^x-1)-(\alpha-\beta)x}.$$

We will associate two exponential Riordan arrays with $g(x;\alpha,\beta)$ in a natural way, and calculate their production matrices, thus throwing light on their structure. In each case, $g(x;\alpha,\beta)$ will be the first element in the pair defining the Riordan arrays, and thus the sequence with n-th term

$$n![x^n]g(x;\alpha,\beta) = n![x^n]e^{\alpha(e^x-1)-(\alpha-\beta)x}$$

will be the first column in both cases. We can describe this sequence in terms of the Stirling numbers of the second kind as follows:

Proposition 8.

$$n![x^n]e^{\alpha(e^x-1)-(\alpha-\beta)x} = \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\left\{ k \atop i \right\}\alpha^i(\beta-\alpha)^{n-k}. \tag{10}$$

Proof. We have

$$\begin{aligned}
n![x^n]e^{\alpha(e^x-1)-(\alpha-\beta)x} &= n!\sum_{k=0}^{n}[x^k]e^{\alpha(e^x-1)}[x^{n-k}]e^{(\beta-\alpha)x} \\
&= n!\sum_{k=0}^{n}\frac{1}{k!}\sum_{i=0}^{k}\left\{ k \atop i \right\}\alpha^i\frac{(\beta-\alpha)^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\left\{ k \atop i \right\}\alpha^i(\beta-\alpha)^{n-k}.
\end{aligned}$$

□

Note that we have used the identity

$$[x^n]u(x)v(x) = \sum_{k=0}^{n}[x^k]u(x)[x^{n-k}]v(x)$$

in the above calculation. We now define

$$\mathrm{Bell}(n;\alpha,\beta) = \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\left\{ k \atop i \right\}\alpha^i(\beta-\alpha)^{n-k}$$

to be the generalized (α,β)-Bell numbers. We have

$$\mathrm{Bell}(n;1,1) = \mathrm{Bell}(n).$$

Re-interpreting equation (10) in terms of the binomial transform, we see that $\text{Bell}(n;\alpha,\beta)$ represents the $(\beta-\alpha)$-th binomial transform of the Bell polynomial $\sum_{k=0}^n {n \brace k}\alpha^k$.

In a similar fashion we define the (α,β)-Stirling numbers of the second kind to be the elements of the lower-triangular invertible matrix given by the exponential Riordan array

$$S(\alpha,\beta)=[g(x;\alpha,\beta),e^x-1]=[e^{\alpha(e^x-1)-(\alpha-\beta)x},e^x-1].$$

We have

$$S=S(0,0).$$

Proposition 9. *The general term of the (α,β)-Stirling matrix $S(\alpha,\beta)$ is given by*

$${n \brace k}_{(\alpha,\beta)}=\sum_{i=0}^n\binom{n}{i}{n-i \brace k}\sum_{l=0}^i\binom{i}{l}\sum_{j=0}^l{l \brace j}\alpha^j(\beta-\alpha)^{i-l}.$$

Proof. The general (n,k)-th term of the exponential Riordan array $S(\alpha,\beta)$ is given by

$$\begin{aligned}\frac{n!}{k!}[x^n]g(x;\alpha,\beta)(e^x-1)^k &= \frac{n!}{k!}\sum_{i=0}^n[x^i]g(x;\alpha,\beta)[x^{n-i}](e^x-1)^k\\ &= \frac{n!}{k!}\sum_{i=0}^n\frac{1}{i!}\sum_{l=0}^i\binom{i}{l}\sum_{j=0}^l{l \brace j}\alpha^j(\beta-\alpha)^{i-l}\frac{k!}{(n-i)!}{n-i \brace k}\\ &= \sum_{i=0}^n\binom{n}{i}{n-i \brace k}\sum_{l=0}^i\binom{i}{l}\sum_{j=0}^l{l \brace j}\alpha^j(\beta-\alpha)^{i-l}.\end{aligned}$$

□

Proposition 10. *The production matrix of $S(\alpha,\beta)$ is tri-diagonal, given by*

$$A_{S(\alpha,\beta)}=\begin{pmatrix}\beta & 1 & 0 & 0 & 0 & 0 & \cdots\\ \alpha & \beta+1 & 1 & 0 & 0 & 0 & \cdots\\ 0 & 2\alpha & \beta+2 & 1 & 0 & 0 & \cdots\\ 0 & 0 & 3\alpha & \beta+3 & 1 & 0 & \cdots\\ 0 & 0 & 0 & 4\alpha & \beta+4 & 1 & \cdots\\ 0 & 0 & 0 & 0 & 5\alpha & \beta+5 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Proof. We calculate the bivariate generating function of the production matrix. We have $f(x)=e^x-1$ and thus $\bar{f}(x)=\ln(1+x)$ and $f'(x)=e^x$. Thus

$$r(x)=f'(\bar{f}(x))=e^{\ln(1+x)}=1+x.$$

Now

$$g'(x)=(\alpha e^x-(\alpha-\beta))g(x)$$

and hence

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{g'(\ln(1+x))}{g(\ln(1+x))} = (\alpha e^{ln(1+x)} - (\alpha-\beta)) = \alpha x + \beta.$$

Thus the g.f. of the production matrix is given by

$$e^{xy}(\alpha x + \beta + (1+x)y)$$

as required. ☐

We define the (α, β)-Stirling numbers of the first kind to be the elements of the matrix $s(\alpha, \beta) = S(\alpha, \beta)^{-1}$. We have

$$s(\alpha, \beta) = [e^{\alpha(e^x-1)-(\alpha-\beta)x}, e^x - 1]^{-1} = [e^{-\alpha x + (\alpha-\beta)\ln(1+x)}, \ln(1+x)].$$

Using for instance the results of [3], we then obtain the following results.

Corollary 11. *For $\alpha \neq 0$, $\beta \neq 0$, the matrix $s(\alpha, \beta)$ of the (α, β)-Stirling numbers of the first kind is the coefficient array of the family of orthogonal polynomials $P_n^{\alpha,\beta}(x)$ defined by the three-term recurrence*

$$P_{n+1}^{\alpha,\beta}(x) = (x - (n+\beta))P_n^{\alpha,\beta}(x) - \alpha n P_{n-1}^{\alpha,\beta}(x).$$

For $\alpha \neq 0$, $\beta \neq 0$, the (α, β)-Bell numbers $\mathrm{Bell}(n; \alpha, \beta)$ *are the moments of the orthogonal polynomials $P_n^{\alpha,\beta}(x)$ defined above.*

We note that in the case $\alpha = 0$, $\beta = 0$, these polynomials become the polynomials $P_n(x) = (x)_n$ (which are not orthogonal).

Corollary 12. *The (α, β)-Bell numbers have ordinary generating function given by the continued fraction*

$$g_o(x; \alpha, \beta) = \cfrac{1}{1 - \beta x - \cfrac{\alpha x^2}{1 - (1+\beta)x - \cfrac{2\alpha x^2}{1 - (2+\beta)x - \cfrac{3\alpha x^2}{1 - \cdots}}}}.$$

Corollary 13. *The Hankel transform of the (α, β)-Bell numbers* $\mathrm{Bell}(n; \alpha, \beta)$ *is given by*

$$h_n = \alpha^{\binom{n+1}{2}} \prod_{k=1}^{n} k!$$

Proof. This follows since

$$h_n = \prod_{k=1}^{n} (\alpha k)^{n-k+1}.$$

☐

We note that this result is also a direct consequence of the known fact that the Hankel transform of the Bell polynomials $\sum_{k=0}^n \left\{ {n \atop k} \right\} \alpha^k$ is given by $\alpha^{\binom{n+1}{2}} \prod_{k=1}^n k!$ [21], and the invariance of the Hankel transform under the binomial transform.

We now define our second exponential Riordan array associated to the (α, β)-Bell numbers. This is the exponential Riordan array

$$[g(x;\alpha,\beta), x] = [e^{\alpha(e^x-1)-(\alpha-\beta)x}, x].$$

Proposition 14.

$$[g(x;\alpha,\beta), x] = S(\alpha,\beta) \cdot S^{-1}.$$

Proof. We have

$$[g(x;\alpha,\beta), x]^{-1} \cdot [g(x;\alpha,\beta), e^x-1] = \left[\frac{1}{g(x;\alpha,\beta)}, x\right] \cdot [g(x;\alpha,\beta), e^x-1] = [1, e^x-1].$$

□

We now calculate the production matrix of $[g(x;\alpha,\beta), x]$.

Proposition 15. *The production matrix of $[g(x;\alpha,\beta), x]$ is given by*

$$\begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 & \cdots \\ \alpha & \beta & 1 & 0 & 0 & 0 & \cdots \\ \alpha & 2\alpha & \beta & 1 & 0 & 0 & \cdots \\ \alpha & 3\alpha & 3\alpha & \beta & 1 & 0 & \cdots \\ \alpha & 4\alpha & 6\alpha & 4\alpha & \beta & 1 & \cdots \\ \alpha & 5\alpha & 10\alpha & 10\alpha & 5\alpha & \beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. We calculate the bivariate g.f. of the production matrix. We have $f(x) = x$ and so $\bar{f}(x) = x$ and $f'(x) = 1$. Hence $r(x) = f'(\bar{f}(x)) = 1$. $g(x) = g(x;\alpha,\beta) = e^{\alpha(e^x-1)-(\alpha-\beta)x}$ and so $g'(x) = (\alpha e^x - (\alpha-\beta))g(x)$. Thus

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{g'(x)}{g(x)} = \alpha e^x - (\alpha - \beta).$$

Thus the bivariate g.f. of the production matrix is given by

$$e^{xy}(\alpha e^x + (\beta - \alpha) + y),$$

as required. □

Corollary 16. *If $\beta = \alpha$ the production matrix of $[g(x;\alpha,\beta), x]$ is given by*

$$\alpha \mathbf{B} + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 17. *If $\alpha = \beta = 1$ then the production matrix of $[g(x;1,1), x]$ is given by*

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & 1 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

In this case, the matrix $[g(x;1,1), x]$ is A056857, which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 5 & 6 & 3 & 1 & 0 & 0 & \dots \\ 15 & 20 & 12 & 4 & 1 & 0 & \dots \\ 52 & 75 & 50 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

with first column equal to the Bell numbers. The Riordan array $[g(x;1,1), e^x - 1]$ is the array $[e^{e^x-1}, e^x - 1]$, A049020, [1]. This matrix begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 3 & 1 & 0 & 0 & 0 & \dots \\ 5 & 10 & 6 & 1 & 0 & 0 & \dots \\ 15 & 37 & 31 & 10 & 1 & 0 & \dots \\ 52 & 151 & 160 & 75 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Again, we see that the first column gives the Bell numbers. The production matrix of this exponential Riordan array is particularly simple:

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4 & 5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The corresponding orthogonal polynomials then have coefficient array given by

$$[e^{e^x-1}, e^x - 1]^{-1} = \left[\frac{1}{1+x}, \ln(1+x)\right],$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 1 & 0 & 0 & 0 & \dots \\ -1 & 8 & -6 & 1 & 0 & 0 & \dots \\ 1 & -24 & 29 & -10 & 1 & 0 & \dots \\ -1 & 89 & -145 & 75 & -15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These are a version of the Charlier polynomials (see A094816 for an unsigned version of this array). The production matrix of this orthogonal polynomial coefficient array is of interest in itself, as it is given by

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -3 & 1 & 0 & 0 & \dots \\ 0 & -1 & 3 & -4 & 1 & 0 & \dots \\ 0 & 1 & -4 & 6 & -5 & 1 & \dots \\ 0 & -1 & 5 & -10 & 10 & -6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4 Final comments

As pointed out by a reviewer, the Bell numbers are often presented as the row sums of the matrix of the Stirling numbers of the second kind. With this in mind, we define

$$B(n;\alpha,\gamma) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \left\{ {k \atop i} \right\} \alpha^i \gamma^{n-k}.$$

Then we have

$$\mathrm{Bell}(n;\alpha,\beta) = B(n;\alpha,\beta-\alpha).$$

It is easy to see that $B(n;\alpha+1,\beta-\alpha)$ is given by the row sums of $S(\alpha,\beta)$, and thus provides another related generalization of the Bell numbers.

We finish this note by directing the reader to [5, 7, 18, 20] for some alternative generalizations of the Stirling and Bell numbers.

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