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Combinatorial Polynomials as Moments, Hankel Transforms, and Exponential Riordan Arrays

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Abstract

In the case of two combinatorial polynomials, we show that they can exhibited as moments of paramaterized families of orthogonal polynomials, and hence derive their Hankel transforms. Exponential Riordan arrays are the main vehicles used for this.

1 Introduction

Let $[n] = 1, 2, \ldots, n$, and let SP_n be the set of set-partitions of $[n]$. For a set-partition $\pi \in \mathsf{SP}_n$, let $|\pi|$ be the number of parts in π. Then the n-th exponential polynomial, also known as the n-th Touchard polynomial (and sometimes called the n-th Bell polynomial [23]), is given by

$$e_n(z) = \sum_{\pi} z^{|\pi|} = \sum_{k=0}^{n} S(n,k) z^k,$$

where

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

is the general element of the exponential Riordan array

$$[1, e^x - 1].$$

This is the matrix of Stirling numbers of the second kind A008277, which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 3 & 1 & 0 & 0 & \dots \\ 0 & 1 & 7 & 6 & 1 & 0 & \dots \\ 0 & 1 & 15 & 25 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is well known [11, 15, 18] that the Hankel transform of these polynomials is given by

$$z^{\binom{n+1}{2}} \prod_{k=1}^{n} k!.$$

Now let

$$A(n,k) = \sum_{k=0}^{k} (-1)^j (k-j)^n \binom{n+1}{j}$$

be the general term of the triangle of Eulerian numbers. The matrix of these numbers A008292 begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 4 & 1 & 0 & 0 & \dots \\ 0 & 1 & 11 & 11 & 1 & 0 & \dots \\ 0 & 1 & 26 & 66 & 26 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$A(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k excedances. The Eulerian polynomials $\mathsf{EU}_n(z)$ are defined by

$$\mathsf{EU}_n(z) = \sum_{k=0}^{n} A(n,k) z^k.$$

It is shown in [18] that the Hankel transform of these polynomials is given by

$$z^{\binom{n+1}{2}} \prod_{k=1}^{n} k!^2.$$

These two results are consequences of the following two theorems.

Theorem 1. *The polynomials $e_n(z)$ are moments of the family of orthogonal polynomials whose coefficient array is given by the inverse of the exponential Riordan array*

$$\left[e^{z(e^x-1)}, e^x - 1\right].$$

Theorem 2. *The polynomials* $\mathsf{EU}_n(z)$ *are moments of the family of orthogonal polynomials whose coefficient array is given by the inverse of the exponential Riordan array*

$$\left[\frac{e^{zx}(1-z)}{e^{zx}-ze^x}, \frac{e^x-e^{zx}}{e^{zx}-ze^x}\right].$$

Note that in the case $z=1$, the above matrix is taken to be $\left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, whose inverse is the coefficient array of the Laguerre polynomials [2].

While partly expository in nature, this note assumes a certain familiarity with integer sequences, generating functions, orthogonal polynomials [5, 10, 22], Riordan arrays [17, 21], production matrices [9, 14], and the integer Hankel transform [4, 6, 13]. Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [19, 20]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.
The plan of the paper is as follows:

1. This Introduction
2. Integer sequences, Hankel transforms, exponential Riordan arrays, orthogonal polynomials
3. Proof of Theorem 1
4. Proof of Theorem 2

2 Integer sequences, Hankel transforms, exponential Riordan arrays, orthogonal polynomials

In this section, we recall known results on integer sequences, Hankel transforms, exponential Riordan arrays and orthogonal polynomials that will be useful for the sequel.

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f_o(x)=\sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence, while $f_e(x)=\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is called the *exponential generating function* or e.g.f. of the sequence. a_n is thus the coefficient of x^n in $f_o(x)$. We denote this by $a_n=[x^n]f_o(x)$. Similarly, $a_n=n![x^n]f_e(x)$. For instance, $F_n=[x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $n!=n![x^n]\frac{1}{1-x}$, which says that $\frac{1}{1-x}$ is the e.g.f. of $n!$ A000142. For a power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ with $f(0)=0$ and $f'(0)\neq 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)=f^{[-1]}(x)$ such that $f(\bar{f}(x))=x$. We sometimes write $\bar{f}=\text{Rev}f$.

The *Hankel transform* [13] of a given sequence $A=\{a_0,a_1,a_2,...\}$ is the sequence of Hankel determinants $\{h_0,h_1,h_2,\dots\}$ where $h_n=|a_{i+j}|_{i,j=0}^n$, i.e

$$A=\{a_n\}_{n\in\mathbb{N}_0} \quad\rightarrow\quad h=\{h_n\}_{n\in\mathbb{N}_0}: \quad h_n=\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \tag{1}$$

The Hankel transform of a sequence a_n and its binomial transform are equal.

In the case that a_n has g.f. $g(x)$ expressible in the form

$$g(x) = \cfrac{a_0}{1-\alpha_0 x - \cfrac{\beta_1 x^2}{1-\alpha_1 x - \cfrac{\beta_2 x^2}{1-\alpha_2 x - \cfrac{\beta_3 x^2}{1-\alpha_3 x - \cdots}}}}$$

(with $\beta_i \neq 0$ for all i) then we have [11, 12, 24]

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-1}^2 \beta_n = a_0^n \prod_{k=1}^{n} \beta_k^{n-k+1}. \tag{2}$$

Note that this is independent from α_n. In general α_n and β_n are not integers. Such a continued fraction is associated to a monic family of orthogonal polynomials which obey the three term recurrence

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The terms appearing in the first column of the inverse of the coefficient array of these polynomials are the moments of the family.

The *exponential Riordan group* [1, 7, 9], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \ldots$ and $f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. It is *monic* if $g_0 = 1$. The group law is given by

$$[g, f] * [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group. If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}$ (x), then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ are given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

Example 3. The *binomial matrix* is the matrix with general term $\binom{n}{k}$. It is realized by Pascal's triangle. As an exponential Riordan array, it is given by $[e^x, x]$. We further have

$$([e^x, x])^m = [e^{mx}, x].$$

Example 4. We have

$$\left[e^{z(e^x-1)}, e^x - 1\right] = \left[e^{z(e^x-1)}, x\right] \cdot [1, e^x - 1].$$

A more interesting factorization is given by

Proposition 5. *The general term of the matrix* $\mathbf{L} = \left[e^{z(e^x-1)}, e^x - 1\right]$ *is given by*

$$L_{n,k} = \sum_{j=0}^{n} S(n,j)\binom{j}{k} z^{j-k}.$$

Proof. A straight-forward calculation shows that

$$\left[e^{z(e^x-1)}, e^x - 1\right] = [1, e^x - 1] \cdot [e^{zx}, x].$$

The assertion now follows since the general term of $[1, e^x - 1]$ is $S(n,k)$ and that of $[e^{zx}, x]$ is $\binom{n}{k} z^{n-k}$. □

As an example of the calculation of an inverse, we have the following proposition.

Proposition 6.

$$\left[e^{z(e^x-1)}, e^x - 1\right]^{-1} = \left[e^{-zx}, \ln(1+x)\right].$$

Proof. This follows since with

$$f(x) = e^x - 1$$

we have

$$\bar{f}(x) = \ln(1+x).$$

□

Proposition 7.

$$\left[\frac{e^{zx}(1-z)}{e^{zx} - ze^x}, \frac{e^x - e^{zx}}{e^{zx} - ze^x}\right]^{-1} = \left[1 + zx, \frac{1}{z-1}\ln\left(\frac{1+zx}{1+x}\right)\right].$$

Note that in the case $z = 1$*, we have*

$$\left[\frac{1}{1-x}, \frac{x}{1-x}\right]^{-1} = \left[\frac{1}{1+x}, \frac{x}{1+x}\right].$$

Proof. This follows since with

$$f(x) = \frac{e^{zx}(1-z)}{e^{zx} - ze^x}$$

we have

$$\bar{f}(x) = \frac{1}{z-1}\ln\left(\frac{1+zx}{1+x}\right).$$

□

An important concept for the sequel is that of production matrix. The concept of a *production matrix* [8, 9] is a general one, but for this note we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting $\mathbf{r}_0$ be the row vector

$$\mathbf{r}_0 = (1, 0, 0, 0, \ldots),$$

we define $\mathbf{r}_i = \mathbf{r}_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P. If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \begin{pmatrix} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{pmatrix}$$

and

$$DA_P = A_P P$$

where $D = (\delta_{i,j+1})_{i,j\geq 0}$ (where δ is the usual Kronecker symbol). In [14] P is called the Stieltjes matrix associated to A_P. In [9], we find the following result concerning matrices that are production matrices for exponential Riordan arrays.

Proposition 8. *Let* $A = (a_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ *be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \ldots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \ldots \tag{3}$$

be two formal power series that that

$$r(f(x)) = f'(x) \tag{4}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{5}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{6}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!}\sum_{i\geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{7}$$

or, defining $c_{-1} = 0$,

$$a_{n+1,k} = \frac{1}{k!}\sum_{i\geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{8}$$

Conversely, starting from the sequences defined by (3), the infinite array $(a_{n,k})_{n,k\geq 0}$ *defined by (8) is an exponential Riordan array.*

A consequence of this proposition is that $P = (p_{i,j})_{i,j\geq 0}$ where

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{r-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate exponential generating function

$$\phi_P(t,z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,z) = e^{tz}(c(z) + tr(z)).$$

Note in particular that we have

$$r(x) = f'(\bar{f}(x))$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

Example 9. We consider the exponential Riordan array $[\frac{1}{1-x}, x]$, A094587. This array [2] has elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and general term $[k \le n]\frac{n!}{k!}$ with inverse

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the array $[1-x, x]$. In particular, we note that the row sums of the inverse, which begin $1, 0, -1, -2, -3, \ldots$ (that is, $1-n$), have e.g.f. $(1-x)\exp(x)$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1-x)$ (which is the sequence starting $1, -1, 0, 0, 0, \ldots$). In order to calculate the production matrix $\mathbf{P}$ of $[\frac{1}{1-x}, x]$ we note that $f(x) = x$, and hence we have $f'(x) = 1$ so $f'(\bar{f}(x)) = 1$. Also $g(x) = \frac{1}{1-x}$ leads to $g'(x) = \frac{1}{(1-x)^2}$, and so, since $\bar{f}(x) = x$, we get

$$\frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{1}{1-x}.$$

Thus the generating function for $\mathbf{P}$ is

$$e^{tz}\left(\frac{1}{1-z} + t\right).$$

Thus $\mathbf{P}$ is the matrix $[\frac{1}{1-x}, x]$ with its first row removed.

Example 10. We consider the exponential Riordan array $[1, \frac{x}{1-x}]$. The general term of this matrix [2] may be calculated as follows:

$$\begin{aligned}
T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\
&= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\
&= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\
&= \frac{n!}{k!}\binom{n-1}{n-k}.
\end{aligned}$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1,..,n\}$ into any number of lists, where a list means an ordered subset'. Its general term is equal to $(n-1)!L_{n-1}(1,-1)$. The inverse of $\left[1, \frac{x}{1-x}\right]$ is the exponential Riordan array $\left[1, \frac{x}{1+x}\right]$, A111596. The row sums of this sequence have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884. To calculate the production matrix of $\left[1, \frac{x}{1+x}\right]$ we note that $g'(x) = 0$, while $\bar{f}(x) = \frac{x}{1+x}$ with $f'(x) = \frac{1}{(1+x)^2}$. Thus

$$f'(\bar{f}(x)) = (1+x)^2,$$

and so the generating function of the production matrix is given by

$$e^{tz}t(1+z)^2.$$

The production matrix of the inverse begins

$$\left(\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 2 & 1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 4 & 1 & 0 & 0 & \ldots \\
0 & 0 & 6 & 6 & 1 & 0 & \ldots \\
0 & 0 & 0 & 12 & 8 & 1 & \ldots \\
0 & 0 & 0 & 0 & 20 & 10 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

Example 11. The exponential Riordan array $\mathbf{A} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

has general term

$$T_{n,k} = \frac{n!}{k!}\binom{n}{k}.$$

Its inverse is $\left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ with general term $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. This is A021009, the triangle of coefficients of the Laguerre polynomials $L_n(x)$. The production matrix $\left[\frac{1}{1-x}, \frac{x}{1-x}\right]$ is given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 5 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9 & 7 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16 & 9 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Example 12. The exponential Riordan array $\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 8 & 6 & 1 & 0 & 0 & \cdots \\ 1 & 24 & 29 & 10 & 1 & 0 & \cdots \\ 1 & 89 & 145 & 75 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

is the coefficient array for the polynomials

$${}_2F_0(-n, x; -1)$$

which are an unsigned version of the Charlier polynomials (of order 0) [10, 16, 22]. This is A094816. It is equal to

$$[e^x, x]\left[1, \ln\left(\frac{1}{1-x}\right)\right],$$

or the product of the binomial array $\mathbf{B}$ and the array of (unsigned) Stirling numbers of the first kind. The production matrix of the inverse of this matrix is given by

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4 & -5 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5 & -6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which indicates the orthogonal nature of these polynomials. We can prove this as follows. We have

$$\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]^{-1} = \left[e^{-(1-e^{-x})}, 1-e^{-x}\right].$$

Hence $g(x) = e^{-(1-e^{-x})}$ and $f(x) = 1 - e^{-x}$. We are thus led to the equations

$$\begin{aligned} r(1-e^{-x}) &= e^{-x}, \\ c(1-e^{-x}) &= -e^{-x}, \end{aligned}$$

with solutions $r(x) = 1 - x$, $c(x) = x - 1$. Thus the bivariate generating function for the production matrix of the inverse array is

$$e^{tz}(z - 1 + t(1-z)),$$

which is what is required.

3 Proof of Theorem 1

Proof. We show first that with $\mathbf{L} = \left[e^{z(e^x-1)}, e^x - 1\right]$, the matrix $\mathbf{L}^{-1}$ which is given by

$$\mathbf{L}^{-1} = \left[e^{z(e^x-1)}, e^x - 1\right]^{-1} = \left[e^{-zx}, \ln(1+x)\right],$$

is the coefficient array of a family of orthogonal polynomials. To this end, we calculate the production array of $\left[e^{z(e^x-1)}, e^x - 1\right]$. We have $f(x) = e^x - 1$, $f'(x) = e^x$ and $\bar{f}(x) = \ln(1+x)$. Thus

$$c(x) = f'(\bar{f}(x)) = 1 + x.$$

Similarly, for $g(x) = e^{z(e^x-1)}$, we have $g'(x) = ze^{z(e^x-1)+x}$ and so

$$r(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{ze^{zx}(1+x)}{e^{zx}} = z(1+x).$$

Thus the production matrix sought has generating function

$$e^{tw}(c(w) + tr(w)) = e^{tw}(1 + w + t(z(1+w))).$$

Thus the production array $\mathbf{P_L}$ is tri-diagonal, beginning

$$\begin{pmatrix} z & 1 & 0 & 0 & 0 & 0 & \cdots \\ z & z+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2z & z+2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3z & z+3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4z & z+4 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5z & z+5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now it is well known that

$$\sum_{k=0}^{n} \frac{e_k(z)}{k!} x^k = e^{z(e^x-1)},$$

and hence the polynomials $e_n(z)$ are the moments of the family of orthogonal polynomials whose coefficient array is $\mathbf{L}^{-1}$. $\square$

Corollary 13. *The Hankel transform of $e_n(z)$ is $z^{\binom{n+1}{2}} \prod_{k=1}^{n} k!$.*

Proof. From the above, we have that the generating function of $e_n(z)$ is given by the continued fraction

$$\cfrac{1}{1-zx-\cfrac{zx^2}{1-(z+1)x-\cfrac{2zx^2}{1-(z+2)x-\cfrac{3zx^2}{1-\cdots}}}}.$$

In other words, $\beta_n = nz$. Thus the Hankel transform of $e_n(z)$ is given by

$$\prod_{k=1}^{n} \beta_k^{n-k+1} = \prod_{k=1}^{n} (kz)^{n-k+1} = z^{\binom{n+1}{2}} \prod_{k=1}^{n} k!.$$

$\square$

We note that the Hankel transform of the row sums of $\mathbf{L} = \left[e^{z(e^x-1)}, e^x - 1\right]$ is equal to

$$(z+1)^{\binom{n+1}{2}} \prod_{k=1}^{n} k!.$$

Note also that if we take $z = e^t$, we obtain a solution to the restricted Toda chain [3].

4 Proof of Theorem 2

Proof. We show first that with $\mathbf{L} = \left[\frac{e^{zx}(1-z)}{e^{zx}-ze^x}, \frac{e^x-e^{zx}}{e^{zx}-ze^x}\right]$, the matrix $\mathbf{L}^{-1}$ which is given by

$$\mathbf{L}^{-1} = \left[\frac{e^{zx}(1-z)}{e^{zx}-ze^x}, \frac{e^x-e^{zx}}{e^{zx}-ze^x}\right]^{-1} = \left[1+zx, \frac{1}{z-1} \ln\left(\frac{1+zx}{1+x}\right)\right],$$

is the coefficient array of a family of orthogonal polynomials. To this end, we calculate the production array of $\mathbf{L} = \left[\frac{e^{zx}(1-z)}{e^{zx}-ze^x}, \frac{e^x-e^{zx}}{e^{zx}-ze^x}\right]$. We have $f(x) = \frac{e^x-e^{zx}}{e^{zx}-ze^x}$, $\bar{f}(x) = \frac{1}{z-1}\ln\left(\frac{1+zx}{1+x}\right)$ and

$$f'(x) = \frac{(1-z)^2 e^{x(1+z)}}{(e^{zx}-ze^x)^2}.$$

Thus

$$c(x) = f'(\bar{f}(x)) = (1+x)(1+zx).$$

Also $g(x) = \frac{e^{zx}(1-z)}{e^{zx}-ze^x}$, which implies that $g'(x) = xf'(x)$ and so

$$r(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = z(1+x).$$

Thus the generating function of $\mathbf{P_L}$ is given by

$$e^{tw}(c(w)+tr(w)) = e^{tw}((1+w)(1+zw)+t(z(1+w))).$$

Thus the production array $\mathbf{P_L}$ is tri-diagonal, beginning

$$\left(\begin{array}{ccccccc} z & 1 & 0 & 0 & 0 & 0 & \cdots \\ z & 2z+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4z & 3z+2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9z & 4z+3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16z & 5z+4 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25z & 6z+5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Now it is known that

$$\sum_{k=0}^{n} \frac{\mathsf{EU}_k(z)}{k!}x^k = \frac{e^{zx}(1-z)}{e^{zx}-ze^x},$$

and hence the polynomials $e_n(z)$ are the moments of the family of orthogonal polynomials whose coefficient array is $\mathbf{L}^{-1}$. □

Corollary 14. *The Hankel transform of* $\mathsf{EU}_n(z)$ *is* $z^{\binom{n+1}{2}}\prod_{k=1}^n k!^2$.

Proof. From the above, we have that the generating function of $\mathsf{EU}_n(z)$ is given by the continued fraction

$$\cfrac{1}{1-zx-\cfrac{zx^2}{1-(2z+1)x-\cfrac{4zx^2}{1-(3z+2)x-\cfrac{9zx^2}{1-\cdots}}}}.$$

In other words, $\beta_n = n^2 z$. Thus the Hankel transform of $\mathsf{EU}_n(z)$ is given by

$$\prod_{k=1}^n \beta_k^{n-k+1} = \prod_{k=1}^n (k^2 z)^{n-k+1} = z^{\binom{n+1}{2}}\prod_{k=1}^n k!^2.$$

□

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