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On the Central Coefficients of Bell Matrices

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Abstract

We use the Lagrange inversion theorem to characterize the central coefficients of matrices in the Bell subgroup of the Riordan group of matrices. We give examples of how by using different means of calculating these coefficients we can deduce the generating functions of interesting sequences.

1 Introduction

Example 1. The most well-known example of a Bell matrix (see later) is the Binomial matrix $\mathbf{B}$ with general element $T_{n,k} = \binom{n}{k}$. As a member of the Riordan group of matrices, this is

$$\mathbf{B} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

By the *central coefficients* of this matrix we understand the terms $T_{2n,n}$. In this case, we have

$$T_{2n,n} = \binom{2n}{n} = [x^n]\frac{1}{\sqrt{1-4x}}.$$

A natural question to ask is how does the defining element

$$\frac{1}{1-x}$$

of $\mathbf{B}$ relate to $\frac{1}{\sqrt{1-4x}}$? The answer is in fact quite simple. In this case, we form the associated Bell matrix

$$(1-x, x(1-x))$$

and take its inverse. This is

$$(c(x), xc(x)),$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers. Then we have

$$\binom{2n}{n} = (n+1)[x^n]c(x),$$

or equivalently,

$$[x^n]c(x) = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}T_{2n,n}.$$

We now note that

$$xc(x) = \overline{x(1-x)}(x) = \overline{\left(\frac{x}{\frac{1}{1-x}}\right)}(x),$$

where $^{-}$ denotes the reversion or compositional inverse. Finally, we have

$$T_{2n,n} = \binom{2n}{n} = (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{\frac{1}{1-x}}\right)}(x).$$

The above result can be generalized to all elements of the Bell subgroup of the Riordan group of matrices. We have the following theorem, which we shall prove in section **3**.

Theorem 3. Let $(g(x), xg(x))$ be an element of the Bell subgroup of the Riordan group of matrices $\mathcal{R}$. If $T_{n,k}$ denotes the n,k-th element of this matrix, then we have

$$T_{2n,n} = (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{g(x)}\right)}(x).$$

Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [8, 9]. Sequences are frequently referred to by their A$nnnnnn$ OEIS number. In the next section, we will review known results concerning integer sequences, Riordan arrays and the Bell subgroup of the group of Riordan arrays.

2 Preliminaries on integer sequences, Riordan arrays and the Bell subgroup

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. The properties and examples of use of the notation $[x^n]$ can be found in [6].

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ and $f'(0) \neq 0$ we define the *reversion* or *compositional inverse* of f to be the power series $\bar{f}(x)$ (also written as $f^{[-1]}(x)$) such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev}f$.

The *Riordan group* [2, 7, 10], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \ldots$ and

$f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$ [10]. The corresponding matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f). The group law, which corresponds to matrix multiplication, is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f. We denote by $\mathcal{R}$ the group of Riordan matrices. If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 2. The so-called *binomial matrix* $\mathbf{B}$ A007318 is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k} m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The *Bell* subgroup of $\mathcal{R}$ is the set of matrices of the form

$$(g(x), xg(x)).$$

Note that

$$(g(x), xg(x))^{-1} = \left(\frac{\overline{xg}}{x}, \overline{xg} \right).$$

We shall call a member of the Bell subgroup of $\mathcal{R}$ a Bell matrix. An interesting sequence characterization of Bell matrices may be found in [3].

3 Proof of the theorem

We restate the theorem and prove it in this section.

Theorem 3. *Let $(g(x), xg(x))$ be an element of the Bell subgroup of the Riordan group of matrices $\mathcal{R}$. If $T_{n,k}$ denotes the n, k-th element of this matrix, then we have*

$$T_{2n,n} = (n+1)[x^n] \frac{1}{x} \overline{\left(\frac{x}{g(x)} \right)}(x).$$

Proof. We let $T_{n,k}$ denote the general element of the Bell matrix $(g(x), xg(x))$. Then we have

$$\begin{aligned} T_{2n,n} &= [x^{2n}] g(x)(xg(x))^n \\ &= [x^{2n}] x^n g(x)^{n+1} \\ &= [x^n] g(x)^{n+1} \\ &= (n+1) \frac{1}{n+1} [x^n] g(x)^{n+1}. \end{aligned}$$

We now recall the Lagrange inversion theorem [5]. This says that if we have

$$h(w) = \frac{w}{g(w)}, \quad g(0) \neq 0,$$

then

$$[x^n]\bar{h}(x) = \frac{1}{n}[w^{n-1}]g(w)^n.$$

Letting $n \to n+1$ gives

$$[x^{n+1}]\overline{\left(\frac{x}{g(x)}\right)}(x) = \frac{1}{n+1}[w^n]g(w)^{n+1},$$

or

$$[x^n]\frac{1}{x}\overline{\left(\frac{x}{g(x)}\right)}(x) = \frac{1}{n+1}[w^n]g(w)^{n+1}.$$

Thus we deduce that

$$T_{2n,n} = (n+1)\frac{1}{n+1}[x^n]g(x)^{n+1} = (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{g(x)}\right)}(x).$$

□

Corollary 4. *The central coefficients $T_{2n,n}$ of the Bell matrix $(g(x), xg(x))$ are given by $(n+1)$ times the elements of the first column of the Bell matrix*

$$\left(\frac{1}{g(x)}, \frac{x}{g(x)}\right)^{-1}.$$

Proof. By the theory of the Bell subgroup, we have

$$\left(\frac{1}{g(x)}, \frac{x}{g(x)}\right)^{-1} = \left(\frac{1}{x}\overline{\left(\frac{x}{g(x)}\right)}, \overline{\left(\frac{x}{g(x)}\right)}\right).$$

Since $T_{2n,n} = (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{g(x)}\right)}$ by the Theorem, the result follows. □

4 Examples

In this section, we show how, by calculating $T_{2n,n}$ in different ways, we can show that certain sequences of interest have a generating function of a given form. Normally these sequences will have the form $\frac{1}{n+1}T_{2n,n}$. We start with the best-known example, the Catalan numbers.

Example 5. The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

For this example, we consider, as in the introduction, the Riordan array

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

We can calculate $T_{2n,n}$ directly as follows.

$$\begin{aligned}
T_{2n,n} &= [x^{2n}]\frac{1}{1-x}\frac{x^n}{(1-x)^n} \\
&= [x^n](1-x)^{-n-1} \\
&= [x^n]\sum_{k=0}^{\infty}\binom{-n-1}{k}(-1)^k x^k \\
&= [x^n]\sum_{k=0}\binom{n+k}{k}x^k \\
&= \binom{2n}{n},
\end{aligned}$$

as expected. Now by the theorem, we look at the first column of the matrix $(1-x, x(1-x))^{-1}$. We obtain

$$T_{2n,n} = (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{\frac{1}{1-x}}\right)} = (n+1)[x^n]\frac{1}{x}\overline{x(1-x)}(x).$$

To carry out the reversion we solve $u(1-u) = x$ to obtain the result (with $u(0) = 0$) given by

$$u = \frac{1-\sqrt{1-4x}}{2}.$$

Thus

$$T_{2n,n} = (n+1)[x^n]\frac{1-\sqrt{1-4x}}{2x}.$$

Since $T_{2n,n} = \binom{2n}{n}$, we obtain

$$C_n = \frac{1}{n+1}\binom{2n}{n} = [x^n]\frac{1-\sqrt{1-4x}}{2x}.$$

Example 6. The generalized Catalan numbers $\frac{1}{2n+1}\binom{3n}{n}$. We wish to prove the following result about the generating function of $\frac{1}{2n+1}\binom{3n}{n}$:

$$\frac{1}{2n+1}\binom{3n}{n} = [x^n]\frac{2}{\sqrt{3x}}\sin\left(\frac{\arcsin\left(\sqrt{\frac{27x}{4}}\right)}{3}\right).$$

In order to do this, we calculate the central coefficients $T_{2n,n}$ of the Catalan triangle $(c(x), xc(x))$ A033184 in two ways. First of all, by the theorem, they are given by $(n+1)$ times the first column of

$$(1/c(x), x/c(x))^{-1}.$$

Thus

$$\begin{aligned}
T_{2n,n} &= (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{c(x)}\right)}(x) \\
&= (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x(1+\sqrt{1-4x})}{2}\right)}(x).
\end{aligned}$$

Taking the solution

$$u = \frac{2\sqrt{x}}{\sqrt{3}} \sin\left(\frac{\arcsin\left(\sqrt{\frac{27x}{4}}\right)}{3}\right)$$

of the equation

$$\frac{u(1+\sqrt{1-4u})}{2} = x,$$

we obtain

$$T_{2n,n} = (n+1)[x^n]\frac{2}{\sqrt{3x}} \sin\left(\frac{\arcsin\left(\sqrt{\frac{27x}{4}}\right)}{3}\right).$$

Secondly, we now note that directly, we have

$$T_{2n,n} = [x^{2n}]c(x)(xc(x))^n = [x^n]c(x)^{n+1}.$$

Then by Formula (2.4) of [5], used backwards, we obtain

$$T_{2n,n} = \frac{n+1}{n+1+2n}\binom{n+1+2n}{n} = \frac{n+1}{3n+1}\frac{(3n+1)(3n)!}{(2n+1)(2n)!n!} = \frac{n+1}{2n+1}\binom{3n}{n}.$$

The result follows immediately by comparing the two expressions for $T_{2n,n}$.

Example 7. We wish to show that the sequence A001002 with general term

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+k}{k}\binom{k}{n-k}$$

has g.f. given by

$$\frac{1}{3x}\left(4\sin\left(\frac{\arcsin\left(\frac{27x+11}{6}\right)}{3}\right) - 1\right).$$

To this end, we consider the central coefficients of the Riordan array

$$\left(\frac{1}{1-x-x^2}, \frac{x}{1-x-x^2}\right).$$

By the theorem, $T_{2n,n}$ for this array is given by $(n+1)$ times the first column of

$$((1-x-x^2), x(1-x-x^2))^{-1}.$$

Thus

$$\begin{aligned} T_{2n,n} &= (n+1)[x^n]\frac{1}{x}\overline{(x(1-x-x^2))}(x) \\ &= (n+1)[x^n]\frac{1}{x}\frac{1}{3}\left(4\sin\left(\frac{\arcsin\left(\frac{27x+11}{6}\right)}{3}\right) - 1\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
T_{2n,n} &= [x^{2n}]\frac{1}{1-x-x^2}\frac{x^n}{(1-x-x^2)^n} \\
&= [x^n]\frac{1}{(1-x-x^2)^{n+1}} \\
&= [x^n]\sum_{k=0}\binom{-n-1}{k}(-x)^k(1+x)^k \\
&= [x^n]\sum_{k=0}\binom{n+k}{k}x^k\sum_{j=0}^{k}\binom{k}{j}x^j \\
&= \sum_{k=0}\binom{n+k}{k}\binom{k}{n-k}.
\end{aligned}$$

This is A038112. A comparison of both expressions for $T_{2n,n}$ now yields the result.

Example 8. For our next example, we will use the following result.

Lemma 9. *Let* $f(x) = \frac{x}{1+ax^2+bx^2}$. *Then*

1.
$$\bar{f}(x) = \frac{1-ax-\sqrt{1-2ax+(a^2-4b)x^2}}{2bx},$$

2.
$$[x^n]\bar{f}(x) = \sum_{k=0}\binom{n-1}{2k}C_k a^{n-2k-1}b^k.$$

Proof. The first result follows by solving the equation

$$\frac{u}{1+au+bu^2} = x$$

and taking the determination for which $u(0) = 0$. The second result is a direct application of Lagrange inversion. We have

$$[x^n]\bar{f} = \frac{1}{n}[x^{n-1}](1+ax+bx^2)^n.$$

Expansion of the binomial now gives the result [1]. □

We note that this implies that

$$[x^n]\frac{1}{x}\bar{f}(x) = \sum_{k=0}\binom{n}{2k}C_k a^{n-2k}b^k.$$

We now wish to study the sequence A007440 with general term

$$\frac{1}{n+1}\sum_{k=0}\binom{n+1}{k}\binom{k}{n-k}(-1)^k = \sum_{k=0}\binom{n}{k}\binom{k}{n-k}\frac{(-1)^k}{n-k+1}.$$

For this, we consider the matrix

$$(1-x-x^2, x(1-x-x^2)).$$

For this matrix, $T_{2n,n}$ is given by $(n+1)$ times the first column of

$$\left(\frac{1}{1-x-x^2}, \frac{x}{1-x-x^2}\right)^{-1}.$$

Equivalently

$$T_{2n,n} = (n+1)[x^n]\frac{1}{x}\overline{\left(\frac{x}{1-x-x^2}\right)}(x).$$

We obtain

$$T_{2n,n} = (n+1)[x^n]\frac{\sqrt{1+2x+5x^2}-x-1}{2x^2} = [x^n]\frac{\sqrt{1+2x+5x^2}-x-1}{2x^2\sqrt{1+2x+5x^2}}.$$

By the lemma, we also have

$$T_{2n,n} = (n+1)\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}C_k(-1)^{n-k}.$$

This is essentially A104506. We now calculate $T_{2n,n}$ directly.

$$\begin{aligned}
T_{2n,n} &= [x^{2n}](1-x-x^2)x^n(1-x-x^2)^n \\
&= [x^n](1-x(1+x))^{n+1} \\
&= [x^n]\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k x^k \sum_{j=0}^{k}\binom{k}{j}x^j \\
&= \sum_{k=0}\binom{n+1}{k}\binom{k}{n-k}(-1)^k.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{n+1}\sum_{k=0}\binom{n+1}{k}\binom{k}{n-k}(-1)^k &= \sum_{k=0}\binom{n}{k}\binom{k}{n-k}\frac{(-1)^k}{n-k+1} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}C_k(-1)^{n-k} \\
&= [x^n]\frac{\sqrt{1+2x+5x^2}-x-1}{2x^2}.
\end{aligned}$$

This is the shifted reversion of the Fibonacci numbers. We remark in passing that the Hankel transform [4] of the Fibonacci reversion is $-(-1)^{\binom{n+1}{2}}F(n)$. It is also noteworthy that the exponential generating function of $T_{2n,n} = \sum_{k=0}\binom{n+1}{k}\binom{k}{n-k}(-1)^k$ is given by

$$\frac{d}{dx}e^{-x}I_1(2ix)/i, \quad i=\sqrt{-1}.$$

The Hankel transform of $T_{2n,n}$ begins

$$1, -4, -4, 11, 11, -29, -29, 76, 76, -199, -199, ...$$

where the sequence A002878 that begins $1, 4, 11, 29, 76, ...$ has general term L_{2n+1}.

Example 10. In our last example we wish to show that the sequence A078531 with general term

$$\frac{4^n}{n+1}\binom{\frac{3n-1}{2}}{n}$$

has g.f. given by

$$\frac{1}{x}\left(\frac{1}{12}+\frac{1}{6}\sin\left(\frac{\arcsin(216x^2-1)}{3}\right)\right).$$

To this end we consider the central coefficients of the Riordan array

$$\left(\frac{1}{\sqrt{1-4x}}, \frac{x}{\sqrt{1-4x}}\right).$$

The central coefficients $T_{2n,n}$ of this matrix will then be given by the first column of the Riordan array

$$\left(\sqrt{1-4x}, x\sqrt{1-4x}\right)^{-1}.$$

We find that

$$T_{2n,n} = (n+1)[x^n]\frac{1}{x}\left(\frac{1}{12}+\frac{1}{6}\sin\left(\frac{\arcsin(216x^2-1)}{3}\right)\right).$$

Now we can also calculate $T_{2n,n}$ directly as:

$$\begin{aligned} T_{2n,n} &= [x^{2n}]\frac{1}{\sqrt{1-4x}}\frac{x^n}{\sqrt{1-4x}^n} \\ &= [x^n](1-4x)^{-\frac{n+1}{2}} \\ &= [x^n]\sum_{k=0}\binom{-\frac{n+1}{2}}{k}(-1)^k 4^k x^k \\ &= [x^n]\sum_{k=0}\binom{\frac{n+1}{2}+k-1}{k}4^k x^k \\ &= \binom{\frac{3n-1}{2}}{n}4^n. \end{aligned}$$

Comparison of the expressions for $T_{2n,n}$ now gives the result.

Note that $T_{2n,n} = (n+1)$A078531(n) =A091527(n).

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