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# Asymptotic Expansions of Certain Sums Involving Powers of Binomial Coefficients

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**Abstract**

In this paper, we investigate the computation of sums involving binomial coefficients. We give asymptotic expansions of certain sums concerning powers of binomial coefficients by using the multi-Laplace asymptotic integral theorem.

## 1 Introduction

Sums related to binomial coefficients are involved in many subjects such as combinatorial analysis, graph theory, number theory, statistics and probability. For example, there is an application of sums involving inverses of binomial coefficients (see [4]). For the computation of sums concerning inverses of binomial coefficients, we may see for instance [10, 12, 13, 14, 15, 16]. Hence, it is important to discuss the computation of sums concerning binomial coefficients. In this paper, we study the computation of certain sums involving powers of binomial coefficients.

For convenience, we first give some notation and definitions involved in this paper. The binomial coefficient $\binom{n}{m}$ is defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & n \geq m; \\ 0, & n < m. \end{cases}$$

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where n and m are nonnegative integers.

Throughout this paper, $H_n(n \geq 1)$ stands for the harmonic number, and the definition of $H_n(n \geq 0)$ is

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n = 1, 2, \ldots.$$

The generating function of $H_n$ is

$$\sum_{n=1}^{\infty} H_n t^n = -\frac{\ln(1-t)}{1-t}.$$

The harmonic numbers have been generalized by several authors (see [1, 2, 5, 6, 7, 8, 9]):

$$\begin{aligned}
&H_n^{[0]} = \frac{1}{n}, \quad H_n^{[r]} = \sum_{k=1}^{n} H_k^{[r-1]} \quad (n, r \geq 1),\\
&H_{n,0} = 1,\ H_{n,r} = \sum_{1 \leq n_1 < \cdots < n_r \leq n} \frac{1}{n_1 n_2 \cdots n_r} \quad (n, r \geq 1),\\
&H(n, r) = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r} \quad (n \geq 1, r \geq 0).
\end{aligned}$$

It is clear that $H(n, 0) = H_n = H_{n,0} = H_n^{[1]}$. The generating functions of the above generalized harmonic numbers are as follows:

$$\sum_{n=1}^{\infty} H_n^{[r]} t^n = \frac{-\ln(1-t)}{(1-t)^r}, \tag{1}$$

$$\sum_{n=r+1}^{\infty} H_{n,r} t^n = \frac{(-1)^r (\ln(1-t))^r}{r!(1-t)}, \tag{2}$$

$$\sum_{n=r+1}^{\infty} H(n, r) t^n = \frac{(-1)^{r+1} (\ln(1-t))^{r+1}}{1-t}. \tag{3}$$

In this paper, we denote the Stirling numbers of the first kind and the second kind by $s(n, k)$, $S(n, k)$, respectively. The generating functions of the two kinds of Stirling numbers are as follows:

$$\sum_{n=k}^{\infty} s(n, k) \frac{t^n}{n!} = \frac{\ln^k(1+t)}{k!}, \qquad \sum_{n=k}^{\infty} S(n, k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.$$

It is well known that it is difficult to give the accurate values of sums involving inverses of binomial coefficients. However, the asymptotic expansions of certain sums related to inverses of binomial coefficients can be obtained. For the computation of sums related to inverses of binomial coefficients, integral representations is an effective method. We know that $\binom{n}{m}^{-1}$ is related to an integral (see [13]):

$$\binom{n}{m}^{-1} = (n+1)\int_0^1 x^m(1-x)^{n-m}dx. \tag{4}$$

We will derive the main results by using (4) and the next lemma (see[17]):

**Lemma 1.** *Let $D$ be a continuous bounded domain, and let $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$ and $f(x) = f(x_1, x_2, \ldots, x_n)$ be the real functions defined in $D$ and satisfy the following conditions:*
*(i) The second-order partial derivatives $f_{ik}$ $\left(\frac{\partial^2 f}{\partial x_i \partial x_k}\ (i,k = 1,2,\ldots,n)\right)$ exist and $f(x) > 0$ $(x \in D)$;*
*(ii) The product $\phi(x)[f(x)]^\lambda$ $(\lambda > 0)$ is absolutely integrable in $D$;*
*(iii) The function $f(x)$ reaches the effective maximum at the interior point $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ in $D$, i.e., $f_k(\varepsilon) = 0$, $H_k[-f(\varepsilon)] > 0$ $(k = 1,2,\ldots,n)$, where, $H_k[-f(x)] = det(-f_{ij}(x))_{1\le i,j\le k}$;*
*(iv) The function $\phi(x)$ is continuous at point $\varepsilon$ and $\phi(\varepsilon) \neq 0$.*
*Then when $\lambda \to \infty$,*

$$\iint\limits_{D}\cdots\int \phi(x)[f(x)]^\lambda\, dx_1 dx_2 \cdots dx_n \sim \frac{\phi(\varepsilon)[f(\varepsilon)]^{\lambda+\frac{n}{2}}}{\sqrt{H_n[-f(\varepsilon)]}}\left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}}.$$

In this paper, we discuss the computation of certain sums related to powers of binomial coefficients. In Section 2, we study the asymptotic expansions of sums such as

$$\sum_{n=0}^{\infty} \frac{1}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2},$$

and in Section 3, we investigate the asymptotic expansions of sums involving binomial coefficients and generalized harmonic numbers.

## 2 Asymptotic Expansions of a Class of Sums Involving Binomial Coefficients

In this section, we give the asymptotic expansions of sums involving inverses of binomial coefficients.

**Theorem 2.** *Let $t$, $p$, $j$ and $k$ be integers with $p > 0$, $j \geq 0$ and $k \geq 0$, we have the asymptotic expansions as follows:*
*(i) when $k$ is fixed,*

$$\textstyle\sum_{n=0}^{p} \frac{\binom{p}{n}}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2} \sim \frac{17^{p+1}\pi}{4^{2k+2p+1}p} \quad (p \to \infty), \tag{5}$$

*(ii) when $p$ is fixed,*

$$\sum_{n=0}^{p} \frac{\binom{p}{n}}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2} \sim \frac{17^p\pi}{4^{2k+2p+1}k} \quad (k\to\infty), \tag{6}$$

*(iii)*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2} \sim \frac{\pi}{4^{2k-1}15k} \quad (k\to\infty), \tag{7}$$

*(iv) when $k$ is fixed and $k \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{\binom{n+k-1}{n}}{(2n+2j+1)^2\binom{2n+2j}{n+j}^2} \sim \frac{\pi}{4^{2j-2k+1}15^k j} \quad (j\to\infty), \tag{8}$$

*(v)*

$$\sum_{n=0}^{\infty} \frac{\binom{n+k}{n}}{(2n+1)^2\binom{2n}{n}^2} \sim \frac{4^{2k+1}\pi}{15^k(k+1)} \quad (k\to\infty), \tag{9}$$

*(vi)*

$$\sum_{n=0}^{\infty} \frac{n\binom{n+k}{n}}{(2n+1)^2\binom{2n}{n}^2} \sim \frac{4^{2k+1}\pi(k+1)}{15^{k+1}(k+2)} \quad (k\to\infty). \tag{10}$$

*Proof.* In the proof, we will use the following formulas:

$$\sum_{k=1}^{n}\binom{n}{k}x^k = (1+x)^n, \tag{11}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \tag{12}$$

$$\sum_{n=0}^{\infty}\binom{n+k-1}{n}x^n = \frac{1}{(1-x)^k}, \tag{13}$$

$$\sum_{n=0}^{\infty}\binom{n+k}{n}x^n = \frac{1}{(1-x)^{k+1}}, \tag{14}$$

$$\sum_{n=0}^{\infty} n\binom{n+k}{n}x^n = \frac{(k+1)x}{(1-x)^{k+2}}. \tag{15}$$

It follows from (4) and (11)–(15) that

$$
\begin{aligned}
&\sum_{n=0}^{p} \frac{\binom{p}{n}}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2} \\
&= \sum_{n=0}^{p} \binom{p}{n} \int_0^1 x^{n+k}(1-x)^{n+k}\,dx \int_0^1 y^{n+k}(1-y)^{n+k}\,dy \\
&= \sum_{n=0}^{p} \binom{p}{n} \int_0^1 \int_0^1 [xy(1-x)(1-y)]^{n+k}\,dxdy \\
&= \int_0^1 \int_0^1 (xy(1-x)(1-y))^k \sum_{n=0}^{p} \binom{p}{n} (xy(1-x)(1-y))^n\,dxdy \\
&= \int_0^1 \int_0^1 (xy(1-x)(1-y))^k (1+xy(1-x)(1-y))^p\,dxdy,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2} \\
&= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 [xy(1-x)(1-y)]^{n+k}\,dxdy \\
&= \int_0^1 \int_0^1 \frac{(xy(1-x)(1-y))^k}{1-xy(1-x)(1-y)}\,dxdy,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\binom{n+k-1}{n}}{(2n+2j+1)^2\binom{2n+2j}{n+j}^2} \\
&= \int_0^1 \int_0^1 \sum_{n=0}^{\infty} \binom{n+k-1}{n} (xy(1-x)(1-y))^{n+j}\,dxdy \\
&= \int_0^1 \int_0^1 \frac{(xy(1-x)(1-y))^j}{(1-xy(1-x)(1-y))^k}\,dxdy,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\binom{n+k}{n}}{(2n+1)^2\binom{2n}{n}^2} \\
&= \int_0^1 \int_0^1 \sum_{n=0}^{\infty} \binom{n+k}{n} (xy(1-x)(1-y))^n\,dxdy \\
&= \int_0^1 \int_0^1 \frac{1}{(1-xy(1-x)(1-y))^{k+1}}\,dxdy,
\end{aligned}
$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{n\binom{n+k}{n}}{(2n+1)^2\binom{2n}{n}^2} \\
=&\int_0^1\int_0^1 \sum_{n=0}^{\infty} n\binom{n+k}{n}(xy(1-x)(1-y))^n\,dxdy \\
=&\int_0^1\int_0^1 \frac{(k+1)x(1-x)y(1-y)}{(1-xy(1-x)(1-y))^{k+2}}\,dxdy.
\end{aligned}$$

(i) Let $\phi(x,y) = (xy(1-x)(1-y))^k,\ f(x,y) = 1 + xy(1-x)(1-y)$. It is evident that $f(x,y) > 0$ for $(x,y) \in [0,1]\times[0,1]$,

$$\begin{aligned}
f_x &= y - 2xy - y^2 + 2xy^2, \\
f_y &= x - x^2 - 2xy + 2x^2y, \\
f_{xx} &= 2y^2 - 2y, \\
f_{yy} &= 2x^2 - 2x, \\
f_{xy} &= 1 - 2x - 2y + 4xy.
\end{aligned}$$

When $f_x = f_y = 0$, we get $x = \frac{1}{2},\ y = \frac{1}{2}$. Then we have

$$\begin{aligned}
H_1\Big[-f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] &= -f_{xx}\Big(\frac{1}{2},\frac{1}{2}\Big) \\
&= \frac{1}{2}, \\
H_2\Big[-f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] &= \begin{vmatrix} -f_{xx}\left(\frac{1}{2},\frac{1}{2}\right) & -f_{xy}\left(\frac{1}{2},\frac{1}{2}\right) \\ -f_{yx}\left(\frac{1}{2},\frac{1}{2}\right) & -f_{yy}\left(\frac{1}{2},\frac{1}{2}\right) \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} \\
&= \frac{1}{4}, \\
\phi\Big(\frac{1}{2},\frac{1}{2}\Big) &= 4^{-2k}, \\
f\Big(\frac{1}{2},\frac{1}{2}\Big) &= \frac{17}{16}.
\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}
&\int_0^1\int_0^1 (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p\,dxdy \\
\sim&\frac{4^{-2k}(17/16)^{p+1}}{1/2}\times\frac{2\pi}{p} \qquad (p\to\infty) \\
=&\frac{17^{p+1}\pi}{4^{2k+2p+1}p} \qquad (p\to\infty).
\end{aligned}$$

(ii) Let $\phi(x,y)=(1+xy(1-x)(1-y))^p,\ f(x,y)=xy(1-x)(1-y)$. It is evident that

$$\begin{aligned}
f_x &= y-2xy-y^2+2xy^2,\\
f_y &= x-x^2-2xy+2x^2y,\\
f_{xx} &= 2y^2-2y,\\
f_{yy} &= 2x^2-2x,\\
f_{xy} &= 1-2x-2y+4xy,\\
f(x,y) &> 0, \quad \Big((x,y)\in[\eta,1-\eta]\times[\eta,1-\eta], \quad 0<\eta<\frac{1}{16}\Big).
\end{aligned}$$

When $f_x=f_y=0$, we get $x=\frac{1}{2},\ y=\frac{1}{2}$. Then we have

$$\begin{aligned}
H_1\Big[-f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] &= \frac{1}{2},\\
H_2\Big[-f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] &= \frac{1}{4},\\
\phi\Big(\frac{1}{2},\frac{1}{2}\Big) &= \Big(\frac{17}{16}\Big)^p,\\
f\Big(\frac{1}{2},\frac{1}{2}\Big) &= \Big(\frac{1}{2}\Big)^4.
\end{aligned}$$

By using Lemma 1, we can derive

$$\begin{aligned}
&\int_\eta^{1-\eta}\int_\eta^{1-\eta}(xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p\,dxdy\\
\sim&\frac{(17/16)^p4^{-2k-2}}{1/2}\times\frac{2\pi}{k} \qquad (k\to\infty)\\
=&\frac{17^p\pi}{4^{2k+2p+1}k} \qquad (k\to\infty).
\end{aligned}$$

When $0<\eta<\frac{1}{16}$, it is obvious that

$$\begin{aligned}
&\int_0^1\int_0^1 (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy\\
=&\int_0^\eta\int_0^1 (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy\\
&+\int_{1-\eta}^1\int_0^1 (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy\\
&+\int_\eta^{1-\eta}\int_0^\eta (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy\\
&+\int_\eta^{1-\eta}\int_{1-\eta}^1 (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy\\
&+\int_\eta^{1-\eta}\int_\eta^{1-\eta} (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy\\
=&O(\eta^k(1-\eta)^k)+\int_\eta^{1-\eta}\int_\eta^{1-\eta} (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy.
\end{aligned}$$

When $0<\eta<\frac{1}{16}$, $\eta^k(1-\eta)^k=o\left(\frac{17^p\pi}{4^{2k+2p+1}k}\right)\quad(k\to\infty)$. Then we can obtain

$$\sum_{n=0}^{p}\frac{\binom{p}{n}}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2}\sim\frac{17^p\pi}{4^{2k+2p+1}k}\quad(k\to\infty).$$

(iii) Let $f(x,y)=xy(1-x)(1-y)$, $\phi(x,y)=\frac{1}{1-xy(1-x)(1-y)}$. By means of the method of proving (6), we derive

$$\sum_{n=0}^{\infty}\frac{1}{(2n+2k+1)^2\binom{2n+2k}{n+k}^2}\sim\frac{\pi}{4^{2k-1}15k}\quad(k\to\infty).$$

The proofs of (8)–(10) are similar to that of (5) and (6), and are omitted here. □

Now, we compare the accurate values with the asymptotic values. Let

$$A_{k,p}=\int_0^1\int_0^1 (xy(1-x)(1-y))^k(1+xy(1-x)(1-y))^p dxdy,\quad B_{k,p}=\frac{17^{p+1}\pi}{4^{2k+2p+1}p}.$$

From the above table, we note that $\frac{A_{1,p}}{B_{1,p}}$ is close to 1 with the increase of $p$.

By using the method of proving Theorem 2, we have

**Theorem 3.** *Let $t$, $p$, $j$ and $k$ be integers with $p>0$, $j\geq 0$ and $k\geq 0$, we have the asymptotic expansions as follows:*
*(i) when $k$ is fixed,*

$$\sum_{n=0}^{p}\frac{\binom{p}{n}}{(2n+2k+1)^3\binom{2n+2k}{n+k}^3}\sim\frac{65^{p+\frac{3}{2}}\pi^{\frac{3}{2}}}{2^{6k+6p+3}p^{\frac{3}{2}}}\quad(p\to\infty),$$

| p | $A_{1,p}$ | $B_{1,p}$ | $A_{1,p}/B_{1,p}$ |
|---|---|---|---|
| 10 | 0.0018 | 0.0096 | 0.1832 |
| 50 | 0.0122 | 0.0216 | 0.5640 |
| 100 | 3.0777 | 3.5835 | 0.8589 |
| 150 | 44.8969 | 49.5072 | 0.9069 |
| 200 | 715.9654 | 769.4432 | 0.9305 |
| 240 | 6828.1 | 7246.9 | 0.9422 |

Table 1: some values of $A_{1,p}$ and $B_{1,p}$

*(ii) when p is fixed,*

$$\sum_{n=0}^{p} \frac{\binom{p}{n}}{(2n+2k+1)^3\binom{2n+2k}{n+k}^3} \sim \frac{65^p \pi^{\frac{3}{2}}}{2^{6k+6p+3}k^{\frac{3}{2}}} \quad (k \to \infty),$$

*(iii)*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+2k+1)^3\binom{2n+2k}{n+k}^3} \sim \frac{\pi^{\frac{3}{2}}}{63k^{\frac{3}{2}}2^{6k-3}} \quad (k \to \infty),$$

*(iv) when k is fixed and $k \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{\binom{n+k-1}{n}}{(2n+2j+1)^3\binom{2n+2j}{n+j}^3} \sim \frac{\pi^{\frac{3}{2}}}{2^{6j-6k+3}63^k j^{\frac{3}{2}}} \quad (j \to \infty),$$

*(v)*

$$\sum_{n=0}^{\infty} \frac{\binom{n+k}{n}}{(2n+1)^3\binom{2n}{n}^3} \sim \frac{\pi^{\frac{3}{2}}2^{6k+3}}{63^{k-\frac{1}{2}}(k+1)^{\frac{3}{2}}} \quad (k \to \infty),$$

*(vi)*

$$\sum_{n=0}^{\infty} \frac{n\binom{n+k}{n}}{(2n+1)^3\binom{2n}{n}^3} \sim \frac{\pi^{\frac{3}{2}}2^{6k+3}(k+1)}{63^{k+\frac{1}{2}}(k+2)^{\frac{3}{2}}} \quad (k \to \infty).$$

## 3 Asymptotic Expansions of Certain Sums Involving Binomial Coefficients and Generalized Harmonic Numbers

There are many identities between the harmonic number and the binomial coefficient. For example (see [11]),

$$\sum_{n=1}^{\infty} \frac{H_n}{\binom{n+k}{k}} = \frac{k}{(k-1)^2} \quad (k > 1).$$

Now we give asymptotic expansions of certain sums involving the generalized harmonic numbers and the binomial coefficients.

**Theorem 4.** *Let $r$ be a positive integer, when $r \to \infty$, we have the following asymptotic expansions:*
*(i)*

$$\textstyle\sum_{n=1}^{\infty} H_n^{[r]} \frac{1}{(2n+1)^2 \binom{2n}{n}^2} \sim \frac{16^{r-1} 4\pi}{15^{r-1} r} \ln \frac{16}{15}, \tag{16}$$

*(ii)*

$$\textstyle\sum_{n=r+1}^{\infty} H_{n,r} \frac{(-1)^n}{(2n+1)^2 \binom{2n}{n}^2} \sim \frac{(-1)^r 4\pi}{r r!} \left(\ln \frac{17}{16}\right)^{r+1}, \tag{17}$$

*(iii)*

$$\textstyle\sum_{n=r+1}^{\infty} H(n,r) \frac{(-1)^n}{(2n+1)^2 \binom{2n}{n}^2} \sim \frac{(-1)^{r+1} 4\pi}{r+1} \left(\ln \frac{17}{16}\right)^{r+2}. \tag{18}$$

*Proof.* It follows from (1)–(4) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} H_n^{[r]} \frac{1}{(2n+1)^2 \binom{2n}{n}^2} \\
=&\sum_{n=1}^{\infty} H_n^{[r]} \int_0^1 \int_0^1 [xy(1-x)(1-y)]^n \, dxdy \\
=&\int_0^1 \int_0^1 \frac{-\ln\left(1-xy(1-x)(1-y)\right)}{\left(1-xy(1-x)(1-y)\right)^r} \, dxdy,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=r+1}^{\infty} H_{n,r} \frac{(-1)^n}{(2n+1)^2 \binom{2n}{n}^2} \\
=&\sum_{n=r+1}^{\infty} H_{n,r} (-1)^n \int_0^1 \int_0^1 [xy(1-x)(1-y)]^n \, dxdy \\
=&\frac{(-1)^r}{r!} \int_0^1 \int_0^1 \frac{\ln^r\left(1+xy(1-x)(1-y)\right)}{1+xy(1-x)(1-y)} \, dxdy,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=r+1}^{\infty} H(n,r) \frac{(-1)^n}{(2n+1)^2 \binom{2n}{n}^2} \\
=&\int_0^1 \int_0^1 \sum_{n=r+1}^{\infty} H(n,r) (-xy(1-x)(1-y))^n \, dxdy \\
=&(-1)^{r+1} \int_0^1 \int_0^1 \frac{\ln^{r+1}\left(1+xy(1-x)(1-y)\right)}{1+xy(1-x)(1-y)} \, dxdy.
\end{aligned}$$

Now we only give the detailed proof of (17), the others can be proved by the same method. Let

$$f(x,y)=\ln\left(1+xy(1-x)(1-y)\right),\quad \phi(x,y)=\frac{1}{1+xy(1-x)(1-y)}.$$

When $(x,y)\in[\eta,1-\eta]\times[\eta,1-\eta]$ $(0<\eta<\frac{1}{2})$, $f(x,y)>0$, $f_x$, $f_y$, $f_{xx}$, $f_{yy}$, $f_{xy}$ can be computed easily. When $f_x=0,\ f_y=0$, we get $x=\frac{1}{2},\ y=\frac{1}{2}$. Then we have

$$\begin{aligned}
H_1\Big[-f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] &= \frac{8}{17},\\
H_2\Big[-f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] &= \left(\frac{8}{17}\right)^2,\\
\phi\Big(\frac{1}{2},\frac{1}{2}\Big) &= \frac{16}{17},\\
f\Big(\frac{1}{2},\frac{1}{2}\Big) &= \ln\frac{17}{16}.
\end{aligned}$$

By using Lemma 1, we obtain

$$\begin{aligned}
&\int_\eta^{1-\eta}\int_\eta^{1-\eta}\frac{\ln^r\left(1+xy(1-x)(1-y)\right)}{1+xy(1-x)(1-y)}\,dxdy\\
\sim&\frac{(16/17)(\ln\frac{17}{16})^{r+1}}{8/17}\times\frac{2\pi}{r}\quad(r\to\infty)\\
=&\frac{4\pi}{r}\left(\ln\frac{17}{16}\right)^{r+1}\quad(r\to\infty).
\end{aligned}$$

We can easily derive that

$$\begin{aligned}
&\int_0^1\int_0^1\frac{\ln^r\left(1+xy(1-x)(1-y)\right)}{1+xy(1-x)(1-y)}\,dxdy-\int_\eta^{1-\eta}\int_\eta^{1-\eta}\frac{\ln^r\left(1+xy(1-x)(1-y)\right)}{1+xy(1-x)(1-y)}\,dxdy\\
<&(\ln\left(1+\eta^2(1-\eta)^2\right))^r4\eta(1-\eta)\\
=&(4\eta-4\eta^2)\ln^r\left(1+\eta^2(1-\eta)^2\right).
\end{aligned}$$

Let $\eta=\frac{1}{16}$, when $r\to\infty$,

$$(4\eta-4\eta^2)\ln^r\left(1+\eta^2(1-\eta)^2\right)=o\left(\frac{4\pi}{r}\left(\ln\frac{17}{16}\right)^{r+1}\right).$$

Hence

$$\sum_{n=r+1}^{\infty}H_{n,r}\frac{(-1)^n}{(2n+1)^2\binom{2n}{n}^2}\sim\frac{(-1)^r4\pi}{rr!}\left(\ln\frac{17}{16}\right)^{r+1}\qquad(r\to\infty).$$

□

In the final of this section, we give some asymptotic expansions of sums involving the Stirling numbers and the powers of binomial coefficients. We know that the Stirling numbers and the binomial coefficients satisfy (see[3]):

$$s(n,k) = \sum_{0\le h\le n-k} (-1)^h \binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h} S(n-k+h,h)$$
$$= \sum_{0\le j\le h\le n-k} (-1)^{j+h} \binom{h}{j}\binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h} \frac{(h-j)^{n-k+h}}{h!}.$$

For two kinds of Stirling numbers and the powers of binomial coefficients, we have

**Theorem 5.** *when $k \to \infty$, we have the following asymptotic expansions:*
*(i)*

$$\sum_{n=k}^{\infty} \frac{s(n,k)}{n!(2n+1)^2\binom{2n}{n}^2} \sim \frac{17\pi}{4kk!}\left(\ln\frac{17}{16}\right)^{k+1},$$

*(ii)*

$$\sum_{n=k}^{\infty} \frac{S(n,k)}{n!(2n+1)^2\binom{2n}{n}^2} \sim \frac{4\pi(e^{\frac{1}{16}}-1)^{k+1}}{kk!e^{\frac{1}{32}}}.$$

The proof of this theorem is similar to that of Theorem 4, and is omitted here.

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