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Relatively Prime Sets and a Phi Function for Subsets of $\{1, 2, \ldots, n\}$

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Abstract

A nonempty subset A of $\{1, 2, \ldots, n\}$ is said to be relatively prime if $\gcd(A) = 1$. Let $f(n)$ and $f_k(n)$ denote respectively the number of relatively prime subsets and the number of relatively prime subsets of cardinality k of $\{1, 2, \ldots, n\}$. Let $\Phi(n)$ and $\Phi_k(n)$ denote respectively the number of nonempty subsets and the number of subsets of cardinality k of $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n. In this paper, we obtain some properties of these functions.

1 Introduction

A nonempty subset A of $\{1, 2, \ldots, n\}$ is said to be *relatively prime* if $\gcd(A) = 1$. Nathanson [5] defined $f(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$, and for $k \geq 1$, he defined $f_k(n)$ to be the number of relatively prime subsets of cardinality k of $\{1, 2, \ldots, n\}$. Nathanson [5] defined $\Phi(n)$ and $\Phi_k(n)$, respectively, to be the number of nonempty subsets and the number of subsets of cardinality k of $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n. Sloane's sequence A085945 enumerates $f(n)$ and A027375 enumerates $\Phi(n)$. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, $\varphi(n)$ the Euler phi function and $\mu(n)$ the Möbius function. Nathanson [5] obtained the following explicit formulas for these functions.

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Theorem 1. *The following hold:*

(*a*) *For all positive integers n,*

$$f(n) = \sum_{d=1}^{n} \mu(d)\big(2^{\lfloor n/d \rfloor} - 1\big). \tag{1}$$

(*b*) *For all positive integers $n \geq 2$,*

$$\Phi(n) = \sum_{d|n} \mu(d) 2^{n/d}. \tag{2}$$

(*c*) *For all integers n and k,*

$$f_k(n) = \sum_{d=1}^{n} \mu(d) \binom{\lfloor n/d \rfloor}{k}. \tag{3}$$

(*d*) *For all integers n and k,*

$$\Phi_k(n) = \sum_{d|n} \mu(d) \binom{n/d}{k}. \tag{4}$$

Generalizations may be found in [2, 3, 4]. In 2009, Ayad and Kihel [1] studied some properties of the functions $f(n)$ and $\Phi(n)$. They showed that $f(n)$ is never a square if $n \geq 2$, and for any prime $l \neq 3$, $f(n)$ is not periodic modulo l. Moreover, they proved the following equality.

Theorem 2. *For any integer $n \geq 1$, we have*

$$f(n+1) - f(n) = \frac{1}{2}\Phi(n+1). \tag{5}$$

In this paper, we give a new simple proof of the above result and obtain some properties of these functions.

Theorem 3. *For all integers n and k, we have*

$$\Phi_k(n+1) = f_k(n+1) - f_k(n) + f_{k+1}(n+1) - f_{k+1}(n). \tag{6}$$

Remark 4. Note that for any positive integers n, $f_1(n+1) = f_1(n) = 1$. By Theorem 3, we have $f_2(n+1) - f_2(n) = \Phi_1(n+1) = \varphi(n+1)$. Thus for $n \geq 2$, we have $f_2(n+1) - f_2(n) \equiv 0 \pmod 2$, and $f_2(2) = 1$, thus $f_2(n) \equiv 1 \pmod 2$ for all $n \geq 2$.

Remark 5. By Theorem 3, for all $n \geq 2$ we have $f_2(n) - f_2(n-1) = \varphi(n)$. Thus

$$\sum_{2 \leq i \leq n} \varphi(i) = \sum_{2 \leq i \leq n} \Big(f_2(i) - f_2(i-1)\Big) = f_2(n),$$

hence

$$f_2(n) = \frac{3n^2}{\pi^2} + O(n \log n).$$

Remark 6. Let p, q be primes. Note that $\Phi(p) = 2^p - 2$ and $\Phi(pq) = 2^{pq} - 2^p - 2^q + 2$. Thus $\Phi(p) \equiv \Phi(pq) \equiv 2 \pmod 4$. And $\Phi(n) \equiv 0 \pmod 3$ for all $n \geq 3$ (see [1]); thus $\Phi(p) \equiv \Phi(pq) \equiv 6 \pmod{12}$. Hence $\Phi(p)$ and $\Phi(pq)$ are neither a square nor a cube.

To prove Theorem 2 and 3, we need the following lemma.

Lemma 7. *For all integers n and k,*

$$\left\lfloor \frac{n+1}{k} \right\rfloor - \left\lfloor \frac{n}{k} \right\rfloor = \begin{cases} 1, & \text{if } k \mid (n+1); \\ 0, & \text{otherwise.} \end{cases}$$

2 Proof of Theorem 2.

By (1) we have

$$\begin{aligned} f(n+1) - f(n) &= \sum_{d=1}^{n+1} \mu(d)\big(2^{\lfloor \frac{n+1}{d} \rfloor} - 1\big) - \sum_{d=1}^{n} \mu(d)\big(2^{\lfloor n/d \rfloor} - 1\big) \\ &= \sum_{d=1}^{n} \mu(d)\big(2^{\lfloor \frac{n+1}{d} \rfloor} - 2^{\lfloor n/d \rfloor}\big) + \mu(n+1). \end{aligned}$$

By (2) and Lemma 7,

$$\begin{aligned} f(n+1) - f(n) &= \sum_{\substack{d=1 \\ d \mid n+1}}^{n} \mu(d) 2^{\lfloor \frac{n}{d} \rfloor} + \mu(n+1) \\ &= \sum_{d \mid n+1} \mu(d) 2^{\lfloor n/d \rfloor} \\ &= \frac{1}{2} \sum_{d \mid n+1} \mu(d) 2^{\frac{n+1}{d}} = \frac{1}{2} \Phi(n+1). \end{aligned}$$

This completes the proof of Theorem 2.

3 Proof of Theorem 3.

Case 1: $k = 1$. $f_1(n+1) = f_1(n) = 1$. By (3) we have

$$\begin{aligned} f_2(n+1) - f_2(n) &= \sum_{d=1}^{n+1} \mu(d) \binom{\lfloor \frac{n+1}{d} \rfloor}{2} - \sum_{d=1}^{n} \mu(d) \binom{\lfloor n/d \rfloor}{2} \\ &= \sum_{d=1}^{n} \mu(d) \left(\binom{\lfloor \frac{n+1}{d} \rfloor}{2} - \binom{\lfloor n/d \rfloor}{2} \right). \end{aligned}$$

By (2) and Lemma 7, we have

$$
\begin{aligned}
f_2(n+1) - f_2(n) &= \sum_{\substack{d=1 \\ d|n+1}}^{n} \mu(d) \binom{\frac{n+1}{d} - 1}{1} \\
&= \sum_{\substack{d=1 \\ d|n+1}}^{n} \mu(d) \left(\frac{n+1}{d} - 1 \right) \\
&= \sum_{d|n+1} \mu(d) \left(\frac{n+1}{d} - 1 \right) \\
&= \sum_{d|n+1} \mu(d) \frac{n+1}{d} = \Phi_1(n+1).
\end{aligned}
$$

Case 2: $k \geq 2$. By (3) and Lemma 7, we have

$$
\begin{aligned}
f_k(n+1) - f_k(n) &= \sum_{d=1}^{n+1} \mu(d) \binom{\lfloor \frac{n+1}{d} \rfloor}{k} - \sum_{d=1}^{n} \mu(d) \binom{\lfloor n/d \rfloor}{k} \\
&= \sum_{d=1}^{n} \mu(d) \left(\binom{\lfloor \frac{n+1}{d} \rfloor}{k} - \binom{\lfloor n/d \rfloor}{k} \right) \\
&= \sum_{d|n+1} \mu(d) \binom{\lfloor \frac{n}{d} \rfloor}{k-1}.
\end{aligned}
$$

Thus by (4) and Lemma 7, we have

$$
\begin{aligned}
f_k(n+1) - f_k(n) + f_{k+1}(n+1) - f_{k+1}(n) &= \sum_{d|n+1} \mu(d) \left(\binom{\lfloor \frac{n}{d} \rfloor}{k-1} + \binom{\lfloor \frac{n}{d} \rfloor}{k} \right) \\
&= \sum_{d|n+1} \mu(d) \binom{\lfloor \frac{n}{d} \rfloor + 1}{k} \\
&= \sum_{d|n+1} \mu(d) \binom{\frac{n+1}{d}}{k} = \Phi_k(n+1).
\end{aligned}
$$

This completes the proof of Theorem 3.

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