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The Terms in Lucas Sequences Divisible by Their Indices

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Abstract

For Lucas sequences of the first kind $(u_n)_{n\geq 0}$ and second kind $(v_n)_{n\geq 0}$ defined as usual by $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, $v_n = \alpha^n + \beta^n$, where α and β are either integers or conjugate quadratic integers, we describe the sets $\{n \in \mathbb{N} : n \text{ divides } u_n\}$ and $\{n \in \mathbb{N} : n \text{ divides } v_n\}$. Building on earlier work, particularly that of Somer, we show that the numbers in these sets can be written as a product of a so-called *basic* number, which can only be 1, 6 or 12, and particular primes, which are described explicitly. Some properties of the set of all primes that arise in this way is also given, for each kind of sequence.

1 Introduction

Given integers P and Q, let α and β be the roots of the equation

$$x^2 - Px + Q = 0.$$

Then the well-known *Lucas sequence of the first kind* (or *generalised Fibonacci sequence*) $(u_n)_{n\geq 0}$ is given by $u_0 = 0, u_1 = 1$ and $u_{n+2} = Pu_{n+1} - Qu_n$ for $n \geq 0$, or explicitly by Binet's formula

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

when $\Delta = (\alpha-\beta)^2 = P^2 - 4Q \neq 0$, and $u_n = n\alpha^{n-1}$ when $\Delta = 0$. In this latter case α is an integer, and so n divides u_n for all $n \geq 1$. In Theorem 1 below we describe, for all pairs (P, Q), the set $S = S(P, Q)$ of all $n \geq 1$ for which n divides u_n.

Corresponding to Theorem 1 we have a similar result (Theorem 12 below) for the *Lucas sequence of the second kind* $(v_n)_{n\geq 0}$, given by $v_0 = 2, v_1 = P$ and $v_{n+2} = Pv_{n+1} - Qv_n$ for $n \geq 0$, or explicitly by the formula

$$v_n = \alpha^n + \beta^n,$$

finding the set $T = T(P, Q)$ of all $n \geq 1$ for which n divides v_n. The results for the set T are given in Section 4.

For $n \in S$, define $\mathcal{P}_{S,n}$ to be the set of primes p such that $np \in S$. We call an element n of S *(first kind) basic* if there is no prime p such that n/p is in S. We shall see that, for given P, Q, there are at most two basic elements of S. It turns out that all elements of S are generated from basic elements using primes from these sets.

Theorem 1. *(a) For $n \in S$, the set $\mathcal{P}_{S,n}$ is the set of primes dividing $u_n\Delta$.*

(b) Every element of S can be written in the form $bp_1 \ldots p_r$ for some $r \geq 0$, where $b \in S$ is basic and, for $i = 1, \ldots, r$, the numbers $bp_1 \ldots p_{i-1}$ are also in S, and p_i is in $\mathcal{P}_{S,bp_1\ldots p_{i-1}}$.

(c) The (first kind) basic elements of S are

- 1 *and* 6 *if* $P \equiv 3 \pmod 6$, $Q \equiv \pm 1 \pmod 6$*;*
- 1 *and* 12 *if* $P \equiv \pm 1 \pmod 6$, $Q \equiv -1 \pmod 6$*;*
- 1 *only, otherwise.*

Note that the primes in part (b) need not be distinct.

Somer [20, Theorem 4] has many results in the direction of this theorem. In particular, he already noted the importance of 6 and 12 for this problem. Walsh [23, unpublished] gave an equivalent categorization of $S(1, -1)$ (the Fibonacci numbers case), a case where 1 and 12 are the basic elements of $S(1, -1)$.

Note that if α and β are integers, then at least one of P, Q is even, so that 1 is the only basic element in this case. In this case, too, it is known (see André-Jeannin [2]) that $S = \{n : n \mid \alpha^n - \beta^n\}$. (His result is stated assuming that $\gcd(n, \alpha\beta) = 1$, and his proof given for n square-free). This follows straight from Proposition 11 below. Further, for α and β integers with $\gcd(\alpha, \beta) = 1$, Győry [10] proved that, for a fixed integer r, the number of elements of S with r prime factors was finite, and described how to find them. See also [11] for the more general problem of the divisibility of $\alpha^n - \beta^n$ (α, β integers) by powers of n.

Now let $\mathcal{P}_S$ be the set of primes p that divide some n in S. It is easy to see that $\mathcal{P}_S = \cup_{n\in S}\mathcal{P}_{S,n}$. It is interesting to compare $\mathcal{P}_{S,n}$ and $\mathcal{P}_{S,np}$ for n and np in S. Write $u_n = u_n(\alpha, \beta)$ to show the dependence of u_n on α and β, and denote $u_n(\alpha^k, \beta^k)$ by $u_n^{(k)}$. Then since

$$u_{kn} = u_k^{(n)} u_n, \tag{1}$$

we have $u_n \mid u_{np}$, so that $\mathcal{P}_{S,n} \subset \mathcal{P}_{S,np}$ by Theorem 1(b). Thus when we multiply $n \in S$ by a succession of primes according to Theorem 1(b) to stay within S, the associated set $\mathcal{P}_{S,n}$ does not lose any primes. Hence we obtain the following consequence of Theorem 1(a).

Corollary 2. *If $n \in S$ and all prime factors of m divide $u_n\Delta$, then $nm \in S$.*

This is a strengthening of the known result (see e.g., [20, Theorem 5(i)]) that if $n \in S$ and all prime factors of m divide $n\Delta$, then $nm \in S$. In particular $(n = 1)$ $\Delta \in S$ and, for $n \in S$, both $u_n = n \cdot (u_n/n) \in S$ and $u_n\Delta \in S$.

In Section 7 we give the conditions on P and Q that make S, $\mathcal{P}_S$, T or $\mathcal{P}_T$ finite. In Section 8 we compare $\mathcal{P}_S$ with the set $\mathcal{P}_{\rm 1st}$ of primes that divide some u_n and the set $\mathcal{P}_T$ with the set $\mathcal{P}_{\rm 2nd}$ of primes that divide some v_n with $n \geq 1$. In Section 9 we briefly discuss divisibility properties of the sequences S and T. These properties are useful for generating the sequences efficiently.

It is of interest to estimate $\{n \in S : n \leq x\}$ and $\{n \in T : n \leq x\}$. It is planned to do this in a forthcoming paper of Shparlinski and the author. For $\mathcal{P}_S$ infinite (and not the set $\mathcal{P}$ of all primes!) it would also be of interest to estimate the relative density of $\mathcal{P}_S$ in $\mathcal{P}$. But this seems to be a more difficult problem (as does the corresponding problem for T).

A basic reference for Lucas numbers is the monograph of Williams [24]. See also Dickson [8, Chapter 17], and Ribenboim [17]. For a more general reference on recurrence sequences see the book [9] by Everest, van der Poorten, Shparlinski, and Ward.

2 Preliminary results for S.

While Theorem 1(b) allows us to multiply $n \in S$ by the primes in $\mathcal{P}_{S,n}$ to stay within S, a vital ingredient in proving Theorem 1(c) is to be able to do the opposite: to divide $n \in S$ by a prime and stay within S. This is provided by the following significant result, due to Somer, generalising special cases due to Jarden [13, Theorem E], Hoggatt and Bergum [12] and Walsh [23] for the Fibonacci sequence (i.e., $P = 1$, $Q = -1$) and André-Jeannin [2] for $\gcd(P, Q) = 1$.

Theorem 3 (Somer [20, Theorem 5(iv)])**.** *Let $n \in S$, $n > 1$, with $p_{\max}$ its largest prime factor. Then, except in the case that P is odd and n is of the form $2^\ell \cdot 3$ for some $\ell \geq 1$, we have $n/p_{\max} \in S$.*

We produce a variant of this result to cover all but two of the exceptional cases, as follows.

Proposition 4. *If P is odd and $n = 2^\ell \cdot 3 \in S$, where $\ell \geq 3$, then $n/2 \in S$.*

The idea of the proof of Theorem 3 is roughly (i.e., ignoring some details) as follows. Let n have prime factorization $n = \prod_p p^{k_p}$, with $\omega(n)$, the *rank of appearance of* n, being the least integer k such that $n \mid u_k$. Then $n \mid u_n$ is equivalent to $\omega(n) \mid n$. Since $\omega(n) = \mathrm{lcm}_p\, \omega\left(p^{k_p}\right)$, and every $\omega\left(p^{k_p}\right)$ is of the form $p^{k'_p}\ell_p$, where $k'_p < k_p$ and $\ell_p \mid (p^2 - 1)$, it follows that $n \mid u_n$ is equivalent to

$$\mathrm{lcm}_{p|n}\left(p^{k'_p}\ell_p\right) \mid n = \prod_{p|n} p^{k_p}. \tag{2}$$

But since for $p > 2$ all prime factors of $p^2 - 1$ are less than p, and $2^2 - 1 = 3$, if equation (2) holds, it will still hold with n replaced by $n/p_{\max}$ when $p_{\max} > 3$ or $p_{\max} = 3$ and (n odd or $2 \mid n$ with $\ell_2 = 1$). When $p_{\max} = 3$ and $2 \mid n$ with $\ell_2 = 3$, (2) will still hold with n replaced by $n/3$ as long as $n/3$ is divisible by 3.

For the proof of Theorem 1, we first need the following, which dates back to Lucas [15, p. 295] and Carmichael [7, Lemma II]. It is the special case $n = 1$ of Theorem 1(a).

Lemma 5. *For any prime p, p divides u_p if and only if p divides Δ.*

Proof. Now $u_2 = P$ and $\Delta = P^2 - 4Q \equiv u_2 \pmod 2$, so the result is true for $p = 2$. The result is trivial for $\Delta = 0$. Now for $\Delta \neq 0$ and $p \geq 3$,

$$\begin{aligned}\Delta^{(p-1)/2} &= \frac{(\alpha-\beta)^p}{(\alpha-\beta)} \\ &= u_p + \sum_{j=1}^{p-1} \binom{p}{j} \alpha^{p-j}(-\beta)^j/(\alpha-\beta) \\ &= u_p + \sum_{j=1}^{(p-1)/2} \binom{p}{j} (-1)^j Q^j u_{p-2j} \\ &\equiv u_p \pmod p,\end{aligned}$$

giving the result. $\square$

We have the following.

A prime is called *irregular* if it divides Q but not P. Clearly $p \nmid \Delta$ for p irregular. A prime that is not irregular is called *regular*.

Lemma 6 (Lucas [15, pp. 295–297], Carmichael [7, Theorem XII], Somer [20, Proposition 1(viii)], Williams [24, pp. 83–84])**.** *If p is an odd prime with $p \nmid Q$, $p \nmid \Delta$, then $p \mid u_{p-\varepsilon}$, where ε is the Legendre symbol $\left(\frac{\Delta}{p}\right)$. On the other hand, if p is irregular then it does not divide any u_n, $n \geq 1$.*

The following result follows easily from Lemmas 5 and 6.

Corollary 7. *The set $\mathcal{P}_{\rm 1st}$ of primes that divide some u_n, $n \geq 1$ consists precisely of the regular primes.*

Lemma 8 (Somer [20, Theorem 5(ii)])**.** *If $m, n \in S$ then $\operatorname{lcm}(m, n) \in S$.*

Proof. Put $\ell = \operatorname{lcm}(m, n)$. From (1) we have $u_n \mid u_\ell$, $u_m \mid u_\ell$, so $n \mid u_n$, $m \mid u_m$ and hence $\ell \mid u_\ell$. $\square$

Lemma 9. *We have*

(i) If P is odd and $2^\ell \mid u_{12}$, where $\ell \geq 1$, then $2^{\ell-1} \mid u_6$;

(ii) If $3 \mid u_{8k}$ then $3 \mid u_{4k}$.

Proof. Using the notation

$$P^{(k)} = P(\alpha^k, \beta^k) = \alpha^k + \beta^k = v_k, \qquad Q^{(k)} = Q(\alpha^k, \beta^k) = Q^k,$$

we have $P^{(2)} = P^2 - 2Q$ and

$$P^{(4)} = (P^2 - 2Q)^2 - 2Q^2 = P^4 - 4P^2Q + 2Q^2. \tag{3}$$

(i) Take P odd. Then

$$P^{(2)} \equiv \begin{cases} 1 \pmod 4, & \text{if } Q \text{ even;} \\ -1 \pmod 4, & \text{if } Q \text{ odd,} \end{cases}$$

and so $P^{(4)} \equiv P^{(2)} \pmod 4$ and

$$v_6 = P^{(2)}(P^{(4)} - Q^2) \equiv \begin{cases} 1 \pmod 4, & \text{if } Q \text{ even;} \\ 2 \pmod 4, & \text{if } Q \text{ odd.} \end{cases}$$

Since $u_{12} = u_6 v_6$ by (1) and $2 \nmid u_{12}$ for Q even, we get the result.

(ii) Since $u_{4k} = u_k^{(4)} u_4$, it is enough to prove that if $3 \mid u_{2k}^{(4)}$ and $3 \nmid u_4$ then $3 \mid u_k^{(4)}$. Now, working modulo 3, $P^{(4)} \equiv P^2(1 - Q) - Q^2$, using (3) and $P^4 \equiv P^2$. Thus

$$\binom{P^{(4)}}{Q^{(4)}} = \begin{cases} \binom{0}{0}, & \text{if } P \equiv Q \equiv 0; \\ \binom{1}{0}, & \text{if } P \equiv \pm 1, Q \equiv 0; \\ \binom{1}{1}, & \text{if } P \equiv \pm 1, Q \equiv -1; \\ \binom{-1}{1}, & \text{otherwise.} \end{cases}$$

The result holds in the first case because $u_4 \equiv 0$, and in the second case because $u_n^{(4)} \equiv 1$ for all $n \geq 1$. In the other two cases, $u_n^{(4)} \equiv 0$ precisely when $3 \mid n$, so the result holds also in these cases.

□

Proposition 10. *If P is odd and $2^\ell \cdot 3 \in S$, where $\ell \geq 3$, then $2^{\ell-1} \cdot 3 \in S$. In particular, then $12 \in S$.*

Proof. Take P odd. Then $P^{(2)} = P^2 - 2Q$ is also odd, and hence so are all $P^{(2^\ell)} = v_{2^\ell}$ for $\ell \geq 0$. Then for $\ell \geq 3$, using (1) and $u_{2k} = u_k v_k$ we have

$$u_{2^\ell . 3} = u_{12}^{(2^{\ell-2})} u_{2^{\ell-2}} = u_{12}^{(2^{\ell-2})} v_{2^{\ell-3}} v_{2^{\ell-4}} \ldots v_2 v_1.$$

So if $2^\ell \mid u_{2^\ell . 3}$ then $2^\ell \mid u_{12}^{(2^{\ell-2})}$ so, by Lemma 9(i), $2^{\ell-1} \mid u_6^{(2^{\ell-2})}$. Hence

$$2^{\ell-1} \mid u_6^{(2^{\ell-2})} u_{2^{\ell-2}} = u_{2^{\ell-1} . 3}.$$

Also, if $3 \mid u_{2^\ell . 3}$ where $\ell \geq 3$ then $3 \mid u_{2^{\ell-1} . 3}$, by Lemma 9(ii). Thus we have proved that if $\ell \geq 3$ and $2^\ell \cdot 3 \in S$ then $2^{\ell-1} \cdot 3 \in S$. Then $12 \in S$ follows. □

Proposition 11. *For any positive integer n and distinct integers a, b,*

$$n \mid a^n - b^n \Longrightarrow n \mid \frac{a^n - b^n}{a - b}.$$

Proof. For any prime p, suppose that $p^\ell \| a - b$ and $p^r \| n$. It is clearly enough to prove that $p^{r+\ell} \mid a^n - b^n$ whenever $\ell > 0$. Put $a = b + \lambda p^\ell$. Then

$$\begin{aligned} a^n - b^n &= \sum_{k=1}^{n} \binom{n}{k} \lambda^k p^{\ell k} b^{n-k} \\ &= \sum_{k=1}^{n} \frac{n}{k} \binom{n-1}{k-1} \lambda^k p^{\ell k} b^{n-k} \\ &\equiv 0 \pmod{p^L}, \end{aligned}$$

where

$$\begin{aligned} L &\geq r + \min_{k=1}^{n} (\ell k - \lfloor \log_p k \rfloor) \\ &\geq r + \ell + \min_{k=1}^{n} (\ell(k-1) - \log_2 k) \\ &\geq r + \ell + \min_{k=1}^{n} ((k-1) - \log_2 k) \\ &= r + \ell. \end{aligned}$$

□

3 Proof of Theorem 1.

To prove part (a), take $n \in S$ and p prime. First note that, from (1), $u_{np} = u_p^{(n)} u_n$. Now suppose that $np \mid u_{np}$. Then either $p \mid u_n$, or, by Lemma 5, we have $p \mid \Delta^{(n)}$, where $\Delta^{(n)} = (\alpha^n - \beta^n)^2 = u_n^2 \Delta$. Hence $p \mid u_n \Delta$.

Conversely, suppose $p \mid u_n \Delta$. Then $p \mid \Delta^{(n)}$, so that, by Lemma 5, $p \mid u_p^{(n)}$, giving $pn \mid u_p^{(n)} u_n = u_{np}$.

To prove (b), take $n \in S$, $n \neq 1, 6$ or 12. If $3 \in S$ then $3/3 = 1 \in S$. Otherwise, by Theorem 3 and Proposition 10, we have $n/p \in S$ for some prime factor p of n. Thus we obtain a sequence $n, n/p, (n/p)/p', \ldots$ of elements of S, which stops only at 1, 6 or 12. But clearly 6 and 12 cannot both be basic, so the process will stop at either 1 (always basic!) or at most one of 6 and 12. This shows that this sequence, written backwards, must be of the form $b, bp_1, bp_1p_2, \ldots, bp_1 \ldots p_r$, say, as required. By (a), we know that p_i is in $\mathcal{P}_{S, bp_1 \ldots p_{i-1}}$.

To prove (c), we just need to find for which P, Q the numbers 6 or 12 are basic.

The case $\boldsymbol{6 \in S, 3 \notin S, 2 \notin S}$. Since $u_2 = P$, we know that $2 \in S$ iff P is even. Hence P is odd. Also

$$u_6 = u_3 v_3 = (P^2 - Q)(P^2 - 3Q)P. \tag{4}$$

As $6 \mid u_6$ and $3 \nmid u_3 = P^2 - Q$, we have $3 \mid P$, and so $Q \equiv \pm 1 \pmod 3$. Also Q must be odd, so $P \equiv 3 \pmod 6$ and $Q \equiv \pm 1 \pmod 6$.

The case $12 \in S, 6 \notin S, 4 \notin S$. Since $2 \notin S$ by Corollary 2, we have P odd, as above. Now $u_{12} = u_6 v_6$ and

$$v_6 = v_3^{(2)} = (P^2 - 2Q)((P^2 - 2Q)^2 - 3Q^2). \tag{5}$$

If Q were even, then by (4) and (5) u_6, v_6, and u_{12} would all be odd. So Q is odd. As u_6 is then even, $3 \nmid u_6$, and we have $P \equiv \pm 1 \pmod 3$ and $Q \equiv 0$ or $-1 \pmod 3$. As $3 \mid u_{12}$, also $3 \mid v_6 \equiv (P^2 - 2Q)^3 \pmod 3$, giving $Q \equiv -1 \pmod 3$. Hence $P \equiv \pm 1 \pmod 6$ and $Q \equiv -1 \pmod 6$.

The converse for both of these cases is easily checked.

4 The set T

The results for the set $T = \{n \in \mathbb{N} : n \mid v_n\}$ differ slightly from those for S. Essentially, this is because of difficulties at the prime 2: v_n divides v_{np} for p odd, but not in general for $p = 2$. The main result is the following. For $n \in T$, define $\mathcal{P}_{T,n}$ to be the set of primes p such that $np \in T$. A prime is said to be *special* if it divides both P and Q. It is clear from applying the recurrence relation that all v_n for $n \geq 1$ are divisible by $\gcd(P, Q)$, and so by all special primes. We say that an element n of T is *(second kind) basic* if there is no prime p such that n/p is in T.

Theorem 12. *(a) For $n \in T$, the set $\mathcal{P}_{T,n}$ is the set of odd primes dividing v_n, with the possible inclusion of 2. Specifically, the prime 2 is in $\mathcal{P}_{T,n}$ if and only if n is a product of special primes and either*

- *P is even;*

  or

- *Q is odd and $3 \mid n$.*

(b) Every element of T can be written in the form $bp_1 \dots p_r$ for some $r \geq 0$, where $b \in T$ is (second kind) basic and, for $i = 1, \dots, r$, the numbers $bp_1 \dots p_{i-1}$ are also in T, and p_i is in $\mathcal{P}_{bp_1 \dots p_{i-1}}$.

(c) The (second kind) basic elements of T are

- *1 and 6 if $P \equiv \pm 1 \pmod 6$, $Q \equiv -1 \pmod 6$;*
- *1 only, otherwise.*

As in Theorem 1, the primes in part (b) of Theorem 12 need not be distinct. Note that parts (a) and (b) of the theorem imply that, unless 2 is special, no element of T is divisible by 4. Again, Somer [21, Theorem 4] had many results concerning the set T. In particular, he already noted the importance of 6 for its structure.

We now compare $\mathcal{P}_{T,n}$ and $\mathcal{P}_{T,np}$, as we did $\mathcal{P}_{S,n}$ and $\mathcal{P}_{S,np}$. But, in this case, the prime 2 is, unsurprisingly, anomalous.

Corollary 13. *(a) For an odd prime p in $\mathcal{P}_{T,n}$, we have $p \in \mathcal{P}_{T,np}$;*

(b) For q an odd prime with $q \in \mathcal{P}_{T,n}$, we have $q \in \mathcal{P}_{T,2n}$ if and only if $q \mid Q$;

(c) For $2 \in \mathcal{P}_{T,n}$, we have $2 \in \mathcal{P}_{T,2n}$ if and only if 2 is special.

Proof. Part (a) follows from the fact that for p odd $v_n \mid v_{np}$, combined with Theorem 12(a). For (b), we know from Theorem 12(a) that $q \mid v_n$. Then from $v_{2n} = v_n^2 - 2Q^n$ we see that $q \mid v_{2n}$ iff $q \mid Q$. For (c), we know from Theorem 12(a) that for $2 \in \mathcal{P}_{T,2n}$ all prime divisors of $2n$ are special, so 2 is special. Conversely, if 2 is special, then all prime factors of $2n$ are special, and P is even, so that, by Theorem 12(a), $2 \in \mathcal{P}_{T,2n}$. □

Corollary 14. *If $n \in T$ and*

- *all odd prime factors of m divide v_n;*

  and

- *if m is even then every prime divisor of nm is special;*

then $nm \in T$.

Proof. On successively multiplying n by first the odd and then the even prime divisors of m, we see from Theorem 12(a) that the stated conditions ensure that we stay within T while doing this. □

This result extends Theorem 5(i) of Somer [21], which has the condition that 'm is a product of special primes or divides n' instead of 'all odd prime factors of m divide v_n'.

5 Preliminary results for T.

We first quote the important result of Somer for T, corresponding to his result (Theorem 3 above) for S.

Theorem 15 (Somer [21, Theorem 5])**.** *Theorem 3 holds with the set S replaced by the set T.*

Jarden [13, Theorem E] proved this result for the classical Lucas sequence (i.e., $P = 1$, $Q = -1$) under the restriction $p_{\max} \neq 3$.

Lemma 16. *Suppose q is a special prime. Then $q^{e_n} \mid v_n$, where $e_n \geq \lfloor \log_q n \rfloor$.*

Proof. From the recurrence, it is easy to see that we can take

$$e_n = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{if } q = 2; \\ \left\lfloor \frac{n+1}{2} \right\rfloor, & \text{if } q \geq 3, \end{cases}$$

the slightly higher value for $q = 2$ coming from the fact that $v_0 = 2$. Then use $\lfloor \log_q n \rfloor \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. □

We then immediately obtain the following.

Corollary 17 (Special case of Somer [21, Theorem 5(i)])**.** *If n is a product of special primes then it belongs to T.*

We can now extend Theorem 15 as follows.

Proposition 18. *If $\ell \geq 2$ and $2^\ell \cdot 3 \in T$, then $2^\ell \in T$.*

Proof. Put $L = 2^\ell$. If 2 is special, then, by Corollary 17, $L \in T$ for all $\ell \geq 1$. So we can assume that 2 is not special. We then know that Q must be odd, as if it were even then we would have $2 \mid v_{3L}$ and $v_{3L} \equiv P^{3L} \pmod{Q}$, so P would be even and 2 special.

From $L \mid v_{3L} = v_L(v_L^2 - 3Q^L)$ we see that if v_L were odd then, as L is even, Q^L is a square, and so $v_L^2 - 3Q^L \equiv 2 \pmod{4}$, giving $2^1 \| v_{3L}$, a contradiction. Hence v_L is even, and $L \mid v_L$. □

Next, we consider the set $\mathcal{P}_T$ of primes that divide some $n \in T$. To set our result in context, recall that $\mathcal{P}_{\text{2nd}}$ denotes the set of the primes dividing v_n for some $n \geq 1$. Clearly $\mathcal{P}_T$ is a subset of $\mathcal{P}_{\text{2nd}}$. Our next result, essentially dating back to Lucas [15], describes this set. See also Somer [21, Proposition 2(iv)].

Proposition 19. *The set $\mathcal{P}_{\text{2nd}}$ is a proper subset of $\mathcal{P}_{\text{1st}}$. It consists of*

- *the primes p for which the rank of appearance $\omega(p)$ of p (in (u_n)) is even;*
- *the special primes;*
- *the prime 2, unless P is odd and Q is even.*

Proof. Take a prime p with $p \nmid 2Q$, and let $\omega = \omega(p)$. If ω is even, then the identity $u_{2n} = u_n v_n$ for $n = \omega/2$ shows that $p \mid v_n$, $p \in \mathcal{P}_{\text{2nd}}$. The identity also shows that $\mathcal{P}_{\text{2nd}} \subset \mathcal{P}_{\text{1st}}$.

Conversely, if $p \in \mathcal{P}_{\text{2nd}}$, say $p \mid v_n$, then $p \mid u_{2n}$, so that, by [20, Proposition 1(iv)], $\omega \mid 2n$. However, from the identity

$$u_n^2 - \Delta v_n^2 = 4Q^n, \tag{6}$$

we have $p \nmid u_n$, so that ω is even.

Now take a prime p with $p \mid Q$. Then from $v_n \equiv P^n \pmod{p}$ we see that, for p to be in $\mathcal{P}_{\text{2nd}}$, p must be special. In particular, $2 \notin \mathcal{P}_{\text{2nd}}$ when P is odd and Q is even. Further, if P is even then $v_1 = P$ is even, while if P and Q are both odd then $v_3 = P(P^2 - 3Q)$ is even.

Finally, for $p \nmid 2Q$, choose m odd, and sufficiently large that we can take p to be a primitive prime divisor of u_m. Then we have $\omega(p) = m$, and hence $p \in \mathcal{P}_{\text{1st}} \setminus \mathcal{P}_{\text{2nd}}$.

□

Our next lemma is an easy exercise. Dickson [8, pp. 67 and 271] traces the result back to an 'anonymous writer' in 1830 [25], and also to Lucas [15, p. 229].

Lemma 20. *For p an odd prime and $j = 1, 2, \ldots, (p-1)/2$, the expression $B_j := \binom{p-1}{j} - (-1)^j$ is divisible by p.*

The following result essentially dates back to Lucas [15, p. 210] and Carmichael [6, Theorem X].

Lemma 21. *(i) For $n \in \mathbb{N}$ and any prime p, p divides v_{np} if and only if p divides v_n.*

(ii) For $n \in \mathbb{N}$ and any odd prime p, v_n divides v_{np} and $v_{np}/v_n \equiv v_n^{p-1} \pmod{p}$.

Proof. (i) Now $v_2 = v_1^2 - 2Q$, which is even iff v_1 is even. Also, for $p \geq 3$,

$$v_1^p = (\alpha + \beta)^p = v_p + \sum_{j=1}^{(p-1)/2} \binom{p}{j} Q^j v_{p-2j} \equiv v_p \pmod{p}. \tag{7}$$

Now replace α, β by α^n, β^n.

(ii) Taking p odd and B_j defined as in Lemma 20, we have

$$\begin{aligned} v_p &= (\alpha+\beta)(\alpha^{p-1} - \alpha^{p-2}\beta + \cdots + \beta^{p-1}) \\ &= (\alpha+\beta)((\alpha+\beta)^{p-1} - \sum_{j=1}^{p-2} B_j \alpha^{p-1-j}\beta^j) \\ &= v_1 \left(v_1^{p-1} - \sum_{j=1}^{(p-3)/2} B_j Q^j v_{p-1-2j} - B_{(p-1)/2} Q^{(p-1)/2} \right). \end{aligned}$$

so that the result of p odd follows by replacing α, β by α^n, β^n and using Lemma 20.

□

6 Proof of Theorem 12

We now prove part (a) of Theorem 12. First take p odd and $n \in T$. Then, by Lemma 21(i), if $p \nmid v_n$ then $p \nmid v_{np}$, so $np \notin T$. Conversely, if $p^\lambda \| v_n$ for some $\lambda \geq 1$ then by Lemma 21(ii) $p^{\lambda+1} \mid v_{np}$. Since $n \mid v_n$ and $pv_n \mid v_{np}$ we have $np \in T$.

Now take $p = 2$, and suppose that both n and $2n$ are in T. First note that v_n must be even, as otherwise $v_{2n} = v_n^2 - 2Q^n$ would be odd. Also, we have $n \mid Q^n$, so that every prime factor q of n divides Q. (Note that this works too if $q = 2$, as then $4 \mid v_{2n}$.) But q must also divide P, as otherwise $v_n \equiv P^n \not\equiv 0 \pmod{q}$. Hence q is special, and n is a product of special primes. If n is even, then 2 is special, so P and Q are both even. If n is odd then, because v_n is even, we must have either P even and Q odd or (from the recurrence) P and Q both odd and $3 \mid n$. So we have either P even or Q odd and $3 \mid n$.

Conversely, assume that $n \in T$ is a product of special primes, and either P is even or (Q is odd and $3 \mid n$). We know from Corollary 17 that every product of special primes is in T. So if 2 is special, then $2n \in T$. So we can assume 2 is not special, and hence that n is odd. If P is even, then, from the recurrence, all the v_k, in particular v_n and v_{2n}, are even. Also, if P and Q are both odd and $3 \mid n$, then v_n and $v_{2n} = v_n^2 - 2Q^n$ are both even. Since for every prime factor q of n with $q^\lambda \| n$ we have $\lambda \leq \log_q n < n$, so that $n \mid Q^n$. Hence $2n \mid v_{2n}$, $2n \in T$.

The proof of part (b) is just the same as that for Theorem 1(b).

To prove part (c): we see easily from Theorem 15 and Proposition 18 that the only possible (second kind) basic numbers are 1 and 6. To find the conditions on P and Q that make 6 basic, we assume that $6 \in T$ but $2 \notin T$, $3 \notin T$. Then $v_2 = P^2 - 2Q$ is odd, so P is odd. Also $3 \nmid v_3 = P(P^2 - 3Q)$, so $P \equiv \pm 1 \pmod 6$. From $6 \mid v_6 = v_2(v_2^2 - 3Q^2)$ we have Q odd and $3 \mid v_2 \equiv 1 - 2Q \pmod 3$, so that $Q \equiv -1 \pmod 6$. Conversely, if $P \equiv \pm 1 \pmod 6$ and $Q \equiv -1 \pmod 6$ then it is easily checked that 6 is basic. This proves part (c).

7 Finiteness results for S and for T.

In this section we look at when S, $\mathcal{P}_S$, and T, $\mathcal{P}_T$ are finite. The results given here are essentially reformulations of results of Somer [20], [21].

Theorem 22. *The set S is finite if and only if $\Delta = 1$, in which case $S = \{1\}$. For S infinite, $\mathcal{P}_S$ is finite when $Q = 0$ and $P \neq 0$, in which case $\mathcal{P}_S$ consists of the prime divisors of P. Otherwise, $\mathcal{P}_S$ is also infinite. Furthermore, $\mathcal{P}_S$ is the set $\mathcal{P}$ of all primes if and only if every prime divisor of Q is special. (This includes the case $Q = \pm 1$.)*

For the proof, we note first that when $\Delta = 1$, α and β are consecutive integers, and 1 is the only basic element. But there are no primes p dividing $u_1\Delta = 1$, so $\mathcal{P}_{S,1}$ is empty, and $S = \{1\}$. In all other cases, $|u_1\Delta| > 1$, $\mathcal{P}_{S,1}$ is nonempty, with $p \in \mathcal{P}_{S,1}$ say, and then, by Corollary 2, $p^k \in S$ for all $k \geq 0$, making S infinite.

Now assume S is infinite. We recall that $(u_n)_{n\geq 0}$ is called *degenerate* if $Q = 0$ or α/β is a root of unity. (The latter alternative includes the case $P = 0$, $Q \neq 0$.) We consider the two cases of (u_n) being degenerate or nondegenerate separately. If (u_n) is degenerate, then by [20, Theorem 9] either

- $P \neq 0$ and $Q = 0$, so that then S consists of those n whose prime factors all divide P, and $\mathcal{P}_S = \mathcal{P}_{1\text{st}}$ is the set of prime divisors of P;

  or

- for some $r = 1, 2, 3, 4$ or 6, S has a subset $\{rk : k \in \mathbb{N}\}$ where $u_{rk} = 0$, so that $\mathcal{P}_S = \mathcal{P}_{1\text{st}} = \mathcal{P}$.

Now consider the case of (u_n) nondegenerate. Then, by Somer [20, Theorem 1], all but finitely many u_n have a primitive prime divisor (a prime dividing u_n that do not divide u_m for any $m < n$). So, using Theorem 1(a), $\mathcal{P}_S$ is infinite. Somer's theorem is based on results of Lekkerkerker [14] and Schinzel [18]. In fact Bilu, Hanrot and Voutier [5] have proved that for such sequences with no special primes every u_n with $n > 30$ has a primitive divisor. They also listed exceptions with $n \leq 30$. Hence u_{p^k} has a primitive prime divisor for all sufficiently large k, making $\mathcal{P}_S$ infinite. See Abouzaid [1] for corrections to their list. Also Stewart [22] and Shorey and Stewart [19] gave lower bounds for the largest prime divisor of u_n.

This proof will be complete after we have proved the following. While this result is contained in Somer [20, Theorem 8], we give another proof here.

Proposition 23. *The set $\mathcal{P}_S$ is the whole of $\mathcal{P}$ if and only if all primes are regular.*

Proof. First note that if there are any irregular primes then, by Corollary 7, $\mathcal{P}_S$, being a subset of $\mathcal{P}_{1\text{st}}$, cannot be the whole of $\mathcal{P}$.

Conversely, assume all primes are regular, so that any prime factor p of Q also divides P. Note that then $p \mid \Delta$. To show that all primes belong to $\mathcal{P}_S$, we proceed by induction. We first show that $2 \in \mathcal{P}_S$. If $u_2 = P$ is even, then $2 \in S$, $2 \in \mathcal{P}_S$. So we can take P odd. Then Q must be odd, too, by our assumption. Then $u_3 = P^2 - Q$ is even, and hence so is $u_6 = u_3 v_3$. We claim that either $3 \mid u_6$, in which case $6 \in S$, $2, 3 \in \mathcal{P}_S$, or $12 \in S$, with the same implication.

- If $P \equiv 3 \pmod 6$, $Q \equiv 3 \pmod 6$, then $3 \mid u_n$ for all $n \geq 2$, so that $3 \mid u_6$.
- If $P \equiv 3 \pmod 6$, $Q \equiv \pm 1 \pmod 6$, then 6 is basic, by Theorem 1(c).
- If $P \equiv \pm 1 \pmod 6$, $Q \equiv -1 \pmod 6$, then 12 is basic, by Theorem 1(c).
- If $P \equiv \pm 1 \pmod 6$, $Q \equiv 1 \pmod 6$, then $3 \mid u_3$ and so $3 \mid u_3 v_3 = u_6$.

Hence $2 \in \mathcal{P}_S$, as claimed.

We now assume that $q \in \mathcal{P}_S$ for every prime $q < p$, where p is a prime at least 3. We have just shown that this is true for $p = 3$. By Lemma 8, we know that for any exponents $\varepsilon_q = 0$ or 1 there is a positive integer k such that $k \prod_{q<p} q^{\varepsilon_q} \in S$; hence, by Corollary 2, $k \prod_{q<p} q^{e_q} \in S$ for any exponents $e_q \geq \varepsilon_q$.

By Lemma 6, $p \mid u_{p+\varepsilon}$, for some $\varepsilon = \pm 1$. As $p > 2$, all prime factors of $p + \varepsilon$ are less than p so, by the induction hypothesis, $k(p + \varepsilon) \in S$ for some k. If $p \mid k$ then $p \in \mathcal{P}_S$. If $p \nmid k$ then, using (1), we have

$$u_{pk(p+\varepsilon)} = u_p^{(k(p+\varepsilon))} u_{k(p+\varepsilon)} = u_{pk}^{(p+\varepsilon)} u_{p+\varepsilon},$$

so that $pk(p + \varepsilon) \in S$, $p \in \mathcal{P}_S$. This proves the induction step. $\square$

This completes the proof of Theorem 22.

We now consider the finiteness (or otherwise) of T and $\mathcal{P}_T$.

Theorem 24 (Somer [21, Theorems 8 and 9])**.** *The set T is finite in the following two cases:*

- $P = \pm 1$, $Q \not\equiv -1 \pmod 6$, *in which case* $T = \{1\}$;
- $P = \varepsilon_1 2^k$, $Q = 2^{2k-1} + \varepsilon_2$, *where k is a positive integer, and ε_1, $\varepsilon_2 \in \{-1, 1\}$, in which case* $T = \{1, 2\}$.

Otherwise, T is infinite. For T infinite, $\mathcal{P}_T$ is finite precisely when P, Q are not both 0 and either

- $P^2 = Q$, *in which case $\mathcal{P}_T$ is the set of prime divisors of* $2P$

  or

- $P^2 = 4Q$ *or* $Q = 0$, *in which case $\mathcal{P}_T$ is the set of prime divisors of* P.

Otherwise, for T infinite, $\mathcal{P}_T$ is also infinite.

Proof. If T contains an integer n having an odd prime factor p then, by Theorem 12(a), $p^k n \in T$ for all $k \geq 0$. In particular, if $P = \pm 1$ and $Q \equiv -1 \pmod 6$, then $6 \in T$, so that T is infinite. On the other hand, if $P = \pm 1$ and $Q \not\equiv -1 \pmod 6$, then 1 is the only basic element of T, and $v_1 = P$ has no prime factors so that, by Theorem 12(a), $\mathcal{P}_{T,1}$ is empty, and hence $T = \{1\}$.

Again starting with $1 \in T$, we see that T is infinite if P has any odd prime factors. Also, T is infinite if P is $\pm$ a positive power of 2 and 2 is special, as then $2^k \in T$ for all $k \geq 0$, by Theorem 12(a).

It therefore remains only to consider the case of $P = \pm 2^k$, $k \geq 1$ and Q odd, so that 2 is not special. Then $2 \in T$ and $4 \notin T$, by Theorem 12(a). If v_2 has an odd prime factor p, then $2p^k \in T$ for all $k \geq 0$, so that T is again infinite. Finally, if v_2 is $\pm$ a power of 2, then $T = \{1, 2\}$. This happens only when $v_2 = 2^{2k} - 2Q = \pm 2$, so that $Q = 2^{2k-1} \mp 1$, as claimed.

Now take T infinite, with P, Q not both 0. If the sequence (v_n) is degenerate, then, using Somer [21, Theorem 9], we get either $P^2 = Q$, $P^2 = 4Q$ or $Q = 0$, and $\mathcal{P}_T$ being the set of prime divisors of P, as required. On the other hand, if (v_n) is not degenerate then by Somer [21, Theorem 1] for sufficiently large n every v_n has a primitive prime divisor. Hence we can find an infinite sequence of numbers n in T such that np is again in T, where p is a primitive prime divisor of v_n. (Here we are using Theorem 12(a).) Thus $\mathcal{P}_T$ then contains infinitely many primes p. $\square$

8 The sets $\mathcal{P}_S$ and $\mathcal{P}_T$.

From the proof of Theorem 22 we see that $\mathcal{P}_S = \mathcal{P}_{1\text{st}}$ for (u_n) degenerate or all primes being regular. Our next result takes care of the remaining cases. I thank Larry Somer and the referee for pointing out how the proof of this could be completed.

Proposition 25. *If (u_n) is nondegenerate and there are irregular primes, then $\mathcal{P}_S$ is a proper subset of $\mathcal{P}_{1\text{st}}$.*

Proof. Take (u_n) nondegenerate and having an irregular prime f. Then, from the discussion preceding Proposition 23, every u_n for n sufficiently large has a primitive prime divisor. Indeed, if $\gcd(P, Q) = 1$ this is true for $n > 30$. Hence for ℓ sufficiently large, $u_{\ell f}$ has a primitive prime divisor, p say, so that $\omega(p) = \ell f$.

Then if, for some k, kp were in S, we would have $kp \mid u_{kp}$, so that, by [20, Proposition 1(iv)], $\omega(p)$, and hence f, would divide kp. Hence f would divide u_n, contradicting Corollary 7. Thus $p \notin \mathcal{P}_S$. $\square$

We have in fact shown that no prime whose rank of appearance is a multiple of any irregular prime f will belong to $\mathcal{P}_S$. The referee has remarked that, when α/β is rational, the density of such primes has been precisely computed in many cases. For $f > 2$ and α/β not an f-th power, it is $f/(f^2 - 1)$. See Ballot [4, Theorem 3.2.3] and also Moree [16].

Using a similar method, we can also prove the corresponding result for T.

Proposition 26. *The set $\mathcal{P}_T$ is a proper subset of $\mathcal{P}_{2\text{nd}}$.*

Proof. Let f be a primitive prime divisor of u_n for some odd n with $f \nmid 2Q$. Then, by Proposition 19, $f \in \mathcal{P}_{\text{1st}} \setminus \mathcal{P}_{\text{2nd}}$. Now, taking ℓ sufficiently large, let p be a primitive prime divisor of $u_{2\ell f}$. Then, as $u_{2\ell f} = u_{\ell f} v_{\ell f}$, $p \mid v_{\ell f}$. Suppose $p \in \mathcal{P}_T$, so that, for some k, $kp \in T$, and hence $kp \mid v_{kp}$. But then by Somer [21, Proposition 2(vii)], kp is a multiple of ℓf. In particular, $f \mid v_{kp}$, contradicting $f \notin \mathcal{P}_{\text{2nd}}$. So $p \notin \mathcal{P}_T$. □

9 Divisibility properties of S and of T.

From Theorem 1 we can consider S as a graph spanned by a forest of one or two trees, with each node corresponding to an element of S, and the root nodes of the trees being $\{1\}$, $\{1, 6\}$ or $\{1, 12\}$. Each edge can be labelled p; it rises from a node $n \in S$ to a node $np \in S$, where p is some prime divisor of $u_n\Delta$. One spanning forest is obtained by taking only the edges $n \to np$, where p is the largest prime factor of np such that $n \in S$. (By Theorem 3 and Proposition 4, p is either $p_{\max}$ or 2). Thus every node above n in the tree is divisible by n. Next, call a *cutset* of the forest a set C of nodes with the property that every path from a root to infinity must contain some vertex of the cutset. Then we clearly have the following.

Proposition 27. *For a cutset C of S, every element of S either lies below C, or it is divisible by some node of C.*

Judicious choice of a cutset places severe divisibility restrictions on elements of S, and so, using this, one can search for elements of S up to a given bound very efficiently.

The same argument applies equally to T, using Theorem 12, with p being either an odd prime divisor of v_n or, under the conditions described in that theorem, the prime 2. For instance, applying this idea to the sequence T of example 2 below, every element of that sequence except 1, 3, 9, 27 and 81 is divisible either by 171 or 243 or 13203 or 2354697 or 10970073 or 22032887841. See [3] for details.

10 Examples

1. $P = 1, Q = -1$ (the classical Fibonacci and Lucas numbers.) Here $\Delta = 5$,

$$S = 1, 5, 12, 24, 25, 36, 48, 60, 72, 96, 108, 120, 125, 144, 168, 180, \ldots,$$

with 1 and 12 basic (A023172 in Neil Sloane's *Encyclopedia*), while $\mathcal{P}_S$ is the whole of $\mathcal{P}$ (see Theorem 22),

$$T = 1, 6, 18, 54, 162, 486, 1458, 1926, 4374, 5778, 13122, 17334, \ldots,$$

with 1 and 6 basic (A016089), and

$$\mathcal{P}_{\text{2nd}} = 2, 3, 7, 11, 19, 23, 29, 31, 41, 43, 47, 59, 67, 71, 79, 83, 101, 103, 107, 127, \ldots,$$

(A140409) of which $\mathcal{P}_T$ is a subsequence:

$$\mathcal{P}_T = 2, 3, 107, 1283, 8747, 21401, 34667, 46187, \ldots,$$

(A129729).

2. $P = 3, Q = 2$, where $u_n = 2^n - 1$, $v_n = 2^n + 1$. Here $S = \{1\}$ as $\Delta = 1$, and

$$T = 1, 3, 9, 27, 81, 171, 243, 513, 729, 1539, 2187, 3249, \ldots,$$

with 1 basic (A006521). Also

$$\mathcal{P}_{\text{2nd}} = 3, 5, 11, 13, 17, 19, 29, 37, 41, 43, 53, 59, 61, 67, 83, 97, 101, 107, 109, \ldots,$$

(A014662 – see also A091317), of which

$$\mathcal{P}_T = 3, 19, 163, 571, 1459, 8803, 9137, 17497, 41113, \ldots$$

(A057719) is a subsequence. Note too that, by Proposition 11 and the fact that all $n \in T$ are odd, we have $T = S(-1, -2)$. Also $S = T(-1, -2) = \{1\}$.

3. $P = 3$, $Q = 5$, $\Delta = -11$,

$$S = 1, 6, 11, 12, 18, 24, 36, 48, 54, 66, 72, 96, 108, 121, 132, 144, 162, 168, 192, 198, \ldots$$

with 1 and 6 basic, with $\mathcal{P}_{\text{1st}}$ consisting of all primes except the irregular prime 5, and

$$\mathcal{P}_S = 2, 3, 7, 11, 13, 17, 23, 37, 41, 43, 67, 71, 73, 83, 89, 97, 101, 103, 107, 113, \ldots.$$

Also

$$T = 1, 3, 9, 27, 81, 153, 243, 459, 729, 1377, 2187, 2601, 4131, 4401, 6561, 7803, \ldots$$

with only 1 basic,

$$\mathcal{P}_{\text{2nd}} = 2, 3, 7, 13, 17, 19, 23, 37, 43, 47, 53, 67, 73, 79, 83, 97, 103, 107, 113, \ldots$$

and

$$\mathcal{P}_T = 2, 3, 17, 103, 163, 373, 487, 1733, \ldots.$$

11 Additional remarks.

1. It would be interesting to see whether the analysis of the paper could be extended to other second-order recurrence sequences, or indeed to any recurrences of higher order.

2. In [3], what we called 'primitive' solutions of $n \mid 2^n + 1$ should in fact have been called *fundamental* solutions, following Jarden [13, p. 70] and Somer [20, p. 522], [21, p. 482]. However, this definition has been superseded by the notion of a basic element (of S or of T) as in this paper.

3. In example 1 of Section 10 above we have 24 and 25 $\in S = S(1, -1)$. Are these the only consecutive integers in $S(1, -1)$?

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