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Analogues of Up-down Permutations for Colored Permutations

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Abstract

André proved that $\sec x$ is the generating function of all up-down permutations of even length and $\tan x$ is the generating function of all up-down permutation of odd length. There are three equivalent ways to define up-down permutations in the symmetric group S_n. That is, a permutation σ in the symmetric group S_n is an up-down permutation if either (i) the rise set of σ consists of all the odd numbers less than n, (ii) the descent set of σ consists of all even number less than n, or (iii) both (i) and (ii). We consider analogues of André's results for colored permutations of the form (σ, w) where $\sigma \in S_n$ and $w \in \{0, \ldots, k-1\}^n$ under the product order. That is, we define $(\sigma_i, w_i) < (\sigma_{i+1}, w_{i+1})$ if and only if $\sigma_i < \sigma_{i+1}$ and $w_i \leq w_{i+1}$. We then say a colored permutation (σ, w) is (I) an *up-not up* permutation if the rise set of (σ, w) consists of all the odd numbers less than n, (II) a *not down-down* permutation if the descent set of (σ, w) consists of all the even numbers less than n, (III) an *up-down* permutation if both (I) and (II) hold. For $k \geq 2$, conditions (I), (II), and (III) are pairwise distinct. We find p, q-analogues of the generating functions for up-not up, not down-down, and up-down colored permutations.

1 Introduction

Let $\mathbf{P} = \{1, 2, 3, \ldots\}$ denote the set of positive integers, $\mathbf{E} = \{2, 4, 6, \ldots\}$ denote the set of even integers in $\mathbf{P}$, and $\mathbf{O} = \{1, 3, 5, \ldots\}$ denote the set of odd integers in $\mathbf{P}$. Let $\mathbf{P}_n = \{1, \ldots, n\}$, $\mathbf{E}_n = \mathbf{E} \cap \mathbf{P}_n$, and $\mathbf{O}_n = \mathbf{O} \cap \mathbf{P}_n$. Let S_n denote the symmetric group, i.e., the set

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of all permutations of $\mathbf{P}_n$. Then if $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$, we define $\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $\mathrm{Ris}(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. We say that σ is an *up-down permutation* if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 \cdots,$$

or, equivalently, if $\mathrm{Des}(\sigma) = \mathbf{E}_{n-1}$ or $\mathrm{Ris}(\sigma) = \mathbf{O}_{n-1}$. Similarly, we say that σ is an *down-up permutation* if

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 \cdots,$$

or, equivalently, if $\mathrm{Ris}(\sigma) = \mathbf{E}_{n-1}$ or $\mathrm{Des}(\sigma) = \mathbf{O}_{n-1}$. Clearly if $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ is an up-down permutation, then the complement of σ,

$$\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$$

is a down-up permutation. Thus the number of up-down permutations in S_n is equal to the number of down-up permutations in S_n. Let UD_n denote the number of up-down permutations in S_n. Then André [2, 3] proved the following.

$$\sec t = 1 + \sum_{n\in\mathbf{E}} UD_n \frac{t^n}{n!} \text{ and} \tag{1}$$

$$\tan t = \sum_{n\in\mathbf{O}} UD_n \frac{t^n}{n!}. \tag{2}$$

The goal of this paper is to find analogues of André's results for colored permutations. That is, we shall consider pairs of the form (σ, w) where $\sigma \in S_n$ and $w \in \{0, 1, \ldots, k-1\}^n$. Thus if $w = w_1 \cdots w_n$ and $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, then we will say that σ_i is colored with w_i in (σ, w). Alternatively, we can think of (σ, w) as an element of the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n. $C_k \wr S_n$ is the group of $k^n n!$ signed permutations where there are k signs, $1 = \epsilon^0$, ϵ, ϵ^2, ..., ϵ^{k-1} where ϵ is a primitive k-th root of unity. Hence we can think of $\Gamma \in C_k \wr S_n$ as a pair (σ, w) where $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ and the sign of σ_i is ϵ^{w_i} for $i = 1, \ldots, n$. Throughout the paper, we shall abbreviate $S_n \times \{0, \ldots, k-1\}^n$ by $C_k \wr S_n$ even though our results do not use the group structure of $C_k \wr S_n$. .

To define the analogues of up-down permutations in $C_k \wr S_n$, we need to define the analogues of the descent set and the rise set of a colored permutation. That is, suppose $\prec$ is a partial order on the set of pairs $(i, j) \in \{1, \ldots, n\} \times \{0, \ldots, k-1\}$. Then if $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ and $w = w_1 \ldots w_n \in \{0, \ldots, k-1\}^n$, we define

$$\begin{aligned} \mathrm{Des}_\prec((\sigma, w)) &= \{i : (\sigma_i, w_i) \succ (\sigma_{i+1}, w_{i+1})\} \text{ and} \\ \mathrm{Ris}_\prec((\sigma, w)) &= \{i : (\sigma_i, w_i) \prec (\sigma_{i+1}, w_{i+1})\}. \end{aligned}$$

We then say that (σ, w) is an up-down permutation if

$$(\sigma_1, w_1) \prec (\sigma_2, w_2) \succ (\sigma_3, w_3) \prec (\sigma_4, w_4) \succ (\sigma_5, w_5) \cdots. \tag{3}$$

Now if $\prec$ is a total order, then counting the number of up-down permutations is uninteresting. For example, Adin and Roichman [1] used the following total order to define their notion of flag major index on $C_k \wr S_n$:

$$(1, k-1) \prec \cdots \prec (n, k-1) \prec (1, k-2) \prec \cdots \prec (n, k-2) \prec \cdots \prec (1, 0) \prec \cdots \prec (n, 0).$$

If we use such a total order $\prec$ and we pick an assignment of colors, $(1,w_1),\ldots(n,w_n)$, to $1,2,\ldots,n$, then $\prec$ induces a total order on the pairs $(1,w_1),\ldots,(n,w_n)$. Hence for this assignment of colors, there are clearly just UD_n pairs (σ,w) which satisfy (3). Hence relative to a total order $\prec$, there are $k^n UD_n$ colored permutations $(\sigma,w)\in C_k \wr S_n$ that satisfy (3).

However if we use the product order on $\{1,\ldots,n\}\times\{0,\ldots,k-1\}$, then we have a completely different situation. That is, we define the product order $\leq$ on $\{1,\ldots,n\}\times\{0,,\ldots,k-1\}$ by declaring that $(i_1,j_1)\leq(i_2,j_2)$ if and only if $i_1\leq i_2$ and $j_1\leq j_2$. Again we can define the analogue of the descent set and rise set of a colored permutation in $C_k\wr S_n$ as

$$\begin{aligned}\mathrm{Des}((\sigma,w)) &= \{i:(\sigma_i,w_i)>(\sigma_{i+1},w_{i+1})\}\text{ and} &(4)\\ \mathrm{Ris}((\sigma,w)) &= \{i:(\sigma_i,w_i)<(\sigma_{i+1},w_{i+1})\}. &(5)\end{aligned}$$

We can then define three different natural analogues of up-down permutations. That is, we define the following three sets of permutations in $C_k\wr S_n$:

1. $U\text{-}D_{n,k}=\{(\sigma,w)\in C_k\wr S_n:\mathrm{Ris}((\sigma,w))=\mathbf{O}_{n-1}\text{ and }\mathrm{Des}((\sigma,w))=\mathbf{E}_{n-1}\}$,
2. $U\text{-}NU_{n,k}=\{(\sigma,w)\in C_k\wr S_n:\mathrm{Ris}((\sigma,w))=\mathbf{O}_{n-1}\}$, and
3. $ND\text{-}D_{n,k}=\{(\sigma,w)\in C_k\wr S_n:\mathrm{Des}((\sigma,w))=\mathbf{E}_{n-1}\}$.

Here $U\text{-}NU$ stands for "up-not up" and $ND\text{-}D$ stands for "not down-down." Clearly $U\text{-}D_{n,k}$ is contained in both $U\text{-}NU_{n,k}$ and $ND\text{-}D_{n,k}$. However, both of these containments are strict and $U\text{-}NU_{n,k}\neq ND\text{-}D_{n,k}$. For example, if $k=2$ and $n=3$, then

$$\begin{aligned}(1\ 2\ 3,1\ 1\ 0) &\in U\text{-}NU_{3,2}-(U\text{-}D_{3,2}\cup ND\text{-}D_{3,2})\text{ and}\\ (1\ 3\ 2,1\ 0\ 0) &\in ND\text{-}D_{3,2}-(U\text{-}D_{3,2}\cup U\text{-}NU_{3,2}).\end{aligned}$$

Let $u\text{-}d_{n,k}=|U\text{-}D_{n,k}|$, $nd\text{-}d_{n,k}=|ND\text{-}D_{n,k}|$, and $u\text{-}nu_{n,k}=|U\text{-}NU_{n,k}|$. Then the main goal of this paper is to find expressions for the following generating functions:

$$\begin{aligned}A(t) &= \sum_{n\geq0}\frac{u\text{-}d_{2n,k}t^{2n}}{(2n)!}, \qquad B(t)=\sum_{n\geq0}\frac{u\text{-}d_{2n+1,k}t^{2n+1}}{(2n+1)!},\\ C(t) &= \sum_{n\geq0}\frac{u\text{-}nu_{2n,k}t^{2n}}{(2n)!}, \qquad D(t)=\sum_{n\geq0}\frac{u\text{-}nu_{2n+1,k}t^{2n+1}}{(2n+1)!},\\ E(t) &= \sum_{n\geq0}\frac{nd\text{-}d_{2n,k}t^{2n}}{(2n)!},\text{ and } F(t)=\sum_{n\geq0}\frac{nd\text{-}d_{2n+1,k}t^{2n+1}}{(2n+1)!}.\end{aligned}$$

The generating functions $A(t)$ and $B(t)$ are simply Hadamard products of the generating functions for up-down permutations of S_n and the generating functions for sequences of words $w=w_1\ldots w_n\in\{0,\ldots,k-1\}^*$ such that

$$w_1\leq w_2\geq w_3\leq w_4\geq w_5\leq w_6\cdots.$$

The generating function for such words was found by Rawlings [16]. The generating functions for $C(t)$, $D(t)$, $E(t)$ and $F(t)$ are more interesting. For example, we shall show that

$$C(t) = \frac{(k-1)!}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\cos t} \text{ and} \tag{6}$$

$$D(t) = \frac{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\sin t}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\cos t} \tag{7}$$

For $k \le 4$, the generating functions and their initial terms are listed below:

k	egf for $u\text{-}nu_{2n,k}$	egf for $u\text{-}nu_{2n+1,k}$	$\{u\text{-}nu_{n,k}\}_{n\ge 1}$
1	$\sec t$	$\tan t$	$1, 1, 2, 5, 16, 61, \ldots$
2	$\frac{1}{\cos t - t\sin t}$	$\frac{\sin t + t\cos t}{\cos t - t\sin t}$	$2, 3, 14, 49, 376, 1987, \ldots$
3	$\frac{2}{(2-t^2)\cos t - 4t\sin t}$	$\frac{4t\cos t + (2-t^2)\sin t}{(2-t^2)\cos t - 4t\sin t}$	$3, 6, 44, 201, 2436, 16768, \ldots$
4	$\frac{6}{(6-9t^2)\cos t - (18t-t^3)\sin t}$	$\frac{(18t-t^3)\cos t + (6-9t^2)\sin t}{(6-9t^2)\cos t - (18t-t^3)\sin t}$	$4, 10, 100, 565, 9356, 79584, \ldots$

It is easy to see that if $(\sigma, w) = (\sigma_1 \ldots \sigma_{2n+1}, w_1 \ldots w_{2n+1})$ is in $U\text{-}NU_{2n+1,k}$, then the reverse of (σ, w),

$$(\sigma, w)^r = (\sigma_{2n+1} \ldots \sigma_1, w_{2n+1} \ldots w_1),$$

will be in $ND\text{-}D_{2n+1,k}$ so that

$$\sum_{n\ge 0} \frac{nd\text{-}d_{2n+1,k} \cdot t^{2n+1}}{(2n+1)!} = \frac{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\sin t}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\cos t}. \tag{8}$$

Thus $D(t) = F(t)$. However the generating function for $E(t)$ is not the same as for $C(t)$. We shall prove that

$$\begin{aligned} E(t) = \sum_{n\ge 0} \frac{nd\text{-}d_{2n,k} \cdot t^{2n}}{(2n)!} &= \left(\frac{(k-1)!}{2\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1} e^{it}} + \frac{(k-1)!}{2\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1} e^{-it}} \right)^{-1} \\ &= \frac{\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1} e^{it}\right)\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1} e^{-it}\right)}{(k-1)!\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\cos t} \\ &= \frac{P_{k-1}(it)P_{k-1}(-it)}{(k-1)!\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1}\cos t}, \end{aligned}$$

where $P_d(z) = \sum_{m=0}^{d} z^d \binom{d}{m}^2((d-m)!)$. In fact, we shall prove that

$$\begin{aligned} P_d(it)P_d(-it) &= \sum_{s=0}^{d} t^{2s} \sum_{r=0}^{2s} (-1)^{s-r} \binom{d}{r}^2 (d-r)! \binom{d}{2s-r}^2 (d-(2s-r))! \\ &= \sum_{s=0}^{d} \frac{(d!)^2}{(2s)!} \binom{d}{s} \binom{d+s}{s} t^{2s}. \end{aligned} \tag{9}$$

Thus $P_d(it)P_d(-it)$ is a polynomial with integer coefficients of degree $2d$ where only even-degree terms are non-zero. For $k \leq 4$, we get the following results:

k	egf for $nd\text{-}d_{2n,k}$	$P_{k-1}(z)$	$\{nd\text{-}d_{n,k}\}_{n\geq 1}$
1	$\sec t$	1	$1, 1, 2, 5, 16, 61, \ldots$
2	$\frac{1+t^2}{\cos t - t\sin t}$	$1+z$	$2, 10, 14, 85, 376, 3457, \ldots$
3	$\frac{4+12t^2+t^4}{(2-t^2)\cos t - 4t\sin t}$	$2+4z+z^2$	$3, 12, 44, 423, 2436, 35398, \ldots$
4	$\frac{36+216t^2+45t^4+t^6}{(6-9t^2)\cos t-(18t-t^3)\sin t}$	$6+18z+9z^2+z^3$	$4, 22, 100, 1315, 9356, 185804, \ldots$

In fact, we shall show that the generating functions $C(t)$, $D(t)$, $E(t)$ and $F(t)$ are special cases of more general generating functions which keep track of more statistics over more general sets of elements in $C_k \wr S_n$. That is, for any $k \geq 2$, let

$$(C_k \wr S_n)^{(2)} = \{(\sigma, w) \in C_k \wr S_n : \mathbf{O}_{n-1} \subseteq \mathrm{Rise}((\sigma, w))\} \text{ and } \quad (10)$$
$$(C_k \wr S_n)_{(2)} = \{(\sigma, w) \in C_k \wr S_n : \mathbf{O}_{n-1} \cap \mathrm{Des}((\sigma, w)) = \emptyset\}. \quad (11)$$

Thus $(C_k \wr S_n)^{(2)}$ is the set of $(\sigma, w) \in C_k \wr S_n$ which are forced of have rises at all odd positions and $(C_k \wr S_n)_{(2)}$ is the set of $(\sigma, w) \in C_k \wr S_n$ which do not have a descent at an odd position. Then we shall find generating functions for certain statistics on $(C_k \wr S_n)^{(2)}$ which specialize to $C(t)$ and $D(t)$ and generating functions for certain statistics on $(C_k \wr S_n)_{(2)}$ which specialize to $E(t)$ and $F(t)$. The techniques that we shall use to derive our generating functions over $(C_k \wr S_n)^{(2)}$ or $(C_k \wr S_n)_{(2)}$ are based on ideas from a paper by Mendes, Remmel, and Riehl [13] who, for any $k \geq 2$ and $0 \leq j \leq k-1$, found generating functions for permutations $\sigma \in S_n$ such that $\mathrm{Des}(\sigma) = \{j + sk : k \geq 0 \;\&\; j + sk < n\}$. Mendes, Remmel, and Riehl derived their generating functions by applying certain ring homomorphisms defined on the ring of symmetric functions Γ over infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities. We will also find our generating functions over $(C_k \wr S_n)^{(2)}$ or $(C_k \wr S_n)_{(2)}$ by applying ring homomorphisms defined on Λ to simple symmetric function identities. To derive our generating functions over $(C_k \wr S_n)_{(2)}$, we shall also need to find the generating function for $(\sigma, w) \in C_k \wr S_n$ such that $\mathrm{Des}((\sigma, w)) = \emptyset$.

The outline of this paper is as follows. In Section 2, we shall derive the generating functions $A(t)$ and $B(t)$. In Section 3, we shall provide the necessary background on symmetric functions that we shall need to derive the generating functions $C(t)$, $D(t)$, $E(t)$ and $F(t)$. In Section 4, we shall give the derivations of the generating functions that specialize to $C(t)$ and $D(t)$. Finally in Section 5, we shall give the derivations of the generating functions that specialize to $E(t)$ and $F(t)$.

2 The generating functions for up-down permutations

In this section, we shall give expressions for the generating functions for $A(t)$ and $B(t)$. To state our results, we first need some notation. Suppose that $f(t) = \sum_{n\geq 0} f_n t^n$ and $g(t) = \sum_{n\geq 0} g_n t^n$. Then the Hadamard product $f(t) \otimes g(t)$ of f and g is defined by

$$f(t) \otimes g(t) = \sum_{n\geq 0} f_n g_n t^n. \quad (12)$$

Let $\mathbf{P}^*$ denote the set of all words over the alphabet $\mathbf{P}$ and $\mathbf{P}^+$ denote the set of all non-empty words in $\mathbf{P}^*$. We let ϵ denote the empty word. For any $w = w_1w_2 \ldots w_n \in \mathbf{P}^+$, we let $\ell(w) = n$ denote the length of w, $|w| = \sum_{i=1}^n w_i$, and $x(w) = \prod_{i=1}^n x_{w_i}$. For example, if $w = 1\ 2\ 1\ 3\ 2\ 4\ 5\ 4$, then $\ell(w) = 8$, $|w| = 22$, and $x(w) = x_1^2x_2^2x_3x_4^2x_5$. Given $w = w_1w_2 \ldots w_n \in \mathbf{P}^+$, we define the descent set $\mathrm{Des}(w)$, the weak descent set $\mathrm{WDes}(w)$, the rise set $\mathrm{Ris}(w)$, and the weak rise set $\mathrm{WRis}(w)$ as follows:

$$\begin{aligned}
\mathrm{Des}(w) &= \{i : w_i > w_{i+1}\}, &(13)\\
\mathrm{WDes}(w) &= \{i : w_i \geq w_{i+1}\}, &(14)\\
\mathrm{Ris}(w) &= \{i : w_i < w_{i+1}\}, \text{ and} &(15)\\
\mathrm{WRis}(w) &= \{i : w_i \leq w_{i+1}\}. &(16)
\end{aligned}$$

Definition 1. Let $w = w_1w_2 \ldots w_n \in \mathbf{P}^+$.

1. We say that w a *strict up-down* word if $w_1 < w_2 > w_3 < w_4 > w_5 \cdots$, or, equivalently if $\mathrm{Ris}(w) = \mathbf{O}_{n-1}$ and $\mathrm{Des}(w) = \mathbf{E}_{n-1}$.

2. We say that w a *strict down-up* word if $w_1 > w_2 < w_3 > w_4 < w_5 \cdots$, or, equivalently if $\mathrm{Des}(w) = \mathbf{O}_{n-1}$ and $\mathrm{Ris}(w) = \mathbf{E}_{n-1}$.

3. We say that w a *weak up-down* word if $w_1 \leq w_2 \geq w_3 \leq w_4 \geq w_5 \cdots$, or, equivalently if $\mathrm{WRis}(w) = \mathbf{O}_{n-1}$ and $\mathrm{WDes}(w) = \mathbf{E}_{n-1}$.

4. We say that w a *weak down-up* word if $w_1 \geq w_2 \leq w_3 \geq w_4 \leq w_5 \cdots$, or, equivalently if $\mathrm{WDes}(w) = \mathbf{O}_{n-1}$ and $\mathrm{WRis}(w) = \mathbf{E}_{n-1}$.

We let SUP_n, SDU_n, WUD_n, and WDU_n denote set of all words in $\{1, \ldots, n\}^*$ which are strict up-down, strict down-up, weak up-down, and weak down-up, respectively. By convention, the empty word ϵ and all one letter words belong to all four sets. Clearly, if $w = w_1w_2 \ldots w_n \in \mathbf{P}_n^*$, then $w \in SUD_n$ (WUD_n, respectively) if and only if the complement of w relative to n,

$$w^{c,n} = (n+1-w_1)(n+1-w_2)\ldots(n+1-w_n) \in SDU_n \ (WUD_n, \text{ respectively}).$$

We let $SUP_{n,m}$, $SDU_{n,m}$, $WUD_{n,m}$, and $WDU_{n,m}$ denote set of all words in $\mathbf{P}_n^*$ of length m which are strict up-down, strict down-up, weak up-down, and weak down-up, respectively.

Carlitz [4] proved analogues of André's formulas for strict up-down words. In particular, Carlitz [4] proved that

$$1 + \sum_{m \in \mathbf{E}} |SUD_{n,m}| z^m = \frac{1}{Q_n(z)} \text{ and } \sum_{m \in \mathbf{O}} |SUD_{n,m}| z^m = \frac{P_n(z)}{Q_n(z)} \tag{17}$$

where

$$\begin{aligned}
P_n(z) &= \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k+1} z^{2k+1} \text{ and} &(18)\\
Q_n(z) &= \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{2k} z^{2k}. &(19)
\end{aligned}$$

Rawlings [16] proved analogues of (17) for weak down-up words. That is, Rawlings proved that

$$1 + \sum_{m \in \mathbf{E}} |WDU_{n,m}| z^m = \frac{1}{R_n(z)} \text{ and } \sum_{m \in \mathbf{O}} |WDU_{n,m}| z^m = \frac{S_n(z)}{R_n(z)} \tag{20}$$

where

$$\begin{aligned} R_n(z) &= \sum_{k \geq 0} (-1)^k \binom{n+k}{2k} z^{2k} \text{ and} && (21) \\ S_n(z) &= \sum_{k \geq 0} (-1)^k \binom{n+k}{2k+1} z^{2k+1}. && (22) \end{aligned}$$

By our observations above, these are also the generating functions for weak up-down words. Carlitz and Rawlings proved their generating functions by recursions. In fact, Carlitz developed recursions for the up-down words w weighted by $x(w)$ and Rawlings actually proved a generating function for weak down-up words w weighted by $q^{|w|} z^{\ell(w)}$. Recently, Fuller and Remmel [8] showed that the generating functions of words w according to the weight $x(w) z^{\ell(w)}$ of words in either SUD_n and WU_n can be expressed in term of quasi-symmetric functions. Fuller and Remmel proved their results combinatorially via some simple involutions and their methods actually extend to a much broader class of words with regular up-down patterns.

Now it is easy to see that if $(\sigma, w) = (\sigma_1 \cdots \sigma_n, w_1 \cdots w_n) \in C_k \wr S_n$ where $\mathrm{Ris}((\sigma, w)) = \mathbf{O}_{n-1}$ and $\mathrm{Des}((\sigma, w)) = \mathbf{E}_{n-1}$, then it must be the case that $\mathrm{Ris}(\sigma) = \mathbf{O}_{n-1}$ and $\mathrm{Des}(\sigma) = \mathbf{E}_{n-1}$ so that σ is an up-down permutation and $w_1 \leq w_2 \geq w_3 \leq w_4 \cdots$ so that w is a weak up-down word over the alphabet $\{0, \ldots, k-1\}$. It then follows that

$$A(t) = \sum_{n \geq 0} \frac{u\text{-}d_{2n,k} t^{2n}}{(2n)!} = \left(\frac{1}{R_k(t)} \right) \otimes \sec t \tag{23}$$

and

$$B(t) = \sum_{n \geq 0} \frac{u\text{-}d_{2n+1,k} t^{2n+1}}{(2n+1)!} = \left(\frac{S_k(t)}{R_k(t)} \right) \otimes \tan t. \tag{24}$$

We can also define a strong product order $<_s$ on $\mathbf{P} \times \{0, \ldots, k-1\}$ by defining $(i_1, w_1) <_s (i_2, w_2)$ if and only if $i_1 < i_2$ and $w_1 < w_2$. We then define

$$\begin{aligned} \mathrm{Des}_s((\sigma, w)) &= \{i : (\sigma_{i+1}, w_i)\} <_s (\sigma_i, w_{i+1})\} \text{ and} \\ \mathrm{Ris}_s((\sigma, w)) &= \{i : (\sigma_i, w_i) <_s (\sigma_{i+1}, w_{i+1})\}. \end{aligned}$$

We say at that $(\sigma, w) \in C_k \wr S_n$ is a *strong up-down* permutation if $\mathrm{Ris}_s((\sigma, w)) = \mathbf{O}_{n-1}$ and $\mathrm{Des}_s((\sigma, w)) = \mathbf{E}_{n-1}$. We let $su\text{-}d_{n,k}$ denote the number of strong up-down permutations of $C_k \wr S_n$. Then clearly,

$$\bar{A}(t) = \sum_{n \geq 0} \frac{su\text{-}d_{2n,k} t^{2n}}{(2n)!} = \left(\frac{1}{G_k(t)} \right) \otimes \sec t \tag{25}$$

and

$$\bar{B}(t) = \sum_{n\geq 0} \frac{su\text{-}d_{2n+1,k}t^{2n+1}}{(2n+1)!} = \left(\frac{F_k(t)}{G_k(t)}\right) \otimes \tan t. \tag{26}$$

3 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs of the generating functions over $(C_k \wr S_n)^{(2)}$ or $(C_k \wr S_n)_{(2)}$.

Let Λ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \ldots$ with coefficients in the field complex numbers $\mathbf{C}$. The n^{th} elementary symmetric function e_n in the variables $x_1, x_2, \ldots$ is given by

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t)$$

and the n^{th} homogeneous symmetric function h_n in the variables $x_1, x_2, \ldots$ is given by

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}.$$

Thus

$$H(t) = 1/E(-t). \tag{27}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition, that is, λ is a finite sequence of weakly increasing nonnegative integers. Let $\ell(\lambda)$ denote the number of nonzero integers in λ. If the sum of these integers is n, we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions says that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ or that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ. Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, then $\{h_\lambda : \lambda \text{ is a partition}\}$ is also a basis for Λ. Since $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ, we can specify a ring homomorphism θ on Λ by simply defining $\theta(e_n)$ for all $n \geq 0$.

Since the set of elementary symmetric functions e_λ is a basis for Λ, one can expresses $h_n = \sum_{\lambda \vdash n} a_{\lambda,n} e_\lambda$ for any $n > 0$. Up to a sign, the coefficient $a_{\lambda,n}$ equals the size of a certain set of combinatorial objects depending on λ. A *brick tabloid* of shape (n) and type $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a filling of a row of n squares of cells with brick of lengths $\lambda_1, \ldots, \lambda_k$ such that bricks to not overlap. One brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$ is displayed below.

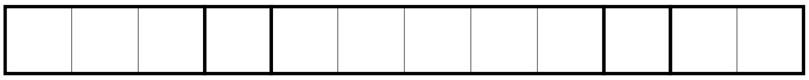

Figure 1: A brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$.

Let $\mathcal{B}_{\lambda,n}$ denote the set of all λ-brick tabloids of shape (n) and let $B_{\lambda,n} = |\mathcal{B}_{\lambda,n}|$. Through simple recursions stemming from (27), Eğecioğlu and Remmel proved in [7] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{28}$$

Next we define a class of symmetric functions $p_{n,\nu}$ which have a relationship with e_λ that is analogous to the relationship between h_n and e_λ. These functions were first introduced in [9] and [11]. Let ν be a function which maps the set of nonnegative integers into the field F. Recursively define $p_{n,\nu} \in \Lambda_n$ by setting $p_{0,\nu} = 1$ and letting

$$p_{n,\nu} = (-1)^{n-1}\nu(n)e_n + \sum_{k=1}^{n-1}(-1)^{k-1}e_k p_{n-k,\nu}$$

for all $n \geq 1$. By multiplying series, this means that

$$\left(\sum_{n\geq 0}(-1)^n e_n t^n\right)\left(\sum_{n\geq 1} p_{n,\nu}t^n\right) = \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} p_{n-k,\nu}(-1)^k e_k\right)t^n = \sum_{n\geq 1}(-1)^{n-1}\nu(n)e_n t^n,$$

where the last equality follows from the definition of $p_{n,\nu}$. Therefore,

$$\sum_{n\geq 1} p_{n,\nu}t^n = \frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)e_n t^n}{\sum_{n\geq 0}(-1)^n e_n t^n} \tag{29}$$

or, equivalently,

$$1 + \sum_{n\geq 1} p_{n,\nu}t^n = \frac{1 + \sum_{n\geq 1}(-1)^n(e_n - \nu(n)e_n)t^n}{\sum_{n\geq 0}(-1)^n e_n t^n}. \tag{30}$$

When taking $\nu(n) = 1$ for all $n \geq 1$, (30) becomes

$$1 + \sum_{n\geq 1} p_{n,1}t^n = \frac{1}{\sum_{n\geq 0}(-1)^n e_n t^n} = 1 + \sum_{n\geq 1} h_n t^n$$

which implies $p_{n,1} = h_n$. Other special cases for ν give well-known generating functions. For example, if $\nu(n) = n$ for $n \geq 1$, then $p_{n,\nu}$ is the power symmetric function $\sum_i x_i^n$. By taking $\nu(n) = (-1)^k\chi(n \geq k+1)$ for some $k \geq 1$, $p_{n,(-1)^k\chi(n\geq k+1)}$ is the Schur function corresponding to the partition $(1^k, n)$.

This definition of $p_{n,\nu}$ is desirable because of its expansion in terms of elementary symmetric functions. The coefficient of e_λ in $p_{n,\nu}$ has a nice combinatorial interpretation similar to that of the homogeneous symmetric functions. Suppose T is a brick tabloid of shape (n) and type λ and that the final brick in T has length ℓ. Define the weight of a brick tabloid $w_\nu(T)$ to be $\nu(\ell)$ and let

$$w_\nu(B_{\lambda,n}) = \sum_{\substack{T \text{ is a brick tabloid}\\ \text{of shape } (n) \text{ and type } \lambda}} w_\nu(T).$$

By the recursions found in the definition of $p_{n,\nu}$, it may be shown that

$$p_{n,\nu} = \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}w_\nu(B_{\lambda,n})e_\lambda$$

in almost the exact same way that (28) was proved in [7].

For $n \geq 1$ and $\lambda \vdash n$, let

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q} = q^0 + q^1 + \cdots + q^{n-1} & [n]_{p,q} &= \frac{p^n - q^n}{p-q} = p^{n-1}q^0 + \cdots + p^0 q^{n-1} \\ [n]_q! &= [n]_q \cdots [1]_q & [n]_{p,q}! &= [n]_{p,q} \cdots [1]_{p,q} \\ \begin{bmatrix} n \\ \lambda \end{bmatrix}_q &= \frac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!} & \begin{bmatrix} n \\ \lambda \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[\lambda_1]_{p,q}! \cdots [\lambda_\ell]_{p,q}!} \end{aligned}$$

be the q- and p,q-analogues of n, $n!$, and $\binom{n}{\lambda}$, respectfully. We shall use the convention that $[0]_q = [0]_{p,q} = 0$ and $[0]_q! = [0]_{p,q}! = 1$. The q- and p,q-analogues for the exponential function are defined by

$$\mathbf{e}_q[t] = \sum_{n \geq 0} \frac{t^n}{[n]_q!} q^{\binom{n}{2}} \qquad \mathbf{e}_{p,q}[t] = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} q^{\binom{n}{2}}.$$

For any permutation $\sigma \in S_n$, we define the number of inversions $\mathrm{inv}(\sigma)$ and the number of coinversions $\mathrm{coinv}(\sigma)$ of σ by

$$\mathrm{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) \text{ and } \mathrm{coinv}(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j)$$

where for any statement A, $\chi(A) = 1$ is A is true and $\chi(A) = 0$ if A is false. Note that $\mathrm{inv}(\sigma)$ and $\mathrm{coinv}(\sigma)$ make sense if σ is any sequence of non-negative integers.

We end this section with three lemmas that will be needed a later sections. All of the lemmas follow from simple codings of a basic result of Carlitz [6] that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\mathcal{R}(1^k 0^{n-k})} q^{\mathrm{inv}(r)}$$

where $\mathcal{R}(1^k 0^{n-k})$ is the number of rearrangements of k 1's and $n-k$ 0's. We start with a lemma from [13]. Fix a brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}$. Let $IF(T)$ denote the set of all fillings of the cells of $T = (b_1, \ldots, b_{\ell(\mu)})$ with the numbers $1, \ldots, n$ so that the numbers increase within each brick reading from left to right. We then think of each such filling as a permutation of S_n by reading the numbers from left to right in each row. For example, Figure 2 pictures an element of $IF(3,6,3)$ whose corresponding permutation is 4 6 12 1 5 7 8 10 11 2 3 9.

4	**6**	**12**	**1**	**5**	**7**	**8**	**10**	**11**	**2**	**3**	**9**

Figure 2: An element of $IF(3,6,3)$

Then the following lemma from [13] gives a combinatorial interpretation to $p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q}$.

Lemma 2. *If $T=(b_1,\ldots,b_{\ell(\mu)})$ is a brick tabloid in $\mathcal{B}_{\mu,n}$, then*

$$p^{\sum_i \binom{b_i}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} = \sum_{\sigma \in IF(T)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}.$$

Let $DF(T)$ denote the set of all fillings of the cells of $T=(b_1,\ldots,b_{\ell(\mu)})$ with the numbers $1,\ldots,n$ so that the numbers decrease within each brick reading from left to right. It is easy to see that if $\sigma \in IF(T)$, then $\sigma^r \in DF((b_{\ell(\mu)},\ldots,b_1))$ and $\mathrm{inv}(\sigma)=\mathrm{coinv}(\sigma^r)$ and $\mathrm{coinv}(\sigma)=\mathrm{inv}(\sigma^r)$. Thus we also have the following lemma.

Lemma 3. *If $T=(b_1,\ldots,b_{\ell(\mu)})$ is a brick tabloid in $\mathcal{B}_{\mu,n}$, then*

$$q^{\sum_i \binom{b_i}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} = \sum_{\sigma \in DF(T)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}.$$

Another well-known combinatorial interpretation for $\begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q$ is that it is equal to sum of the sizes of the partitions that are contained in a $n\times(k-1)$ rectangle. Thus we have the following lemma.

Lemma 4.

$$\sum_{0\le a_1\le\cdots\le a_n\le k-1} q^{a_1+\cdots+a_n} = \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q.$$

4 The generating functions up-not up permutations

In this section, we shall derive two generating functions that can be specialized to give the generating functions for $C(t)$ and $D(t)$ stated in the introduction. If $(\sigma,w)\in C_k\wr S_n$, we let

$$\begin{aligned} \mathrm{Ris}_{\mathbf{E}}((\sigma,w)) &= \{2i : 2i \in Ris((\sigma,w))\} \text{ and} && (31)\\ \mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w)) &= |\mathrm{Ris}_{\mathbf{E}}((\sigma,w))|. && (32) \end{aligned}$$

We let $(C_k\wr S_n)^{(2)}$ denote the set of $(\sigma,w)\in C_k\wr S_n$ such that $\mathbf{O}_{n-1}\subseteq \mathrm{Ris}((\sigma,w))$. Thus if $(\sigma,w)\in (C_k\wr S_n)^{(2)}$, (σ,w) is forced to have rises at all odd positions and $\mathrm{ris}(\sigma)_{\mathbf{E}}(\sigma,w)$ counts how many elements of the form $2i$ are in $\mathrm{Ris}((\sigma,w))$. In particular, if $(\sigma,w)\in(C_k\wr S_n)^{(2)}$ and $\mathrm{ris}(\sigma)_{\mathbf{E}}(\sigma,w)=0$, then $(\sigma,w)\in U\text{-}NU_{n,k}$.

Our first theorem of this section is the following.

Theorem 5. *For all $k\ge 2$,*

$$\sum_{n\ge 0}\frac{t^{2n}}{[2n]_{p,q}!}\sum_{(\sigma,w)\in(C_k\wr S_{2n})^{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))} = \frac{1-x}{1-x+\sum_{m\ge 1}\frac{p^{\binom{m}{2}}(x-1)^m t^{2m}}{[2m]_{p,q}!}\begin{bmatrix} 2m+k-1 \\ k-1 \end{bmatrix}_r} \tag{33}$$

Note that when we set $x=0$ and $p=q=r=1$, then (33) reduces to

$$\begin{aligned}\sum_{n\geq 0}\frac{u\text{-}nu_{2n,k}t^{2n}}{(2n)!} &= \frac{1}{\sum_{m\geq 0}\frac{t^{2m}(-1)^m}{(2m)!}\frac{(2m+k-1)(2m-k-2)\cdots(2m+1)}{(k-1)!}} \\ &= \frac{(k-1)!}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\cos t}. \end{aligned} \tag{34}$$

which is the generating function of $C(t)$ claimed in the introduction.

Proof. Define a ring homomorphism $\theta:\Lambda\to\mathbb{Q}(p,q,r,x)$, where $\mathbb{Q}$ is the set of the rational numbers, by setting

$$\theta(e_{2n}) = (-1)^{2n-1}(x-1)^{n-1}\frac{\begin{bmatrix}2n+k-1\\k-1\end{bmatrix}_r}{[2n]_{p,q}!}p^{\binom{n}{2}} \tag{35}$$

if $n\geq 1$ and

$$\theta(e_{2n+1}) = 0 \tag{36}$$

if $n\geq 0$. Then we claim that

$$\theta(h_{2n+1}) = 0 \tag{37}$$

for all $n\geq 0$ and

$$[2n]_{p,q}!\theta(h_{2n}) = \sum_{(\sigma,w)\in(C_k\wr S_{2n})^{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)\mathbf{E}((\sigma,w))} \tag{38}$$

for all $n\geq 1$. Note that

$$\theta(h_n) = \sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,(n)}\theta(e_\mu). \tag{39}$$

First suppose that n is odd. Then clearly every partition μ of n must have an odd part and, hence, $\theta(h_n)=0$ since $\theta(e_{2k+1})=0$ for all $k\geq 0$. If n is even, then the only μ such that $\theta(e_\mu)\neq 0$ on the RHS of (39) are when all the parts of μ are even. That is, μ must be of the form 2λ where $\lambda=(\lambda_1,\ldots,\lambda_k)$ is partition of n and $2\lambda=(2\lambda_1,\ldots,2\lambda_k)$. Thus

$$\begin{aligned}&[2n]_{p,q}!\theta(h_{2n}) \\ &= [2n]_{p,q}!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)}B_{2\mu,(2n)}\theta(e_{2\mu}) \\ &= [2n]_{p,q}!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)}\sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n)}}\prod_{j=1}^{\ell(\mu)}(-1)^{2b_j-1}(x-1)^{b_j-1}\frac{\begin{bmatrix}2b_j+k-1\\k-1\end{bmatrix}_r}{[2b_j]_{p,q}!}p^{\binom{2b_j}{2}} \\ &= \sum_{\mu\vdash n}\sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n)}}p^{\sum_{j=1}^{\ell(\mu)}\binom{2b_j}{2}}\begin{bmatrix}2n\\2b_1,\ldots,2b_{\ell(\mu)}\end{bmatrix}_{p,q}\prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}\begin{bmatrix}2b_j+k-1\\k-1\end{bmatrix}_r.\end{aligned} \tag{40}$$

Next we want to give a combinatorial interpretation to (40). By Lemma 2 for each brick tabloid $T=(2b_1,\ldots,2b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)}\binom{2b_j}{2}}\begin{bmatrix}2n\\2b_1,\ldots,2b_{\ell(\mu)}\end{bmatrix}_{p,q}$ as the sum of the weights of fillings of T with permutations $\sigma \in S_{2n}$ such that σ is increasing in each brick and we weight σ with $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}$. By Lemma 4, we can interpret the term $\prod_{j=1}^{\ell(\mu)} \begin{bmatrix} 2b_j+k-1 \\ k-1 \end{bmatrix}_r$ as the sum of the weights of fillings $w = w_1 \cdots w_{2n}$ of T where the elements of w are between 0 and $k-1$ and are weakly increasing in each brick and we weight w by $r^{|w|}$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}$ as all ways of picking a label of the even cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call all such objects that can be created by these choices *filled labeled brick tabloids* and we let $\mathcal{F}_{2n}$ denote the set of all filled labeled brick tabloids. Thus a $C \in \mathcal{F}_{2n}$ consists of a brick tabloid T, a permutation $\sigma \in S_{2n}$, a sequence $w \in \{0, \ldots, k-1\}^{2n}$, and a labeling L of the even cells of T with elements from $\{x, 1, -1\}$ such that

1. all the bricks of T have even length,
2. σ is strictly increasing in each brick,
3. w is weakly increasing in each brick,
4. the final cell of each brick is labeled with 1, and
5. each even numbered cell which is not a final cell of a brick is labeled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}$ times the product of all the x labels in L and the sign $\mathrm{sgn}(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 6, 2)$, then Figure 3 pictures such a composite object $C \in \mathcal{F}_{12}$ where $w(C) = q^{23}p^{43}r^{17}x^2$ and $\mathrm{sgn}(C) = -1$.

Thus

$$[2n]_{p,q}!\theta(h_{2n}) = \sum_{C \in \mathcal{F}_{2n}} \mathrm{sgn}(C)w(C). \tag{41}$$

L		x		1		−1		x		1		1
w	0	1	1	3	0	0	1	2	2	3	1	3
σ	2	3	10	11	1	4	6	8	9	12	5	7

Figure 3: A composite object $C \in \mathcal{F}_{12}$.

Next we define a weight preserving sign-reversing involution $I_1 : \mathcal{F}_{2n} \to \mathcal{F}_{2n}$. To define $I_1(C)$, we scan the cells of $C = (T, \sigma, w, L)$ from right to left looking for the leftmost cell $2t$ such that either (i) $2t$ is labeled with -1 or (ii) $2t$ is at the end a brick b_j and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1} and w is weakly increasing in all the cells corresponding to b_j and b_{j+1}. In case (i), $I_1(C) = (T', \sigma', w', L')$ where T' is the result of replacing the brick b in T containing $2t$ by two bricks b^* and b^{**} where b^* contains the cell $2t$ plus all the cells in b to the left of $2t$ and b^{**} contains all the cells of b to the right of $2t$, $\sigma' = \sigma$, $w' = w$, and L' is the labeling that results from L by changing the label of cell $2t$ from -1 to 1. In case (ii), $I_1(C) = (T', \sigma', w', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma' = \sigma$, $w' = w$, and L' is the labeling that results from L by changing the label of cell $2t$ from 1 to -1. If neither case (i) or case (ii) applies, then we let $I_1(C) = C$. For example, if C is the element of $\mathcal{F}_{12}$ pictured in Figure 3, then $I_1(C)$ is pictured in Figure 4.

L		x		1		1		x		1		1
w	0	1	1	3	0	0	1	2	2	3	1	3
σ	2	3	10	11	1	4	6	8	9	12	5	7

Figure 4: $I_1(C)$ for C in Figure 3.

It is easy to see that I_1 is a weight-preserving sign-reversing involution and hence I_1 shows that

$$[2n]_{p,q}!\theta(h_{2n}) = \sum_{C\in\mathcal{F}_{2n}, I_1(C)=C} \operatorname{sgn}(C)w(C). \tag{42}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I_1. First there can be no -1 labels in L so that $sg(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and $2t$ is that last cell of b_j, then it can not be the case that $\sigma_{2t} < \sigma_{2t+1}$ and $w_{2t} \leq w_{2t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we can think of the pair (σ, w) as an element of $C_k \wr S_{2n}$. It follows that if cell $2t$ is at the end of a brick, then $2t \notin \mathrm{Ris}_{\mathbf{E}}((\sigma, w))$. However if $2v$ is a cell which is not at the end of brick, then our definitions force $\sigma_{2v} < \sigma_{2v+1}$ and $w_{2v} \leq w_{2v+1}$ so that $2v \in \mathrm{Ris}_{\mathbf{E}}((\sigma, w))$. Since each such cell $2v$ must be labeled with an x, it follows that $\operatorname{sgn}(C)w(C) = q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))}$. Moreover our definitions force that $\mathbf{O}_{2n-1} \subseteq \mathrm{Ris}((\sigma, w))$ so that $(\sigma, w) \in (C_k \wr S_{2n})^{(2)}$. Such a fixed point is pictured in Figure 5. Vice versa, if $(\sigma, w) \in (C_k \wr S_{2n})^{(2)}$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end at cells of the form $2t$ where $2t \notin \mathrm{Ris}_{\mathbf{E}}((\sigma, w)$ and labeling each cell $2t \in \mathrm{Ris}_{\mathbf{E}}((\sigma, w))$ with x and labeling all other even numbered cells with 1. Thus we have shown that

$$[2n]_{p,q}!\theta(h_2n) = \sum_{(\sigma,w)\in(C_k\wr S_{2n})^{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))}$$

as desired.

L		x		1		x		x		1		1
w	0	1	1	3	0	0	1	2	2	3	1	3
σ	2	3	7	11	1	4	5	6	8	9	10	12

Figure 5: A fixed point of I_1.

Applying θ to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned}
\sum_{n\geq 0} \theta(h_n)t^n &= \sum_{n\geq 0} \frac{t^{2n}}{[2n]_{p,q}!} \sum_{(\sigma,w)\in (C_k\wr S_{2n})^{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{ris}(\sigma)_{\mathbf{E}}(\sigma,w)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^n\theta(e_n)} \\
&= \frac{1}{1+\sum_{m\geq 1}(-1)^{2m}t^{2m}\frac{(-1)^{2m-1}(x-1)^{m-1}p^{\binom{2m}{2}}}{[2m]_{p,q}!}\begin{bmatrix}2m+k-1\\ k-1\end{bmatrix}_r} \\
&= \frac{1-x}{1-x+\sum_{m\geq 1}\frac{p^{\binom{2m}{2}}(x-1)^m t^{2m}}{[2m]_{p,q}!}\binom{2m+k-1}{k-1}}
\end{aligned}$$

which proves (33).

□

Theorem 6. *For all $k \geq 2$,*

$$\sum_{n\geq 0} \frac{t^{2n+1}}{[2n+1]_{p,q}!} \sum_{(\sigma,w)\in (C_k\wr S_{2n+1})^{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))} = \frac{-\sum_{m\geq 1}\frac{p^{\binom{2m-1}{2}}(x-1)^m t^{2m-1}}{[2m-1]_{p,q}}\begin{bmatrix}2m-1+k-1\\ k-1\end{bmatrix}_r}{1-x+\sum_{m\geq 1}\frac{p^{\binom{m}{2}}(x-1)^m t^{2m}}{[2m]_{p,q}!}\begin{bmatrix}2m+k-1\\ k-1\end{bmatrix}_r}. \tag{43}$$

Note that when we set $x = 0$ and $p = q = r = 1$, then (33) reduces to

$$\begin{aligned}
\sum_{n\geq 0}\frac{u\text{-}nu_{2n+1,k}t^{2n+1}}{(2n+1)!} &= \frac{\sum_{m\geq 1}\frac{(-1)^{m-1}t^{2m-1}}{(2m-1)!}\frac{(2m+k-2)\cdots(2m)}{(k-1)!}}{\sum_{m\geq 0}\frac{t^{2m}(-1)^m}{(2m)!}\frac{(2m+k-1)(2m-k-2)\cdots(2m+1)}{(k-1)!}} \\
&= \frac{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\sin t}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\cos t}
\end{aligned} \tag{44}$$

which is the generating function of $D(t)$ claimed in the introduction.

Proof. Let θ be the ring homomorphism defined in Theorem 5. In this case, we will derive (43) by applying θ to the identity

$$\sum_{n\geq 1} p_{n,\nu}t^n = \frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)e_n t^n}{\sum_{n\geq 0}(-1)^n e_n t^n} \tag{45}$$

where

$$\begin{aligned}
\nu(2n) &= \frac{[2n]_{p,q}[2n]_r}{p^{2n-1}[2n+k-1]_r} \\
&= \frac{p^{\binom{2n-1}{2}}}{p^{\binom{2n}{2}}}\frac{[2n]_{p,q}!}{[2n-1]_{p,q}!}\frac{\begin{bmatrix}2n-1+k-1\\ k-1\end{bmatrix}_r}{\begin{bmatrix}2n+k-1\\ k-1\end{bmatrix}_r}
\end{aligned} \tag{46}$$

for $n \geq 1$ and $\nu(2m+1) = 0$ for $m \geq 0$. We have defined ν so that

$$\nu(2n)\theta(e_{2n}) = \frac{(-1)^{2n-1}(x-1)^{n-1}p^{\binom{2n-1}{2}}}{[2n-1]_{p,q}!}\begin{bmatrix}2n-1+k-1\\ k-1\end{bmatrix}_r. \tag{47}$$

Again it is easy to see that

$$\theta(p_{2n+1,\nu}) = 0 \tag{48}$$

for all $n \geq 0$ since the expansion of $p_{2n+1,\nu}$ in terms of the elementary symmetric functions only involves e_μ's where μ contains an odd part. The key fact that we have to prove is that

$$[2n+1]_{p,q}!\theta(p_{2n+2,\nu}) = \sum_{(\sigma,w)\in(C_k\wr S_{2n+1})^{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))} \tag{49}$$

for all $n \geq 0$. Note that

$$\begin{aligned}
&[2n+1]_{p,q}!\theta(p_{2n+2,\nu}) \qquad (50)\\
&= [2n+1]_{p,q}! \sum_{\mu\vdash n+1}(-1)^{2n+2-\ell(\mu)}w_\nu(B_{2\mu,(2n+2)})\theta(e_{2\mu})\\
&= [2n+1]_{p,q}! \sum_{\mu\vdash n+1}(-1)^{2n+2-\ell(\mu)} \sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n+2)}} \nu(2b_{\ell(\mu)}) \times\\
&\quad \prod_{j=1}^{\ell(\mu)}(-1)^{2b_j-1}(x-1)^{b_j-1}\frac{\begin{bmatrix}2b_j+k-1\\ k-1\end{bmatrix}_r}{[2b_j]_{p,q}!}p^{\binom{2b_j}{2}}\\
&= [2n+1]_{p,q}! \sum_{\mu\vdash n+1}\sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n+2)}} \frac{p^{\binom{2b_{\ell(\mu)}-1}{2}}(x-1)^{b_{\ell(\mu)}-1}}{[2b_{\ell(\mu)}-1]_{p,q}!}\begin{bmatrix}2b_{\ell(\mu)}-1+k-1\\ k-1\end{bmatrix}_r \times\\
&\quad \prod_{j=1}^{\ell(\mu)-1}\frac{p^{\binom{2b_j}{2}}(x-1)^{b_j-1}}{[2b_j]_{p,q}!}\begin{bmatrix}2b_j+k-1\\ k-1\end{bmatrix}_r\\
&= \sum_{\mu\vdash n+1}\sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n+2)}} p^{\binom{2b_{\ell(\mu)}-1}{2}}p^{\sum_{j=1}^{\ell(\mu)-1}\binom{2b_j}{2}}\begin{bmatrix}2n+1\\ 2b_1,\ldots,2b_{\ell(\mu)-1},2b_{\ell(\mu)}-1\end{bmatrix}_{p,q} \times\\
&\quad (x-1)^{b_{\ell(\mu)}-1}\begin{bmatrix}2b_{\ell(\mu)}-1+k-1\\ k-1\end{bmatrix}_r \prod_{j=1}^{\ell(\mu)-1}(x-1)^{b_j-1}\begin{bmatrix}2b_j+k-1\\ k-1\end{bmatrix}_r.
\end{aligned}$$

Again we want to give a combinatorial interpretation to (50). By Lemma 2 for each brick tabloid $T = (2b_1,\ldots,2b_{\ell(\mu)})$, we can interpret

$$p^{\binom{2b_{\ell(\mu)}-1}{2}}p^{\sum_{j=1}^{\ell(\mu)-1}\binom{2b_j}{2}}\begin{bmatrix}2n+1\\ 2b_1,\ldots,2b_{\ell(\mu)-1},2b_{\ell(\mu)-1}\end{bmatrix}_{p,q}$$

the sum of the weights of fillings of the first 2n+1 cells of T with a permutation $\sigma \in S_{2n+1}$ such that σ is increasing in each brick and where we weight σ with $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}$. Note that T has $2n+2$ cells so we will assume that the last cell of T is filled in and we will not place anything in that cell. By Lemma 4, we can interpret the term $\left[{2b_{\ell(\mu)}-1+k-1 \atop k-1}\right]_r \prod_{j=1}^{\ell(\mu)-1} \left[{2b_j+k-1 \atop k-1}\right]_r$ as giving the sum of the weights of fillings $w = w_1 \cdots w_{2n+1}$ of the first $2n+1$ cells of T where the elements of w are between 0 and $k-1$ and are weakly increasing in each brick and where we weight w by $r^{|w|}$. Finally, we interpret $(x-1)^{2b_{\ell(\mu)}-1} \prod_{j=1}^{\ell(\mu)-1} (x-1)^{b_j-1}$ as all ways of picking a label of the even cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call all such objects that can be created in this way filled labeled brick tabloids and let $\mathcal{G}_{2n+2}$ denote the set of all filled labeled brick tabloids. Thus a $C \in \mathcal{G}_{2n+2}$ consists of a brick tabloid T, a permutation $\sigma \in S_{2n+1}$, a sequence $w \in \{0, \ldots, k-1\}^{2n+1}$, and a labeling L of the even cells of T with elements from $\{x, 1, -1\}$ such that

1. all the bricks of T have even length,
2. σ is strictly increasing in each brick and fills in the first $2n+1$ cells,
3. w is weakly increasing in each brick and fills in the first $2n+1$ cells,
4. the final cell of each brick is labeled with 1, and
5. each even numbered cell which is not a final cell of a brick is labeled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|}$ times the product of all the x labels in L and the sign $\mathrm{sgn}(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 6, 2)$, then Figure 6 pictures such a composite object $C \in \mathcal{G}_{12}$ where $w(C) = q^{20} p^{35} r^{14} x^2$ and $\mathrm{sgn}(C) = -1$.

Thus

$$[2n+1]_{p,q}! \theta(p_{2n+2,\nu}) = \sum_{C \in \mathcal{G}_{2n+2}} \mathrm{sgn}(C) w(C). \tag{51}$$

L		x		1		−1		x		1		1
w	0	1	1	3	0	0	1	2	2	3	1	
σ	2	3	10	11	1	4	6	7	8	9	5	

Figure 6: A composite object $C \in \mathcal{G}_{12}$.

Next we define a weight preserving sign-reversing involution $I_2 : \mathcal{G}_{2n+2} \to \mathcal{G}_{2n+2}$. I_2 is essentially the same as I_1 of Theorem 6. That is, we scan the cells of $C = (T, \sigma, w, L)$ from right to left looking for the leftmost cell $2t$ such that either (i) $2t$ is labeled with -1 or (ii) $2t$ is at the end a brick b_j and the brick b_{j+1} immediately following b_j has the property that the σ is strictly increasing in all the cells corresponding to b_j and b_{j+1} and w is weakly increasing in all the cells corresponding to b_j and b_{j+1}. In case (i), $I_2(C) = (T', \sigma', w', L')$ where T' is the result of replacing the brick b in T containing $2t$ by two bricks b^* and b^{**} where b^* contains cell $2t$ plus all the cells in b to the left of $2t$ and b^{**} contains all the cells of b to the right of $2t$, $\sigma' = \sigma$, $w' = w$, and L' is the labeling that results from L by changing the label of cell $2t$ from -1 to 1. In case (ii), $I_2(C) = (T', \sigma', w', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma' = \sigma$, $w' = w$, and L' is the labeling that results from L by changing the label of cell $2t$ from 1 to -1. If neither case (i) or case (ii) applies, then we let $I_2(C) = C$. For example, if C is the element of $\mathcal{G}_{12}$ pictured in Figure 6, then $I_2(C)$ is pictured in Figure 7.

L		x		1		1		x		1		1
w	0	1	1	3	0	0	1	2	2	3	1	
σ	2	3	10	11	1	4	6	7	8	9	5	

Figure 7: $I_2(C)$ for C in Figure 6.

Again, it is easy to see that I_2 is a weight-preserving sign-reversing involution so that

$$[2n+1]_{p,q}!\theta(p_{2n+2,\nu}) = \sum_{C \in \mathcal{G}_{2n+2}, I_2(C)=C} \mathrm{sgn}(C)w(C). \tag{52}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I_2. First there can be no -1 labels in L so that $sg(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and $2t$ is that last cell of b_j, then it can not be the case that $\sigma_{2t} < \sigma_{2t+1}$ and $w_{2t} \leq w_{2t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we can think of (σ, w) as an element of $C_k \wr S_{2n+1}$. It follows that if cell $2t$ is at the end of a brick, then $2t \notin \mathrm{Ris}_{\mathbf{E}}((\sigma, w))$. However if $2v$ is a cell which is not at the end of brick, then our definitions force $\sigma_{2v} < \sigma_{2v+1}$ and $w_{2v} \leq w_{2v+1}$ so that $2v \in \mathrm{Ris}_{\mathbf{E}}((\sigma, w))$. Since each such cell $2v$ must be labeled with an x, it follows that $\mathrm{sgn}(C)w(C) = q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))}$. Moreover our definitions force that $\mathbf{O}_{2n-1} \subseteq \mathrm{Ris}((\sigma, w))$ so that $(\sigma, w) \in (C_k \wr S_{2n+1})^{(2)}$. Such a fixed point is pictured in Figure 8. Vice versa, if $(\sigma, w) \in (C_k \wr S_{2n+1})^{(2)}$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end at cells of the form $2t$ where $2t \notin \mathrm{Ris}_{\mathbf{E}}((\sigma, w)$ and labeling each cell $2t \in \mathrm{Ris}_{\mathbf{E}}((\sigma, w))$ with x and labeling all other even numbered cells with 1. Thus we have shown that

$$[2n+1]_{p,q}!\theta(p_{2n+2,\nu}) = \sum_{(\sigma,w) \in (C_k \wr S_{2n+1})^{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))}$$

as desired.

L		x		1		x		x		1		1
w	0	1	1	3	0	0	1	2	2	3	1	
σ	2	3	7	11	1	4	5	6	8	9	10	

Figure 8: A fixed point of I_2.

Applying θ to the identity (45), we get

$$\begin{aligned}
\sum_{n\geq 1}\theta(p_{n,\nu})t^n &= \sum_{n\geq 1}\frac{t^{2n+2}}{[2n+1]_{p,q}!}\sum_{(\sigma,w)\in(C_k\wr S_{2n+1})^{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))}\\
&= \frac{\sum_{m\geq 1}(-1)^{2m-1}t^{2m}\nu(2m)\theta(e_{2m})}{1+\sum_{n\geq 1}(-t)^n\theta(e_n)}\\
&= \frac{\sum_{m\geq 1}t^{2m}\frac{p^{\binom{2m-1}{2}}(x-1)^{m-1}}{[2n-1]_{p,q}}\begin{bmatrix}2n-1+k-1\\ k-1\end{bmatrix}_r}{1+\sum_{m\geq 1}(-1)^{2m}t^{2m}\frac{(-1)^{2m-1}(x-1)^{m-1}p^{\binom{2m}{2}}}{[2m]_{p,q}!}\begin{bmatrix}2m+k-1\\ k-1\end{bmatrix}_r}\\
&= \frac{-\sum_{m\geq 1}t^{2m}\frac{p^{\binom{2n-1}{2}}(x-1)^{m}}{[2n-1]_{p,q}}\begin{bmatrix}2n-1+k-1\\ k-1\end{bmatrix}_r}{1-x+\sum_{m\geq 1}\frac{p^{\binom{2m}{2}}(x-1)^{m}t^{2m}}{[2m]_{p,q}!}\binom{2m+k-1}{k-1}}
\end{aligned}$$

That is, we have shown that

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^{2n+2}}{[2n+1]_q!}\sum_{(\sigma,w)\in(C_k\wr S_{2n+1})^{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))} = \\
&\frac{-\sum_{m\geq 1}t^{2m}\frac{p^{\binom{2n-1}{2}}(x-1)^{m}}{[2n-1]_{p,q}}\begin{bmatrix}2n-1+k-1\\ k-1\end{bmatrix}_r}{1-x+\sum_{m\geq 1}\frac{p^{\binom{m}{2}}(x-1)^{m}t^{2m}}{[2m]_q!}\begin{bmatrix}2m+k-1\\ k-1\end{bmatrix}_r}.
\end{aligned} \tag{53}$$

Then dividing both sides of (53) by t yields (43).

□

5 The generating functions for not down-down permutations

In this section, we shall prove two generating functions which specialize to the generating functions $E(t)$ and $F(t)$ described in the introduction. In particular, we let $(C_k\wr S_n)_{(2)}$ denote the set of all $(\sigma,w)\in C_k\wr S_n$ such that $\mathbf{O}_{n-1}\cap \mathrm{Des}((\sigma,w))=\emptyset$. We let

$$\mathrm{NonDes}_{\mathbf{E}}((\sigma,w)) = \{2i : 2i\notin \mathrm{Des}((\sigma,w))\} \text{ and} \tag{54}$$

$$\mathrm{nondes}_{\mathbf{E}}((\sigma,w)) = |\mathrm{NonDes}_{\mathbf{E}}((\sigma,w))|. \tag{55}$$

It is easy to see that the generating functions

$$1+\sum_{n\geq 1}\frac{t^{2n}}{[2n]_{p,q}!}\sum_{(\sigma,w)\in\ (C_k\wr S_{2n})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))}. \tag{56}$$

and

$$\sum_{n\geq 0}\frac{t^{2n+1}}{[2n+1]_{p,q}!}\sum_{(\sigma,w)\in\ (C_k\wr S_{2n+1})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))}. \tag{57}$$

specialize to $E(t)$ and $F(t)$ respectively when we set $x=0$ and $p=q=r=1$.

Our strategy for finding these generating functions is very similar to finding the generating function (33) and (43). That is, if one reflects on the proof of Theorem 5, the main role of the definition of a ring homomorphism $\theta : \Lambda \to \mathbb{Q}(p,q,r,x)$ was to ensure that $[2n]_{p,q}!\theta(h_{2n})$ could be interpreted as the sum of the weights of labeled fillings (T,L,σ,w) such that (σ,w) was strictly increasing within each brick relative to the product ordering. Then the combinatorics of the involution I_1 showed that

$$[2n]_{p,q}!\theta(h_{2n}) = \sum_{(\sigma,w)\in (C_k\wr S_{2n})^{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))}$$

from which we could find the generating function by applying the ring homomorphism to the identity $H(t) = 1/E(-t)$. Now suppose that we could define a ring homomorphism $\Delta : \Lambda \to \mathbb{Q}(p,q,r,x)$ so that $[2n]_{p,q}!\Delta(h_{2n})$ could be interpreted as the sum of the weights of labeled fillings (T,L,σ,w) such that (σ,w) was non-increasing within each brick relative to the product ordering. Then it is not difficult to see that we can define an analogue of the involution I_1 where replace the condition that (σ,w) is strictly increasing in each brick by the condition that (σ,w) is non-increasing in each brick to show that

$$[2n]_{p,q}!\Delta(h_{2n}) = \sum_{(\sigma,w)\in (C_k\wr S_{2n})_{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))} \tag{58}$$

We shall show at the end of this section that

$$\begin{aligned} Z(p,q,r,t) &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in ND_{n,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} \\ &= \frac{1}{1+\sum_{m\geq 1} \frac{q^{\binom{m}{2}}(-t)^m}{[2m]_{p,q}!} \begin{bmatrix} m+k-1 \\ k-1 \end{bmatrix}_r} \end{aligned} \tag{59}$$

where $ND_{n,k}$ is the set of permutations $\sigma \in C_k \wr S_n$ such that $\mathrm{Des}((\sigma,w)) = \emptyset$. Then clearly

$$\begin{aligned} \frac{Z(p,q,r,t)+Z(p,q,r,-t)}{2} &= 1+\sum_{n\geq 1} \frac{t^{2n}}{[2n]_{p,q}!} \sum_{(\sigma,w)\in ND_{2n,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} \text{ and} \\ \frac{Z(p,q,r,t)-Z(p,q,r,-t)}{2} &= \sum_{n\geq 0} \frac{t^{2n+1}}{[2n+1]_{p,q}!} \sum_{(\sigma,w)\in ND_{2n+1,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|}. \end{aligned}$$

Hence for all $n \geq 1$,

$$\begin{aligned} \frac{Z(p^2,qp,r,t)+Z(p^2,qp,r,-t)}{2}\Big|_{\frac{t^{2n}}{[2n]_{p,q}!}} &= \sum_{(\sigma,w)\in ND_{2n,k}} (pq)^{\mathrm{inv}(\sigma)} (p^2)^{\mathrm{coinv}(\sigma)} r^{|w|} \\ &= p^{\binom{2n}{2}} \sum_{(\sigma,w)\in ND_{2n,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} \end{aligned} \tag{60}$$

since for all $\sigma \in S_{2n}$, $\mathrm{inv}(\sigma)+\mathrm{coinv}(\sigma)=\binom{2n}{2}$. Similarly,

$$\begin{aligned}\frac{Z(p^2,qp,r,t)-Z(p^2,qp,r,-t)}{2}\Big|_{\frac{t^{2n+1}}{[2n+1]_{p,q}!}} &= \sum_{(\sigma,w)\in ND_{2n+1,k}} (pq)^{\mathrm{inv}(\sigma)}(p^2)^{\mathrm{coinv}(\sigma)} r^{|w|} \\ &= p^{\binom{2n+1}{2}} \sum_{(\sigma,w)\in ND_{2n+1,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} \qquad (61)\end{aligned}$$

since for all $\sigma \in S_{2n+1}$, $\mathrm{inv}(\sigma)+\mathrm{coinv}(\sigma)=\binom{2n+1}{2}$.

We define our desired ring homomorphism $\Delta:\Lambda\rightarrow\mathbb{Q}(p,q,r,x)$ by setting

$$\begin{aligned}\Delta(e_{2n}) &= \frac{(-1)^{2n-1}(x-1)^{n-1}}{[2n]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}\Big|_{\frac{t^{2n}}{[2n]_{p,q}!}}\right) \\ &= \frac{(-1)^{2n-1}(x-1)^{n-1}}{[2n]_{p,q}!} p^{\binom{2m}{2}} \sum_{(\sigma,w)\in ND_{2n,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|}. \qquad (62)\end{aligned}$$

if $n\geq 1$ and

$$\Delta(e_{2n+1})=0 \qquad (63)$$

if $n\geq 0$. Again it is easy to see that

$$\Delta(h_{2n+1})=0 \qquad (64)$$

for all $n\geq 0$. Now

$$\begin{aligned}&[2n]_{p,q}!\Delta(h_{2n}) \qquad (65)\\ &= [2n]_{p,q}!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)} B_{2\mu,(2n)}\Delta(e_{2\mu}) \\ &= [2n]_{p,q}!\sum_{\mu\vdash n}(-1)^{2n-\ell(\mu)} \sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n)}} \prod_{j=1}^{\ell(\mu)} \frac{(-1)^{2b_j-1}(x-1)^{b_j-1}}{[2b_j]_{p,q}!}\times \\ &\qquad \left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}\Big|_{\frac{t^{2b_j}}{[2b_j]_{p,q}!}}\right) \\ &= \sum_{\mu\vdash n}\sum_{(2b_1,\ldots,2b_{\ell(\mu)})\in\mathcal{B}_{2\mu,(2n)}} p^{\sum_{j=1}^{\ell(\mu)} p^{\binom{2b_j}{2}}} \begin{bmatrix}2n\\ 2b_1,\ldots,2b_{\ell(\mu)}\end{bmatrix}_{p,q} \prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}\times \\ &\qquad \prod_{j=1}^{\ell(\mu)}\left(\sum_{(\sigma,w)\in ND_{2b_j,k}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}\right).\end{aligned}$$

Next we want to give a combinatorial interpretation to (65). By Lemma 2, for each brick tabloid $T=(2b_1,\ldots,2b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)}\binom{2b_j}{2}}\begin{bmatrix}2n\\ 2b_1,\ldots,2b_{\ell(\mu)}\end{bmatrix}_{p,q}$ as the sum of the weights of fillings of T with a permutation $\tau \in S_{2n}$ such that τ is increasing in each brick and we weight τ with $q^{\mathrm{inv}(\tau)}p^{\mathrm{coinv}(\tau)}$. Next the product

$$\prod_{j=1}^{\ell(\mu)} \sum_{(\sigma,w)\in ND_{2b_j,k}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}$$

can be interpreted as all ways to pick a permutation $(\sigma^{(j)}, w^{(j)}) \in C_k \wr S_{2b_i}$ for each brick $2b_j$ in T such that $\mathrm{Des}(\sigma^{(j)}, w^{(j)})) = \emptyset$ with weight $q^{\mathrm{inv}(\sigma^{(j)},w^{(j)})}p^{\mathrm{coinv}(\sigma^{(j)},w^{(j)}))}r^{|w^{(j)}|}$. For example, Figure 9 pictures such a choice for σ and choices for $(\sigma^{(1)}, w^{(2)}), (\sigma^{(2)}, w^{(2)}), (\sigma^{(3)}, w^{(3)})$ for the brick tabloid $T = (4, 6, 2)$. Here we have written the $(\sigma^{(j)}, w^{(j)})$'s in two line arrays as we did in the previous proofs, namely, the bottom line of the array gives $\sigma^{(j)}$ and the top line of the array gives $w^{(j)}$. We can then combine these two diagrams into single diagram which is pictured at the bottom of Figure 9 by rearranging the elements of σ in each brick b_j according to the permutation $\sigma^{(j)}$ and bringing down the top sequence $w^{(j)}$ to be the top sequence in each brick. The result is a pair (σ, w) such that (σ, w) has no descents that occur between two cells in the same brick and where we weight the pair (σ, w) by $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}$ as all ways of picking a label of the even cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call such objects filled labeled brick tabloids and we let $\mathcal{K}_{2n}$ denote the set of all filled labeled brick tabloids that arise in this way. Thus a $C \in \mathcal{K}_{2n}$ consists of a brick tabloid T, a permutation $\sigma \in S_{2n}$, a sequence $w \in \{0, \ldots, k-1\}^{2n}$, and a labeling L of the even cells of T with elements from $\{x, 1, -1\}$ such that

1. all the bricks of T have even length,
2. if $i \in \mathrm{Des}((\sigma, w))$, then i must be the final cell of some brick,
3. the final cell of each brick is labeled with 1, and
4. each even numbered cell which is not a final cell of a brick is labeled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}$ times the product of all the x labels in L and the sign $\mathrm{sgn}(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 6, 2)$, then the composite object C pictured at the bottom of Figure 9 is an element of $\mathcal{K}_{12}$ where $w(C) = q^{33}p^{33}r^{15}x^2$ and $\mathrm{sgn}(C) = -1$.

Thus

$$[2n]_{p,q}!\Delta(h_{2n}) = \sum_{C\in\mathcal{K}_{2n}} \mathrm{sgn}(C)w(C). \tag{66}$$

At this point, we can follow the analogous steps in Theorem 5 to prove that

$$[2n]_{p,q}!\Delta(h_{2n}) = \sum_{(\sigma,w)\in(C_k\wr S_{2n})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))}.$$

τ	2	3	10	11	1	4	6	7	8	9	5	12

$(\sigma^{(1)}, w^{(1)})$				$(\sigma^{(2)}, w^{(2)})$						$(\sigma^{(3)}, w^{(3)})$	
1	0	1	3	0	3	0	2	1	2	1	1
1	3	2	4	6	3	5	2	4	1	1	2

L		x		1		−1		x		1		1
w	1	0	1	3	0	3	0	2	1	2	1	1
σ	2	10	3	11	9	6	8	4	7	1	5	12

Figure 9: A composite object $C \in \mathcal{K}_{12}$.

Applying Δ to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned}
\sum_{n\geq 0} \Delta(h_n)t^n &= \sum_{n\geq 0} \frac{t^{2n}}{[2n]_{p,q}!} \sum_{(\sigma,\epsilon)\in\ (C_k \wr S_{2n})_{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|\epsilon|} x^{\mathrm{nondes}_{\mathbf{E}}(\sigma,\epsilon)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^n \Delta(e_n)} \\
&= \frac{1}{1+\sum_{m\geq 1}(-1)^{2m} t^{2m} \frac{(-1)^{2m-1}(x-1)^{m-1}}{[2m]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}\Big|_{\frac{t^{2m}}{[2m]_{p,q}!}}\right)} \\
&= \frac{1-x}{1-x+\sum_{m\geq 1}\frac{((x-1)^{1/2}t)^{2m}}{[2m]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}\Big|_{\frac{t^{2m}}{[2m]_{p,q}!}}\right)} \\
&= \frac{1-x}{-x+\frac{Z(p^2,pq,r,(x-1)^{1/2}t)+Z(p^2,pq,r,-(x-1)^{1/2}t)}{2}}.
\end{aligned}$$

Thus we have proved the following.

Theorem 7. *For all $k \geq 2$,*

$$1+\sum_{n\geq 1}\frac{t^{2n}}{[2n]_{p,q}!}\sum_{(\sigma,w)\in(C_k \wr S_{2n})_{(2)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))} =$$

$$\frac{1-x}{-x+\frac{Z(p^2,qp,r,(x-1)^{1/2}t)+Z(p^2,qp,r,-(x-1)^{1/2}t)}{2}}. \quad (67)$$

Note that setting $x = 0$ and $p = q = r = 1$ in (67) and using Corollary 11, we obtain that

$$\begin{aligned}
\sum_{n\geq 0} \frac{nd\text{-}d_{2n,k}t^{2n}}{(2n)!} &= \frac{1}{\frac{Z(1,1,1,it)+Z(1,1,1,-it)}{2}} \\
&= \frac{1}{\frac{1}{2}\left(\frac{1}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{-it}} + \frac{1}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{it}}\right)} \\
&= \frac{(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{-it})(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{it})}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\left(\frac{e^{-it}+e^{it}}{2}\right)} \\
&= \frac{(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{-it})(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{it})}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\cos t}.
\end{aligned}$$

Let

$$P_{k-1}(t) = \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{it} \tag{68}$$

for $k \geq 2$. Observe that if $D = \frac{\mathrm{d}}{\mathrm{d}t}$ is the ordinary differential operator, then

$$D^n(f(t)\cdot g(t)) = \sum_{k=0}^{n}\binom{n}{k}D^k(f(t))D^{n-k}(g(t)). \tag{69}$$

Hence in the special case where $f(t) = e^{it}$ and $g(t) = t^n$, we have that

$$\begin{aligned}
P_n(t) &= \sum_{k=0}^{n}\binom{n}{k}i^k e^{it}(n)\downarrow_{n-k} t^k \\
&= e^{it}\sum_{k=0}^{n}\binom{n}{k}^2(n-k)!i^k t^k
\end{aligned}$$

where $(n)\downarrow_0 = 1$ and $(n)\downarrow_s = n(n-1)\cdots(n-s+1)$ for $s \geq 1$. It follows that

$$\begin{aligned}
P_n(it)P_n(-it) &= \sum_{s=0}^{2n}t^s\sum_{r=0}^{s}\binom{n}{r}^2(n-r)!i^r\binom{n}{s-r}^2(n-(s-r))!(-i)^{s-r} \\
&= \sum_{s=0}^{2n}(i)^s t^s\sum_{r=0}^{s}(-1)^{s-r}\binom{n}{r}^2(n-r)!\binom{n}{s-r}^2(n-(s-r))!.
\end{aligned}$$

Note that when s is odd, then term

$$\sum_{r=0}^{s}(-1)^{s-r}\binom{n}{r}^2(n-r)!\binom{n}{s-r}^2(n-(s-r))! = 0$$

since the r-th term in the sum is the negative of $(s-r)$-th term in the sum. Thus

$$P_n(it)P_n(-it) = \sum_{s=0}^{n}t^{2s}\sum_{r=0}^{2s}(-1)^{s-r}\binom{n}{r}^2(n-r)!\binom{n}{2s-r}^2(n-(2s-r))!. \tag{70}$$

We can rewrite the sum

$$\sum_{r=0}^{2s}(-1)^{s-r}\binom{n}{r}^2(n-r)!\binom{n}{s-r}^2(n-(s-r))!$$

as a hypergeometric series. That is, we can rewrite this sum in terms of rising factorials $(a)_n$ where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$ to obtain that

$$\begin{aligned}
&\sum_{r=0}^{2s}(-1)^{s-r}\binom{n}{r}^2(n-r)!\binom{n}{s-r}^2(n-(s-r))! = \\
&\frac{(-1)^s n!n!(-n)_{2s}}{(2s)!(2s)!}\sum_{r=0}^{2s}\frac{(-n)_r(-2s)_r(-2s)_r}{(n-2s+1)_r(1)_r}\frac{1}{r!} = \\
&\frac{(-1)^s n!n!(-n)_{2s}}{(2s)!(2s)!}\ {}_3F_2(-n,-2s,-2s;n-2s+1,1;1).
\end{aligned} \tag{71}$$

This is a special case of the following hypergeometric series identity found in [15]:

$${}_3F_2(b,c,-2n;1-b-2n,1-c-2n;1) = \frac{(2n)!(b)_n(c)_n(b+c)_{2n}}{n!(b)_{2n}(c)_{2n}(b+c)_n}. \tag{72}$$

Using (72), we see that (71) is equal to

$$\begin{aligned}
&\frac{(-1)^s n!n!(-n)_{2s}}{(2s)!(2s)!}\frac{(2s)!(-n)_s(-2s)_s(-n-2s)_{2s}}{s!(-n)_{2s}(-2s)_{2s}(-n-2s)_s} = \\
&\frac{(-1)^s n!n!}{(2s)!}\frac{(-1)^s n\downarrow_s (-1)^s(2s)\downarrow_s (-1)^{2s}(n+2s)\downarrow_{2s}}{s!(-1)^{2s}(2s)!(-1)^s(n+2s)\downarrow_s} = \\
&\frac{(n!)^2}{(2s)!}\binom{n}{s}\binom{n+s}{s}
\end{aligned}$$

where $a\downarrow_0 = 1$ and $a\downarrow_n = a(a-1)\cdots(a-n+1)$ for $n \geq 1$. Hence

$$P_n(it)P_n(-it) = \sum_{s=0}^{n}\frac{(n!)^2}{(2s)!}\binom{n}{s}\binom{n+s}{s}t^{2s}. \tag{73}$$

Thus we have the following corollary.

Corollary 8. *For $k \geq 2$,*

$$1+\sum_{n\geq 1}\frac{nd\text{-}d_{2n,k}t^{2n}}{(2n)!} = \frac{\sum_{s=0}^{k-1}\frac{((k-1)!)^2}{(2s)!}\binom{k-1}{s}\binom{k-1+s}{s}t^{2s}}{\frac{d^{k-1}}{dt^{k-1}}t^{k-1}\cos t}. \tag{74}$$

We can also prove an analogue of Theorem 6. That is, suppose that we define

$$\begin{aligned}
\nu(2n) &= \frac{1}{\Delta(e_{2n})}\frac{(-1)^{2n-1}(x-1)^{n-1}}{[2n-1]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}\Big|_{\frac{t^{2n-1}}{[2n-1]_{p,q}!}}\right) \\
&= \frac{1}{\Delta(e_{2n})}\frac{(-1)^{2n-1}p^{\binom{2n-1}{2}}(x-1)^{n-1}}{[2n-1]_{p,q}!}\sum_{(\sigma,w)\in ND_{2n-1,k}} q^{\mathrm{inv}(\sigma,w)}p^{\mathrm{coinv}(\sigma,w)}r^{|w|}
\end{aligned} \tag{75}$$

for $n \geq 1$ and $\nu(2m+1) = 0$ for $m \geq 0$. In this case we have defined ν so that

$$\begin{aligned}
\nu(2n)\Delta(e_{2n}) &= \frac{(-1)^{2n-1}(x-1)^{n-1}}{[2n-1]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)-Z(p^2,pq,r,-t)}{2}|_{t^{2n-1}[2n-1]_{p,q}!}\right) \\
&= \frac{(-1)^{2n-1}p^{\binom{2n-1}{2}}(x-1)^{n-1}}{[2n-1]_{p,q}!}\sum_{(\sigma,w)\in ND_{2n-1,k}} q^{\mathrm{inv}(\sigma,w)}p^{\mathrm{coinv}(\sigma,w)}r^{|w|}. \qquad (76)
\end{aligned}$$

We can then follow the same sequence of steps as in the proof of Theorem 6 to prove that

$$[2n+1]_{p,q}!\Delta(p_{2n+2,\nu}) = \sum_{(\sigma,w)\in(C_k\wr S_{2n+1})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))}.$$

Applying Δ to the identity (45), we get

$$\begin{aligned}
\sum_{n\geq 1}\Delta(p_{n,\nu})t^n &= \sum_{n\geq 1}\frac{t^{2n+2}}{[2n+1]_{p,q}!}\sum_{(\sigma,w)\in(C_k\wr S_{2n+1})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))} \\
&= \frac{\sum_{m\geq 1}(-1)^{2m-1}t^{2m}\nu(2m)\Delta(e_{2m})}{1+\sum_{n\geq 1}(-t)^n\Delta(e_n)} \\
&= \frac{\sum_{m\geq 1}(-1)^{2m-1}t^{2m}\frac{(-1)^{2m-1}(x-1)^{m-1}}{[2m-1]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)-Z(p^2,pq,r,-t)}{2}|_{t^{2m-1}[2m-1]_{p,q}!}\right)}{1+\sum_{m\geq 1}(-1)^{2m}t^{2m}\frac{(-1)^{2m-1}(x-1)^{m-1}}{[2m]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}|_{t^{2m}[2m]_{p,q}!}\right)} \\
&= \frac{-t(x-1)^{1/2}\sum_{m\geq 1}\frac{((x-1)^{1/2}t)^{2m-1}}{[2m-1]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)-Z(p^2,pq,r,-t)}{2}|_{t^{2m-1}[2m-1]_{p,q}!}\right)}{1+\sum_{m\geq 1}(t^{2m}\frac{((x-1)^{1/2}t)^{2m}}{[2m]_{p,q}!}\left(\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}|_{t^{2m}[2m]_{p,q}!}\right)} \\
&= \frac{-t(x-1)^{1/2}(\frac{Z(p^2,pq,r,t)-Z(p^2,pq,r,-t)}{2})}{-x+\frac{Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}{2}} \\
&= \frac{-t(x-1)^{1/2}(Z(p^2,pq,r,t)-Z(p^2,pq,r,-t))}{-2x+Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}.
\end{aligned}$$

That is, we have shown that

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^{2n+2}}{[2n+1]_q!}\sum_{(\sigma,w)\in(C_k\wr S_{2n+1})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{ris}(\sigma)_{\mathbf{E}}((\sigma,w))} \qquad (77) \\
&= \frac{-t(x-1)^{1/2}(Z(p^2,pq,r,t)-Z(p^2,pq,r,-t))}{-2x+Z(p^2,pq,r,t)+Z(p^2,pq,r,-t)}
\end{aligned}$$

Then dividing both sides of (77) by t yields the following result.

Theorem 9. *For all $k \geq 2$,*

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^{2n+1}}{[2n+1]_{p,q}!}\sum_{(\sigma,w)\in(C_k\wr S_{2n+1})_{(2)}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{\mathrm{nondes}_{\mathbf{E}}((\sigma,w))} = \\
&\frac{-(x-1)^{1/2}\left(Z(p^2,pq,r,(x-1)^{1/2}t)-Z(p^2,pq,r,-(x-1)^{1/2}t)\right)}{-2x+Z(p^2,pq,r,(x-1)^{1/2}t)+Z(p^2,pq,r,-(x-1)^{1/2}t)} \qquad (78)
\end{aligned}$$

Note that when we set $x=0$ and $p=q=r=1$, then (78) reduces to

$$\begin{aligned}\sum_{n\geq 0}\frac{nd\text{-}d_{2n+1,k}t^{2n+1}}{(2n+1)!} &= \frac{-i(Z(1,1,1,it)-Z(1,1,1,-it))}{Z(1,1,1,it)+Z(1,1,1,-it))}\\ &= \frac{-i\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{-it}-\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{it}\right)}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{-it}+\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}e^{it}}\\ &= \frac{-i\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}(e^{it}-e^{-it})}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}(e^{it}+e^{-it})}\\ &= \frac{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\sin t}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\cos t}.\end{aligned}$$

which is the generating function of $F(t)$ claimed in the introduction.

We note that the generating functions of Theorems 6 and 9 are two different generating functions that can be specialized to

$$\frac{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\sin t}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}t^{k-1}\cos t}.$$

We end this section by proving a result which specializes to (59).

Theorem 10. *For all $k\geq 2$,*

$$\begin{aligned}&1+\sum_{n\geq 1}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{des((\sigma,w))}=\\ &\frac{1-x}{1-x+\sum_{n\geq 1}\frac{q^{\binom{n}{2}}((x-1)t)^n}{[n]_{p,q}!}\begin{bmatrix}n+k-1\\ k-1\end{bmatrix}_r}.\end{aligned} \tag{79}$$

Proof. Define a ring homomorphism $\Gamma:\Lambda\to\mathbb{Q}(p,q,r,x)$ by setting

$$\Gamma(e_n)=(-1)^{n-1}(x-1)^{n-1}\frac{\begin{bmatrix}n+k-1\\ k-1\end{bmatrix}_r}{[n]_{p,q}!}q^{\binom{n}{2}}. \tag{80}$$

Then we claim that

$$[n]_{p,q}!\Gamma(h_n)=\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{|w|}x^{des((\sigma,w))} \tag{81}$$

for all $n \geq 1$. That is,

$$\begin{aligned}
&[n]_{p,q}!\Gamma(h_n) \qquad (82)\\
&= [n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma(e_\mu)\\
&= [n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,(n)}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j-1}(x-1)^{b_j-1} \frac{\begin{bmatrix} b_j+k-1 \\ k-1 \end{bmatrix}_r}{[b_j]_{p,q}!} q^{\binom{b_j}{2}}\\
&= \sum_{\mu \vdash n} \sum_{(2b_1,\ldots,2b_{\ell(\mu)}) \in \mathcal{B}_{2\mu,(2n)}} q^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} \prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1} \begin{bmatrix} b_j+k-1 \\ k-1 \end{bmatrix}_r.
\end{aligned}$$

Next we want to give a combinatorial interpretation to (82). By Lemma 3 for each brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)})$, we can interpret $q^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q}$ as the sum of the weights of all fillings of T with a permutation $\sigma \in S_n$ such that σ is decreasing in each brick and we weight σ with $q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$. By Lemma 4, we can interpret the term $\prod_{j=1}^{\ell(\mu)} \begin{bmatrix} b_j+k-1 \\ k-1 \end{bmatrix}_r$ as the sum of the weights of fillings $w = w_1 \cdots w_n$ where the elements of w are between 0 and $k-1$ and are weakly decreasing in each brick and where we weight w by $r^{|w|}$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call all such objects created in this way filled labeled brick tabloids and let $\mathcal{H}_n$ denote the set of all filled labeled brick tabloids that arise in this way. Thus a $C \in \mathcal{H}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$, a sequence $w \in \{0, \ldots, k-1\}^n$, and a labeling L of the cells of T with elements from $\{x, 1, -1\}$ such that

1. σ is strictly decreasing in each brick,
2. w is weakly decreasing in each brick,
3. the final cell of each brick is labeled with 1, and
4. each cell which is not a final cell of a brick is labeled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|}$ times the product of all the x labels in L and the sign $\mathrm{sgn}(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 3, 3, 2)$, then Figure 10 pictures such a composite object $C \in \mathcal{H}_{12}$ where $w(C) = q^{35} p^{31} r^{17} x^5$ and $\mathrm{sgn}(C) = -1$.

Thus

$$[n]_{p,q}!\Gamma(h_n) = \sum_{C \in \mathcal{H}_n} \mathrm{sgn}(C) w(C). \qquad (83)$$

Next we define a weight preserving sign-reversing involution $I_3 : \mathcal{H}_n \to \mathcal{H}_n$. To define $I_3(C)$, we scan the cells of $C = (T, \sigma, s, L)$ from right to left looking for the leftmost cell t such that either (i) t is labeled with -1 or (ii) t is at the end a brick b_j and the brick b_{j+1} immediately following b_j has the property that the σ is strictly decreasing in all the cells corresponding to b_j and b_{j+1} and w is weakly decreasing in all the cells corresponding to b_j

L	x	x	−1	1	x	−1	1	x	x	1	−1	1
w	3	1	1	0	3	1	1	3	2	0	2	0
σ	11	4	3	2	10	9	6	12	8	1	7	5

Figure 10: A composite object $C \in \mathcal{H}_{12}$.

and b_{j+1}. In case (i), $I_3(C) = (T', \sigma', w', L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma' = \sigma$, $w' = w$, and L' is the labeling that results from L by changing the label of cell t from -1 to 1. In case (ii), $I_3(C) = (T', \sigma', w', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma' = \sigma$, $w' = w$, and L' is the labeling that results from L by changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I_3(C) = C$. For example, if C is the element of $\mathcal{H}_{12}$ pictured in Figure 10, then $I_3(C)$ is pictured in Figure 11.

L	x	x	1	1	x	−1	1	x	x	1	−1	1
w	3	1	1	0	3	1	1	3	2	0	2	0
σ	11	4	3	2	10	9	6	12	8	1	7	5

Figure 11: $I_3(C)$ for C in Figure 10.

It is easy to see that I_3 is a weight-preserving sign-reversing involution and hence I_3 shows that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{C \in \mathcal{H}_n, I_3(C)=C} \operatorname{sgn}(C) w(C). \tag{84}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I_3. First there can be no -1 labels in L so that $sg(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is that last cell of b_j, then it can not be the case that $\sigma_t > \sigma_{t+1}$ and $w_t \geq w_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we can think of (σ, w) as an element of $C_k \wr S_n$. Such a fixed point is pictured in 12. It follows that if cell t is at the end of a brick, then $t \notin \operatorname{Des}((\sigma, w))$. However if v is a cell which is not at the end of brick, then our definitions force $\sigma_v > \sigma_{v+1}$ and $w_v \geq w_{v+1}$ so that $v \in \operatorname{Des}((\sigma, w))$. Since each such cell v must be labeled with an x, it follows that $\operatorname{sgn}(C)w(C) = q^{\operatorname{inv}(\sigma)} p^{\operatorname{coinv}(\sigma)} r^{|w|} x^{des((\sigma,w))}$. Vice versa, if $(\sigma, w) \in C_k \wr S_n$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end as cells of the form t where $t \notin \operatorname{Des}((\sigma, w)$, labeling each cell $t \in \operatorname{Des}((\sigma, w))$ with x, and labeling the remaining cells with 1. Thus we have shown that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{(\sigma,w) \in C_k \wr S_{2n}} q^{\operatorname{inv}(\sigma)} p^{\operatorname{coinv}(\sigma)} r^{|w|} x^{des((\sigma,w))}$$

as desired.

L	x	x	x	1	x	x	1	x	x	1	x	1
w	3	1	1	0	3	1	1	3	2	0	2	0
σ	11	4	3	2	10	9	6	12	8	7	1	5

Figure 12: A fixed point of I_3.

Applying Γ to the identity $H(t) = (E(-t))^{-1}$, we obtain

$$\begin{aligned}
\sum_{n\geq 0} \Gamma(h_n)t^n &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{|w|} x^{des(\sigma,w)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^n \Gamma(e_n)} \\
&= \frac{1}{1+\sum_{m\geq 1}(-1)^m t^m \frac{(-1)^{m-1}(x-1)^{m-1} q^{\binom{m}{2}}}{[m]_{p,q}!} {\begin{bmatrix} m+k-1 \\ k-1 \end{bmatrix}}_r} \\
&= \frac{1-x}{1-x+\sum_{m\geq 1} \frac{q^{\binom{m}{2}}(x-1)^m t^m}{[m]_{p,q}!} {\begin{bmatrix} m+k-1 \\ k-1 \end{bmatrix}}_r}
\end{aligned}$$

which proves (79). □

Observe that if we set $x = 0$ in (79), we obtain (59) as desired. Moreover, if we set $x = 0$ and $r = 1$ in (79), then we obtain that

$$\begin{aligned}
&1 + \sum_{n\geq 1} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in ND_{n,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} = \\
&\frac{1}{1+\sum_{n\geq 1} \frac{q^{\binom{n}{2}}(-t)^n}{[n]_{p,q}!} \frac{(n+k-1)(n+k-2)\cdots(n+1)}{(k-1)!}} = \\
&\frac{(k-1)!}{(k-1)! + \sum_{n\geq 1} \frac{q^{\binom{n}{2}}(-t)^n}{[n]_{p,q}!} (n+k-1)(n+k-2)\cdots(n+1)} = \\
&\frac{(k-1)!}{\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} t^{k-1} e_{p,q}(-t)}.
\end{aligned}$$

Thus we have the following corollary.

Corollary 11. *For all $k \geq 2$,*

$$1 + \sum_{n\geq 1} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in ND_{n,k}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} = \frac{(k-1)!}{\frac{d^{k-1}}{dt^{k-1}} t^{k-1} e_{p,q}(-t)}. \tag{85}$$

and

$$1 + \sum_{n\geq 1} \frac{nd_{n,k} t^n}{n!} = \frac{(k-1)!}{\frac{d^{k-1}}{dt^{k-1}} t^{k-1} e^{-t}} \tag{86}$$

where $nd_{n,k} = |ND_{n,k}|$.

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