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Vacca-Type Series for Values of the Generalized Euler Constant Function and its Derivative

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Abstract

We generalize well-known Catalan-type integrals for Euler's constant to values of the generalized Euler constant function and its derivatives. Using generating functions appearing in these integral representations, we give new Vacca and Ramanujan-type series for values of the generalized Euler constant function and Addison-type series for values of the generalized Euler constant function and its derivative. As a consequence, we get base-B rational series for $\log\frac{4}{\pi}$, $\frac{G}{\pi}$ (where G is Catalan's constant), $\frac{\zeta'(2)}{\pi^2}$ and also for logarithms of the Somos and Glaisher-Kinkelin constants.

1 Introduction

J. Sondow [24] proved the following two formulas:

$$\gamma=\sum_{n=1}^{\infty}\frac{N_{1,2}(n)+N_{0,2}(n)}{2n(2n+1)}, \tag{1}$$

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$$\log\frac{4}{\pi}=\sum_{n=1}^{\infty}\frac{N_{1,2}(n)-N_{0,2}(n)}{2n(2n+1)}, \tag{2}$$

where γ is Euler's constant and $N_{i,2}(n)$ is the number of i's in the binary expansion of n (see sequences A000120 and A023416 in Sloane's database [23]). The series (1) is equivalent to the well-known Vacca series [28]

$$\gamma=\sum_{n=1}^{\infty}(-1)^n\frac{\lfloor\log_2 n\rfloor}{n}=\sum_{n=1}^{\infty}(-1)^n\frac{N_{1,2}\left(\lfloor\frac{n}{2}\rfloor\right)+N_{0,2}\left(\lfloor\frac{n}{2}\rfloor\right)}{n} \tag{3}$$

and both series (1) and (3) may be derived from Catalan's integral [8]

$$\gamma=\int_0^1\frac{1}{1+x}\sum_{n=1}^{\infty}x^{2^n-1}\,dx. \tag{4}$$

To see this it suffices to note that

$$G(x)=\frac{1}{1-x}\sum_{n=0}^{\infty}x^{2^n}=\sum_{n=1}^{\infty}(N_{1,2}(n)+N_{0,2}(n))x^n$$

is a generating function of the sequence $N_{1,2}(n)+N_{0,2}(n)$, (see A070939), which is the binary length of n, rewrite (4) as

$$\gamma=\int_0^1(1-x)\frac{G(x^2)}{x}\,dx$$

and integrate the power series termwise. In view of the equality

$$1=\int_0^1\sum_{n=1}^{\infty}x^{2^n-1}\,dx,$$

which is easily verified by termwise integration, (4) is equivalent to the formula

$$\gamma=1-\int_0^1\frac{1}{1+x}\sum_{n=1}^{\infty}x^{2^n}\,dx \tag{5}$$

obtained independently by Ramanujan (see [5, Corollary 2.3]). Catalan's integral (5) gives the following rational series for γ :

$$\gamma=1-\int_0^1(1-x)G(x^2)\,dx=1-\sum_{n=1}^{\infty}\frac{N_{1,2}(n)+N_{0,2}(n)}{(2n+1)(2n+2)}. \tag{6}$$

Averaging (1), (6) and (4), (5), respectively, we get Addison's series for γ [1]

$$\gamma=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{N_{1,2}(n)+N_{0,2}(n)}{2n(2n+1)(2n+2)}$$

and its corresponding integral

$$\gamma = \frac{1}{2} + \frac{1}{2}\int_0^1 \frac{1-x}{1+x}\sum_{n=1}^{\infty} x^{2^n-1}\,dx, \tag{7}$$

respectively. Integrals (5), (4) were generalized to an arbitrary integer base $B > 1$ by S. Ramanujan and by B. C. Berndt and D. C. Bowman (see [5]):

$$\gamma = 1 - \int_0^1 \left(\frac{1}{1-x} - \frac{Bx^{B-1}}{1-x^B}\right)\sum_{n=1}^{\infty} x^{B^n}\,dx \qquad \text{(Ramanujan)}, \tag{8}$$

$$\gamma = \int_0^1 \left(\frac{B}{1-x^B} - \frac{1}{1-x}\right)\sum_{n=1}^{\infty} x^{B^n-1}\,dx \qquad \text{(Berndt-Bowman)}. \tag{9}$$

Formula (9) implies the generalized Vacca series for γ (see [5, Theorem 2.6]) proposed by L. Carlitz [7]:

$$\gamma = \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n}\lfloor \log_B n \rfloor, \tag{10}$$

where

$$\varepsilon(n) = \begin{cases} B-1, & \text{if } B \text{ divides } n; \\ -1, & \text{otherwise;} \end{cases} \tag{11}$$

and the averaging integral of (8) and (9) produces the generalized Addison series for γ found by Sondow [24]:

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\lfloor \log_B Bn \rfloor P_B(n)}{Bn(Bn+1)\cdots(Bn+B)}, \tag{12}$$

where $P_B(x)$ is a polynomial of degree $B-2$ defined by

$$P_B(x) = (Bx+1)(Bx+2)\cdots(Bx+B-1)\sum_{m=1}^{B-1} \frac{m(B-m)}{Bx+m}. \tag{13}$$

In this paper, we consider the generalized Euler constant function

$$\gamma_{a,b}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{an+b} - \log\left(\frac{an+b+1}{an+b}\right)\right) z^n, \qquad a, b \in \mathbb{N}, \qquad |z| \le 1, \tag{14}$$

which is related to the constants in (1), (2) as $\gamma_{1,1}(1) = \gamma$, $\gamma_{1,1}(-1) = \log\frac{4}{\pi}$. Basic properties of a special case of this function, $\gamma_{1,1}(z)$, were studied earlier in [25, 14]. In Section 2, we show that $\gamma_{a,b}(z)$ admits an analytic continuation to the domain $\mathbb{C} \setminus [1, +\infty)$ in terms of the Lerch transcendent. In Sections 3–4, we generalize Catalan-type integrals (8), (9) to values of the generalized Euler constant function and its derivatives. Using generating functions appearing in these integral representations, we give new Vacca- and Ramanujan-type series for values of $\gamma_{a,b}(z)$ and Addison-type series for values of $\gamma_{a,b}(z)$ and its derivative. In Section

5, we get base-B rational series for $\log \frac{4}{\pi}$, $\frac{G}{\pi}$, (where G is Catalan's constant), $\frac{\zeta'(2)}{\pi^2}$ and also for logarithms of the Somos and Glaisher-Kinkelin constants. We also mention a connection of our approach to summation of series of the form

$$\sum_{n=1}^{\infty} N_{\omega,B}(n) Q(n,B) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{N_{\omega,B}(n) P_B(n)}{Bn(Bn+1)\cdots(Bn+B)},$$

where $Q(n,B)$ is a rational function of B and n

$$Q(n,B) = \frac{1}{Bn(Bn+1)} + \frac{2}{Bn(Bn+2)} + \cdots + \frac{B-1}{Bn(Bn+B-1)}, \tag{15}$$

and $N_{\omega,B}(n)$ is the number of occurrences of a word ω over the alphabet $\{0, 1, \ldots, B-1\}$ in the B-ary expansion of n, considered in [2]. Moreover, we answer some questions posed in [2] concerning possible generalizations of the series (1) and (2) to any integer base $B > 1$. Note that in the above notation, the generalized Vacca series (10) can be written as follows:

$$\gamma = \sum_{k=1}^{\infty} L_B(k) Q(k,B), \tag{16}$$

where $L_B(k) := \lfloor \log_B Bk \rfloor = \sum_{\alpha=0}^{B-1} N_{\alpha,B}(k)$ is the B-ary length of k. Indeed, representing $n = Bk + r$, $0 \le r \le B-1$ and summing in (10) over $k \ge 1$ and $0 \le r \le B-1$ we get

$$\gamma = \sum_{k=1}^{\infty} \lfloor \log_B Bk \rfloor \left(\frac{B-1}{Bk} - \frac{1}{Bk+1} - \cdots - \frac{1}{Bk+B-1} \right) = \sum_{k=1}^{\infty} \lfloor \log_B Bk \rfloor Q(k,B).$$

Using the same notation, the generalized Addison series (12) gives another base-B expansion of Euler's constant

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{L_B(n) P_B(n)}{Bn(Bn+1)\cdots(Bn+B)} = \frac{1}{2} + \sum_{n=1}^{\infty} L_B(n) \left(Q(n,B) - \frac{B-1}{2Bn(n+1)} \right) \tag{17}$$

which converges faster than (16) to γ. Here we used the fact that

$$\sum_{n=1}^{\infty} \sum_{\alpha=0}^{B-1} \frac{N_{\alpha,B}(n)}{n(n+1)} = \frac{B}{B-1},$$

which can be easily checked by [3, Section 3]. On the other hand,

$$Q(n,B) - \frac{B-1}{2Bn(n+1)} = \frac{1}{2} \sum_{m=1}^{B-1} \left(\frac{1}{Bn} - \frac{2}{Bn+m} + \frac{1}{Bn+B} \right)$$
$$= \frac{1}{Bn(Bn+B)} \sum_{m=1}^{B-1} \left(2m - B + \frac{2m(B-m)}{Bn+m} \right) = \frac{P_B(n)}{Bn(Bn+1)\cdots(Bn+B)}.$$

Finally, we give a brief description of some other generalized Euler constants that have appeared in the literature in Section 6.

2 Analytic continuation

We consider the generalized Euler constant function $\gamma_{a,b}(z)$ defined in (14), where a, b are positive real numbers, $z \in \mathbb{C}$, and the series converges when $|z| \leq 1$. We show that $\gamma_{a,b}(z)$ admits an analytic continuation to the domain $\mathbb{C} \setminus [1, +\infty)$. The following theorem is a slight modification of [25, Theorem 3].

Theorem 1. *Let a, b be positive real numbers, $z \in \mathbb{C}$, $|z| \leq 1$. Then*

$$\gamma_{a,b}(z) = \int_0^1 \int_0^1 \frac{(xy)^{b-1}(1-x)}{(1 - zx^a y^a)(-\log xy)}\, dxdy = \int_0^1 \frac{x^{b-1}(1-x)}{1 - zx^a} \left(\frac{1}{1-x} + \frac{1}{\log x} \right) dx. \quad (18)$$

The integrals converge for all $z \in \mathbb{C} \setminus (1, +\infty)$ and give the analytic continuation of the generalized Euler constant function $\gamma_{a,b}(z)$ for $z \in \mathbb{C} \setminus [1, +\infty)$.

Proof. Denoting the double integral in (18) by $I(z)$ and for $|z| \leq 1$, expanding $(1 - zx^a y^a)^{-1}$ in a geometric series we have

$$\begin{aligned} I(z) &= \sum_{k=0}^{\infty} z^k \int_0^1 \int_0^1 \frac{(xy)^{ak+b-1}(1-x)}{(-\log xy)}\, dxdy \\ &= \sum_{k=0}^{\infty} z^k \int_0^1 \int_0^1 \int_0^{+\infty} (xy)^{t+ak+b-1}(1-x)\, dxdydt \\ &= \sum_{k=0}^{\infty} z^k \int_0^{+\infty} \left(\frac{1}{(t+ak+b)^2} - \left(\frac{1}{t+ak+b} - \frac{1}{t+ak+b+1} \right) \right) dt = \gamma_{a,b}(z). \end{aligned}$$

On the other hand, making the change of variables $u = x^a$, $v = y^a$ in the double integral we get

$$I(z) = \frac{1}{a} \int_0^1 \int_0^1 \frac{(uv)^{\frac{b}{a}-1}(1 - u^{\frac{1}{a}})}{(1 - zuv)(-\log uv)}\, dudv.$$

Now by [12, Corollary 3.3], for $z \in \mathbb{C} \setminus [1, +\infty)$ we have

$$I(z) = \frac{1}{a}\Phi\left(z, 1, \frac{b}{a}\right) - \frac{\partial \Phi}{\partial s}\left(z, 0, \frac{b}{a}\right) + \frac{\partial \Phi}{\partial s}\left(z, 0, \frac{b+1}{a}\right),$$

where $\Phi(z, s, u)$ is the Lerch transcendent, a holomorphic function in z and s, for $z \in \mathbb{C} \setminus [1, +\infty)$ and all complex s (see [12, Lemma 2.2]), which is the analytic continuation of the series

$$\Phi(z, s, u) = \sum_{n=0}^{\infty} \frac{z^n}{(n+u)^s}, \qquad u > 0.$$

To prove the second equality in (18), make the change of variables $X = xy$, $Y = y$ and integrate with respect to Y. □

Corollary 2. *Let a, b be positive real numbers, $l \in \mathbb{N}$, $z \in \mathbb{C} \setminus [1, +\infty)$. Then for the l-th derivative we have*

$$\gamma_{a,b}^{(l)}(z) = \int_0^1 \int_0^1 \frac{(xy)^{al+b-1}(x-1)}{(1 - zx^a y^a)^{l+1} \log xy}\, dxdy = \int_0^1 \frac{x^{la+b-1}(1-x)}{(1 - zx^a)^{l+1}} \left(\frac{1}{1-x} + \frac{1}{\log x} \right) dx.$$

From Corollary 2, [12, Cor.3.3, 3.8, 3.9] and [2, Lemma 4] we get

Corollary 3. *Let a, b be positive real numbers, $z \in \mathbb{C} \setminus [1, +\infty)$. Then the following equalities hold:*

$$\gamma_{a,b}(1) = \log\Gamma\Big(\frac{b+1}{a}\Big) - \log\Gamma\Big(\frac{b}{a}\Big) - \frac{1}{a}\psi\Big(\frac{b}{a}\Big),$$

$$\gamma_{a,b}(z) = \frac{1}{a}\Phi\Big(z, 1, \frac{b}{a}\Big) - \frac{\partial\Phi}{\partial s}\Big(z, 0, \frac{b}{a}\Big) + \frac{\partial\Phi}{\partial s}\Big(z, 0, \frac{b+1}{a}\Big),$$

$$\gamma'_{a,b}(z) = -\frac{b}{a^2}\Phi\Big(z, 1, \frac{b}{a}+1\Big) + \frac{1}{a(1-z)} + \frac{b}{a}\frac{\partial\Phi}{\partial s}\Big(z, 0, \frac{b}{a}+1\Big) - \frac{\partial\Phi}{\partial s}\Big(z, -1, \frac{b}{a}+1\Big) -$$

$$\frac{b+1}{a}\frac{\partial\Phi}{\partial s}\Big(z, 0, \frac{b+1}{a}+1\Big) + \frac{\partial\Phi}{\partial s}\Big(z, -1, \frac{b+1}{a}+1\Big),$$

where $\Phi(z, s, u)$ is the Lerch transcendent and $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the logarithmic derivative of the gamma function.

3 Catalan-type integrals for $\gamma^{(l)}_{a,b}(z)$.

Berndt and Bowman [5] demonstrated that for $x > 0$ and any integer $B > 1$,

$$\frac{1}{1-x} + \frac{1}{\log x} = \sum_{k=1}^{\infty} \frac{(B-1) + (B-2)x^{\frac{1}{B^k}} + (B-3)x^{\frac{2}{B^k}} + \cdots + x^{\frac{B-2}{B^k}}}{B^k(1 + x^{\frac{1}{B^k}} + x^{\frac{2}{B^k}} + \cdots + x^{\frac{B-1}{B^k}})}. \tag{19}$$

The special cases $B = 2, 3$ of this equality can be found in Ramanujan's third note book [21, p. 364]. Using this key formula we prove the following generalization of the integral (9).

Theorem 4. *Let a, b, B be positive integers with $B > 1$, l a non-negative integer. If either $z \in \mathbb{C} \setminus [1, +\infty)$ and $l \geq 1$, or $z \in \mathbb{C} \setminus (1, +\infty)$ and $l = 0$, then*

$$\gamma^{(l)}_{a,b}(z) = \int_0^1 \left(\frac{B}{1-x^B} - \frac{1}{1-x}\right) F_l(z, x)\, dx \tag{20}$$

where

$$F_l(z, x) = \sum_{k=1}^{\infty} \frac{x^{(b+al)B^k-1}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}. \tag{21}$$

Proof. First we note that the series of variable x on the right-hand side of (19) converges uniformly on $[0, 1]$, since the absolute value of its general term does not exceed $\frac{B-1}{2B^{k-1}}$. Then for $l \geq 0$, multiplying both sides of (19) by $\frac{x^{la+b-1}(1-x)}{(1-zx^a)^{l+1}}$ and integrating over $0 \leq x \leq 1$ we get

$$\gamma^{(l)}_{a,b}(z) = \sum_{k=1}^{\infty} \int_0^1 \frac{x^{la+b-1}(1-x)}{(1-zx^a)^{l+1}} \cdot \frac{(B-1) + (B-2)x^{\frac{1}{B^k}} + \cdots + x^{\frac{B-2}{B^k}}}{B^k(1 + x^{\frac{1}{B^k}} + x^{\frac{2}{B^k}} + \cdots + x^{\frac{B-1}{B^k}})}\, dx.$$

Replacing x by x^{B^k} in each integral we find

$$\begin{aligned}\gamma_{a,b}^{(l)}(z) &= \sum_{k=1}^{\infty} \int_0^1 \frac{x^{(la+b)B^k-1}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}} \cdot \frac{(B-1)+(B-2)x+\cdots+x^{B-2}}{1+x+x^2+\cdots+x^{B-1}}\,dx \\ &= \int_0^1 \left(\frac{B}{1-x^B} - \frac{1}{1-x}\right) F_l(z,x)\,dx,\end{aligned}$$

as required. □

From Theorem 4 we readily get a generalization of Ramanujan's integral.

Corollary 5. *Let a, b, B be positive integers with $B > 1$, l a non-negative integer. If either $z \in \mathbb{C} \setminus [1, +\infty)$ and $l \geq 1$, or $z \in \mathbb{C} \setminus (1, +\infty)$ and $l = 0$, then*

$$\gamma_{a,b}^{(l)}(z) = \int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}}\,dx + \int_0^1 \left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) F_l(z,x)\,dx. \tag{22}$$

Proof. First we note that the series (21), considered as a sum of functions of the variable x converges uniformly on $[0, 1-\varepsilon]$ for any $\varepsilon > 0$. Then integrating termwise we have

$$\int_0^{1-\varepsilon} F_l(z,x)\,dx = \sum_{k=1}^{\infty} \int_0^{1-\varepsilon} \frac{x^{(b+al)B^k-1}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}\,dx.$$

Making the change of variable $y = x^{B^k}$ in each integral we get

$$\int_0^{1-\varepsilon} F_l(z,x)\,dx = \sum_{k=1}^{\infty} \frac{1}{B^k} \int_0^{(1-\varepsilon)^{B^k}} \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}}\,dy.$$

Since the last series, considered as a series in the variable ε, converges uniformly on $[0, 1]$, letting ε tend to zero we get

$$\int_0^1 F_l(z,x)\,dx = \frac{1}{B-1} \int_0^1 \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}}\,dy. \tag{23}$$

Now from (20) and (23) it follows that

$$\gamma_{a,b}^{(l)}(z) - \int_0^1 \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}}\,dy = \int_0^1 \left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) F_l(z,x)\,dx,$$

and the proof is complete. □

Averaging the formulas (20) and (22), we get the following generalization of the integral (7).

Corollary 6. *Let a, b, B be positive integers with $B > 1$, l a non-negative integer. If either $z \in \mathbb{C} \setminus [1, +\infty)$ and $l \geq 1$, or $z \in \mathbb{C} \setminus (1, +\infty)$ and $l = 0$, then*

$$\gamma_{a,b}^{(l)}(z) = \frac{1}{2} \int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}}\,dx + \frac{1}{2} \int_0^1 \left(\frac{B(1+x^B)}{1-x^B} - \frac{1+x}{1-x}\right) F_l(z,x)\,dx.$$

4 Vacca-type series for $\gamma_{a,b}(z)$ and $\gamma'_{a,b}(z)$.

Theorem 7. *Let a, b, B be positive integers with $B > 1$, $z \in \mathbb{C}$, $|z| \le 1$. Then for the generalized Euler constant function $\gamma_{a,b}(z)$, the following expansion is valid:*

$$\gamma_{a,b}(z) = \sum_{k=1}^{\infty} a_k Q(k,B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B} \rfloor} \frac{\varepsilon(k)}{k},$$

where $Q(k,B)$ is a rational function given by (15), $\{a_k\}_{k=0}^{\infty}$ is a sequence defined by the generating function

$$G(z,x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{x^{bB^k}(1-x^{B^k})}{1-zx^{aB^k}} = \sum_{k=0}^{\infty} a_k x^k \tag{24}$$

and $\varepsilon(k)$ is defined in (11).

Proof. For $l = 0$, rewrite (20) in the form

$$\gamma_{a,b}(z) = \int_0^1 \frac{1-x^B}{x} \left(\frac{B}{1-x^B} - \frac{1}{1-x} \right) G(z,x^B)\, dx$$

where $G(z,x)$ is defined in (24). Then, since $a_0 = 0$, we have

$$\gamma_{a,b}(z) = \int_0^1 (B-1-x-x^2-\cdots-x^{B-1}) \sum_{k=1}^{\infty} a_k x^{Bk-1}\, dx. \tag{25}$$

Expanding $G(z,x)$ in a power series of x,

$$G(z,x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} z^m x^{(am+b)B^k} (1+x+\cdots+x^{B^k-1}),$$

we see that $a_k = O(\ln_B k)$. Therefore, by termwise integration in (25), which can be easily justified by the same way as in the proof of Corollary 5, we get

$$\begin{aligned}\gamma_{a,b}(z) &= \sum_{k=1}^{\infty} a_k \int_0^1 [(x^{Bk-1}-x^{Bk}) + (x^{Bk-1}-x^{Bk+1}) + \cdots + (x^{Bk-1}-x^{Bk+B-2})]\, dx \\ &= \sum_{k=1}^{\infty} a_k Q(k,B).\end{aligned}$$

□

Theorem 8. *Let a, b, B be positive integers with $B > 1$, $z \in \mathbb{C}$, $|z| \le 1$. Then for the generalized Euler constant function, the following expansion is valid:*

$$\gamma_{a,b}(z) = \int_0^1 \frac{x^{b-1}(1-x)}{1-zx^a}\, dx - \sum_{k=1}^{\infty} a_k \widetilde{Q}(k,B),$$

where

$$\begin{aligned}\widetilde{Q}(k,B) &= \frac{B-1}{Bk(k+1)} - Q(k,B)\\ &= \frac{B-1}{(Bk+B)(Bk+1)} + \frac{B-2}{(Bk+B)(Bk+2)} + \cdots + \frac{1}{(Bk+B)(Bk+B-1)}\end{aligned}$$

and the sequence $\{a_k\}_{k=1}^{\infty}$ *is defined in Theorem 7.*

Proof. From Corollary 5 with $l=0$, using the same method as in the proof of Theorem 7, we get

$$\begin{aligned}&\int_0^1 \left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) F_0(z,x) = \int_0^1 \frac{1-x^B}{x}\left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) G(z,x^B)\,dx\\ &= \int_0^1 (Bx^{B-1} - (1+x+\cdots+x^{B-1})) \sum_{k=1}^{\infty} a_k x^{Bk}\,dx\\ &= \sum_{k=1}^{\infty} a_k \int_0^1 [(x^{Bk+B-1} - x^{Bk+B-2}) + \cdots + (x^{Bk+B-1} - x^{Bk+1}) + (x^{Bk+B-1} - x^{Bk})]\,dx\\ &= -\sum_{k=1}^{\infty} a_k \widetilde{Q}(k,B).\end{aligned}$$

□

Theorem 9. *Let a, b, B be positive integers with $B > 1$, $z \in \mathbb{C}$, $|z| \le 1$. Then for the generalized Euler constant function $\gamma_{a,b}(z)$ and its derivative, the following expansion is valid:*

$$\gamma_{a,b}^{(l)}(z) = \frac{1}{2}\int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}}\,dx + \sum_{k=1}^{\infty} a_{k,l} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}, \qquad l = 0, 1,$$

where $P_B(k)$ is a polynomial of degree $B-2$ given by (13), $z \neq 1$ if $l = 1$, and the sequence $\{a_{k,l}\}_{k=0}^{\infty}$ is defined by the generating function

$$G_l(z,x) = \frac{1}{1-x}\sum_{k=0}^{\infty} \frac{x^{(b+al)B^k}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}} = \sum_{k=0}^{\infty} a_{k,l} x^k, \qquad l = 0, 1. \tag{26}$$

Proof. Expanding $G_l(z,x)$ in a power series of x,

$$G_l(z,x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \binom{m+l}{l} z^m x^{(b+al+am)B^k}(1 + x + x^2 + \cdots + x^{B^k-1}),$$

we see that $a_{k,l}=O(k^l\ln_B k)$. Therefore, for $l=0,1$, by termwise integration we get

$$\int_0^1\left(\frac{B(1+x^B)}{1-x^B}-\frac{1+x}{1-x}\right)F_l(z,x)dx=\int_0^1\frac{1-x^B}{x}\left(\frac{B(1+x^B)}{1-x^B}-\frac{1+x}{1-x}\right)G_l(z,x^B)dx$$
$$=\int_0^1[(B-1)-2x-2x^2-\cdots-2x^{B-1}+(B-1)x^B]\sum_{k=1}^{\infty}a_{k,l}x^{Bk-1}\,dx$$
$$=\sum_{k=1}^{\infty}a_{k,l}\left(\frac{B-1}{Bk}-\frac{2}{Bk+1}-\frac{2}{Bk+2}-\cdots-\frac{2}{Bk+B-1}+\frac{B-1}{Bk+B}\right)$$
$$=2\sum_{k=1}^{\infty}a_{k,l}\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $P_B(k)$ is defined in (13) and the last series converges since $\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}=O(k^{-3})$. Now our theorem easily follows from Corollary 6. $\square$

5 Examples of rational series

It is easily seen that the generating function (26) satisfies the following functional equation:

$$G_l(z,x)-\frac{1-x^B}{1-x}G_l(z,x^B)=\frac{x^{b+al}}{(1-zx^a)^{l+1}},\tag{27}$$

which is equivalent to the following identity for series:

$$\sum_{k=0}^{\infty}a_{k,l}x^k-(1+x+\cdots+x^{B-1})\sum_{k=0}^{\infty}a_{k,l}x^{Bk}=\sum_{k=l}^{\infty}\binom{k}{l}z^{k-l}x^{ak+b}.$$

Comparing coefficients of powers of x we get an alternative definition of the sequence $\{a_{k,l}\}_{k=0}^{\infty}$ by means of the recursion

$$a_{0,l}=a_{1,l}=\cdots=a_{al+b-1,l}=0$$

and for $k\ge al+b$,

$$a_{k,l}=\begin{cases}a_{\lfloor\frac{k}{B}\rfloor,l}, & \text{if}\quad k\not\equiv b \pmod a;\\ a_{\lfloor\frac{k}{B}\rfloor,l}+\binom{(k-b)/a}{l}z^{\frac{k-b}{a}-l}, & \text{if}\quad k\equiv b \pmod a.\end{cases}\tag{28}$$

On the other hand, in view of Corollary 3, $\gamma_{a,b}(z)$ and $\gamma'_{a,b}(z)$ can be explicitly expressed in terms of the Lerch transcendent, ψ-function and logarithm of the gamma function. This allows us to sum the series in Theorems 7–9 in terms of these functions.

Example 10. Suppose that ω is a non-empty word over the alphabet $\{0,1,\ldots,B-1\}$. Then obviously ω is uniquely defined by its length $|\omega|$ and its size $v_B(\omega)$ which is the value of ω when interpreted as an integer in base B. Let $N_{\omega,B}(k)$ be the number of (possibly overlapping) occurrences of the block ω in the B-ary expansion of k. Note that for every B and ω, $N_{\omega,B}(0)=0$, since the B-ary expansion of zero is the empty word. If the word

ω begins with 0, but $v_B(\omega) \neq 0$, then in computing $N_{\omega,B}(k)$ we assume that the B-ary expansion of k starts with an arbitrary long prefix of 0's. If $v_B(\omega) = 0$ we take for k the usual shortest B-ary expansion of k.

Now we consider equation (27) with $l = 0$, $z = 1$

$$G(1,x) - \frac{1-x^B}{1-x} G(1,x^B) = \frac{x^b}{1-x^a} \tag{29}$$

and for a given non-empty word ω, set $a = B^{|\omega|}$ in (29) and

$$b = \begin{cases} B^{|\omega|}, & \text{if} \quad v_B(\omega) = 0; \\ v_B(\omega), & \text{if} \quad v_B(\omega) \neq 0. \end{cases}$$

Then by (28), it is easily seen that $a_k := a_{k,0} = N_{\omega,B}(k)$, $k = 1, 2, \ldots$, and by Theorem 7, we get another proof of the following statement (see [2, Sections 3, 4.2]).

Corollary 11. *Let ω be a non-empty word over the alphabet $\{0, 1, \ldots, B-1\}$. Then*

$$\sum_{k=1}^{\infty} N_{\omega,B}(k) Q(k,B) = \begin{cases} \gamma_{B^{|\omega|}, v_B(\omega)}(1), & \textit{if} \quad v_B(\omega) \neq 0; \\ \gamma_{B^{|\omega|}, B^{|\omega|}}(1), & \textit{if} \quad v_B(\omega) = 0. \end{cases}$$

By Corollary 3, the right-hand side of the last equality can be calculated explicitly and we have

$$\sum_{k=1}^{\infty} N_{\omega,B}(k) Q(k,B) = \begin{cases} \log\Gamma\left(\frac{v_B(\omega)+1}{B^{|\omega|}}\right) - \log\Gamma\left(\frac{v_B(\omega)}{B^{|\omega|}}\right) - \frac{1}{B^{|\omega|}}\psi\left(\frac{v_B(\omega)}{B^{|\omega|}}\right), & \text{if } v_B(\omega) \neq 0; \\ \log\Gamma\left(\frac{1}{B^{|\omega|}}\right) + \frac{\gamma}{B^{|\omega|}} - |\omega| \log B, & \text{if } v_B(\omega) = 0. \end{cases} \tag{30}$$

Corollary 12. *Let ω be a non-empty word over the alphabet $\{0, 1, \ldots, B-1\}$. Then*

$$\sum_{k=1}^{\infty} \frac{N_{\omega,B}(k) P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}$$
$$= \begin{cases} \gamma_{B^{|\omega|}, v_B(\omega)}(1) - \frac{1}{2B^{|\omega|}}\left(\psi\left(\frac{v_B(\omega)+1}{B^{|\omega|}}\right) - \psi\left(\frac{v_B(\omega)}{B^{|\omega|}}\right)\right), & \textit{if} \quad v_B(\omega) \neq 0; \\ \gamma_{B^{|\omega|}, B^{|\omega|}}(1) - \frac{1}{2B^{|\omega|}}\psi\left(\frac{1}{B^{|\omega|}}\right) - \frac{\gamma}{2B^{|\omega|}} - \frac{1}{2}, & \textit{if} \quad v_B(\omega) = 0. \end{cases}$$

Proof. The required statement easily follows from Theorem 9, Corollary 11 and the equality

$$\int_0^1 \frac{x^{b-1}(1-x)}{1-x^a}\, dx = \sum_{k=0}^{\infty} \left(\frac{1}{ak+b} - \frac{1}{ak+b+1}\right) = \frac{1}{a}\left(\psi\Big(\frac{b+1}{a}\Big) - \psi\Big(\frac{b}{a}\Big)\right).$$

□

From Theorem 7, (27) and (28) with $a = 1$, $l = 0$ we have

Corollary 13. *Let b, B be positive integers with $B > 1$, $z \in \mathbb{C}$, $|z| \le 1$. Then*

$$\gamma_{1,b}(z) = \sum_{k=1}^{\infty} a_k Q(k, B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B} \rfloor} \frac{\varepsilon(k)}{k},$$

where $a_0 = a_1 = \cdots = a_{b-1} = 0$, $a_k = a_{\lfloor \frac{k}{B} \rfloor} + z^{k-b}$, $k \ge b$.

Similarly, from Theorem 9 we have

Corollary 14. *Let b, B be positive integers with $B > 1$, $z \in \mathbb{C}$, $|z| \le 1$. Then*

$$\gamma_{1,b}(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k}{(k+b)(k+b+1)} + \sum_{k=1}^{\infty} a_k \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $a_0 = a_1 = \cdots = a_{b-1} = 0$, $a_k = a_{\lfloor \frac{k}{B} \rfloor} + z^{k-b}$, $k \ge b$.

Example 15. If in Corollary 13 we take $z = 1$, then we get that a_k is equal to the B-ary length of $\lfloor \frac{k}{b} \rfloor$, i. e.,

$$a_k = \sum_{\alpha=0}^{B-1} N_{\alpha,B}\left(\left\lfloor \frac{k}{b} \right\rfloor\right) = L_B\left(\left\lfloor \frac{k}{b} \right\rfloor\right).$$

On the other hand,

$$\gamma_{1,b}(1) = \log b - \psi(b) = \log b - \sum_{k=1}^{b-1} \frac{1}{k} + \gamma$$

and hence we get

$$\log b - \psi(b) = \sum_{k=1}^{\infty} L_B\left(\left\lfloor \frac{k}{b} \right\rfloor\right) Q(k, B). \tag{31}$$

If $b = 1$, formula (31) gives (16). If $b > 1$, then from (31) and (16) we get

$$\log b = \sum_{k=1}^{b-1} \frac{1}{k} + \sum_{k=1}^{\infty} \left(L_B\left(\left\lfloor \frac{k}{b} \right\rfloor\right) - L_B(k)\right) Q(k, B), \tag{32}$$

which is equivalent to [5, Theorem 2.8]. Similarly, from Corollary 14 we obtain (17) and

$$\log b = \sum_{k=1}^{b-1} \frac{1}{k} - \frac{b-1}{2b} + \sum_{k=1}^{\infty} \frac{\left(L_B(\lfloor \frac{k}{b} \rfloor) - L_B(k)\right) P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}. \tag{33}$$

Example 16. Using the fact that for any integer $B > 1$,

$$L_B\left(\left\lfloor \frac{k}{B} \right\rfloor\right) - L_B(k) = -1,$$

from (30), (16) and (32) we get the following rational series for $\log \Gamma(1/B)$:

$$\log \Gamma\left(\frac{1}{B}\right) = \sum_{k=1}^{B-1} \frac{1}{k} + \sum_{k=1}^{\infty} \Big(N_{0,B}(k) - \frac{1}{B} L_B(k) - 1\Big) Q(k, B).$$

Example 17. Substituting $b = 1$, $z = -1$ in Corollary 13 we get the generalized Vacca series for $\log \frac{4}{\pi}$.

Corollary 18. *Let $B \in \mathbb{N}$, $B > 1$. Then*

$$\log \frac{4}{\pi} = \sum_{k=1}^{\infty} a_k Q(k, B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B} \rfloor} \frac{\varepsilon(k)}{k},$$

where

$$a_0 = 0, \qquad a_k = a_{\lfloor \frac{k}{B} \rfloor} + (-1)^{k-1}, \quad k \geq 1. \tag{34}$$

In particular, if B is even, then

$$\log \frac{4}{\pi} = \sum_{k=1}^{\infty} (N_{odd,B}(k) - N_{even,B}(k)) Q(k, B) = \sum_{k=1}^{\infty} \frac{\left(N_{odd,B}(\lfloor \frac{k}{B} \rfloor) - N_{even,B}(\lfloor \frac{k}{B} \rfloor)\right)}{k} \varepsilon(k), \tag{35}$$

where $N_{odd,B}(k)$ (respectively $N_{even,B}(k)$) is the number of occurrences of the odd (respectively even) digits in the B-ary expansion of k.

Proof. To prove (35), we notice that if B is even, then the sequence $\widetilde{a}_k := N_{odd,B}(k) - N_{even,B}(k)$ satisfies recurrence (34). □

Substituting $b = 1, z = -1$ in Corollary 14 with the help of (33) we get the generalized Addison series for $\log \frac{4}{\pi}$.

Corollary 19. *Let $B > 1$ be a positive integer. Then*

$$\log \frac{4}{\pi} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{\left(L_B(\lfloor \frac{k}{2} \rfloor) - L_B(k) + a_k\right) P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where the sequence a_k is defined in Corollary 18. In particular, if B is even, then

$$\log \frac{4}{\pi} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{\left(L_B(\lfloor \frac{k}{2} \rfloor) - 2N_{even,B}(k)\right) P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}.$$

Example 20. For $t > 1$, the generalized Somos constant σ_t is defined by

$$\sigma_t = \sqrt[t]{1\sqrt[t]{2\sqrt[t]{3\cdots}}} = 1^{1/t} 2^{1/t^2} 3^{1/t^3} \cdots = \prod_{n=1}^{\infty} n^{1/t^n}$$

(see [25, Section 3]). In view of the relation [25, Theorem 8]

$$\gamma_{1,1}\left(\frac{1}{t}\right) = t \log \frac{t}{(t-1)\sigma_t^{t-1}}, \tag{36}$$

by Corollary 13 and formula (32) we get

Corollary 21. *Let $B \in \mathbb{N}$, $B > 1$, $t \in \mathbb{R}$, $t > 1$. Then*

$$\log \sigma_t = \frac{1}{(t-1)^2} + \frac{1}{t-1} \sum_{k=1}^{\infty} \left(L_B\left(\left\lfloor \frac{k}{t} \right\rfloor\right) - L_B\left(\left\lfloor \frac{k}{t-1} \right\rfloor\right) - \frac{a_k}{t} \right) Q(k, B),$$

where $a_0 = 0$, $a_k = a_{\lfloor \frac{k}{B} \rfloor} + t^{1-k}$, $k \geq 1$.

In particular, setting $B = t = 2$ we get the following rational series for Somos's quadratic recurrence constant:

$$\log \sigma_2 = 1 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{b_k}{2k(2k+1)},$$

where $b_1 = 3$, $b_k = b_{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2^{k-1}}$, $k \geq 2$.

From (36), (33) and Theorem 9 we find

Corollary 22. *Let $B \in \mathbb{N}$, $B > 1$, $t \in \mathbb{R}$, $t > 1$. Then*

$$\log \sigma_t = \frac{3t-1}{4t(t-1)^2} + \frac{t+1}{2(t-1)} \sum_{k=1}^{\infty} \left(L_B\left(\left\lfloor \frac{k}{t} \right\rfloor\right) - L_B\left(\left\lfloor \frac{k}{t-1} \right\rfloor\right) - \frac{2a_k}{t(t+1)} \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where the sequence a_k is defined in Corollary 21.

In particular, if $B = t = 2$ we get

$$\log \sigma_2 = \frac{5}{8} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{c_k}{2k(2k+1)(2k+2)},$$

where $c_1 = 4$, $c_k = c_{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2^{k-1}}$, $k \geq 2$.

Example 23. The Glaisher-Kinkelin constant is defined by the limit [11, p.135]

$$A := \lim_{n \to \infty} \frac{1^2 2^2 \cdots n^n}{n^{\frac{n^2+n}{2} + \frac{1}{12}} e^{-\frac{n^2}{4}}} = 1.28242712\cdots.$$

Its connection to the generalized Euler constant function $\gamma_{a,b}(z)$ is given by the formula [25, Corollary 4]

$$\gamma'_{1,1}(-1) = \log \frac{2^{11/6} A^6}{\pi^{3/2} e}. \tag{37}$$

By Theorem 9, since

$$\int_0^1 \frac{x(1-x)}{(1+x)^2} \, dx = 3 \log 2 - 2,$$

we have

$$\log A = \frac{4}{9} \log 2 - \frac{1}{4} \log \frac{4}{\pi} + \frac{1}{6} \sum_{k=1}^{\infty} a_{k,1} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where the sequence $a_{k,1}$ is defined by the generating function (26) with $a = b = l = 1$, $z = -1$, or using the recursion (28):

$$a_{0,1} = a_{1,1} = 0, \qquad a_{k,1} = a_{\lfloor \frac{k}{B} \rfloor,1} + (-1)^k (k-1), \quad k \geq 2.$$

Now by Corollary 19 and (33) we get

Corollary 24. *Let $B > 1$ be a positive integer. Then*

$$\log A = \frac{13}{48} - \frac{1}{36} \sum_{k=1}^{\infty} \left(7L_B(k) - 7L_B\left(\left\lfloor \frac{k}{2} \right\rfloor \right) + b_k \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $b_0 = 0$, $b_k = b_{\lfloor \frac{k}{B} \rfloor} + (-1)^{k-1}(6k+3)$, $k \geq 1$.

In particular, if $B = 2$ we get

$$\log A = \frac{13}{48} - \frac{1}{36} \sum_{k=1}^{\infty} \frac{c_k}{2k(2k+1)(2k+2)},$$

where $c_1 = 16$, $c_k = c_{\lfloor \frac{k}{2} \rfloor} + (-1)^{k-1}(6k+3)$, $k \geq 2$.

Using the formula expressing $\frac{\zeta'(2)}{\pi^2}$ in terms of the Glaisher-Kinkelin constant [11, p. 135],

$$\log A = -\frac{\zeta'(2)}{\pi^2} + \frac{\log 2\pi + \gamma}{12},$$

by Corollaries 14, 19 and 24, we get

Corollary 25. *Let $B > 1$ be a positive integer. Then*

$$\frac{\zeta'(2)}{\pi^2} = -\frac{1}{16} + \frac{1}{36} \sum_{k=1}^{\infty} \left(4L_B(k) - L_B\left(\left\lfloor \frac{k}{2} \right\rfloor \right) + c_k \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $c_0 = 0$, $c_k = c_{\lfloor \frac{k}{B} \rfloor} + (-1)^{k-1} 6k$, $k \geq 1$.

Example 26. First we evaluate $\gamma_{2,1}^{(l)}(-1)$ for $l = 0, 1$. From Corollaries 2, 3 and [12, Examples 3.9, 3.15] we have

$$\gamma_{2,1}(-1) = \int_0^1 \int_0^1 \frac{(x-1)\, dxdy}{(1+x^2y^2) \log xy} = \frac{\pi}{4} - 2 \log \Gamma\left(\frac{1}{4}\right) + \log \sqrt{2\pi^3}$$

and

$$\begin{aligned} \gamma_{2,1}'(-1) = &-\frac{1}{4}\Phi(-1,1,3/2) + \frac{1}{2}\Phi(-1,0,3/2) + \frac{1}{2}\frac{\partial \Phi}{\partial s}(-1,0,3/2) \\ &- \frac{\partial \Phi}{\partial s}(-1,-1,3/2) - \frac{\partial \Phi}{\partial s}(-1,0,2) + \frac{\partial \Phi}{\partial s}(-1,-1,2). \end{aligned}$$

The last expression can be evaluated explicitly (see [12, Section 2]) and we get

$$\gamma'_{2,1}(-1) = \frac{G}{\pi} + \frac{\pi}{8} - \log\Gamma\left(\frac{1}{4}\right) - 3\log A + \log\pi + \frac{1}{3}\log 2,$$

or

$$\frac{G}{\pi} = \gamma'_{2,1}(-1) - \frac{1}{2}\gamma_{2,1}(-1) + \frac{1}{4}\log\frac{4}{\pi} + 3\log A - \frac{7}{12}\log 2. \tag{38}$$

On the other hand, by Theorem 9 and (28) we have

$$\gamma_{2,1}(-1) = \frac{\pi}{8} - \frac{1}{4}\log 2 + \sum_{k=1}^{\infty} a_{k,0}\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}, \tag{39}$$

where $a_{0,0} = 0$, $a_{2k,0} = a_{\lfloor\frac{2k}{B}\rfloor,0}$, $k \geq 1$, $a_{2k+1,0} = a_{\lfloor\frac{2k+1}{B}\rfloor,0} + (-1)^k$, $k \geq 0$, and

$$\gamma'_{2,1}(-1) = \frac{\pi}{16} - \frac{1}{4}\log 2 + \sum_{k=1}^{\infty} a_{k,1}\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}, \tag{40}$$

where $a_{0,1} = 0$, $a_{2k,1} = a_{\lfloor\frac{2k}{B}\rfloor,1}$, $k \geq 1$, $a_{2k+1,1} = a_{\lfloor\frac{2k+1}{B}\rfloor,1} + (-1)^{k-1}k$, $k \geq 0$. Now from (38) – (40), (33) and Corollary 19 we get the following expansion for G/π.

Corollary 27. *Let $B > 1$ be a positive integer. Then*

$$\frac{G}{\pi} = \frac{11}{32} + \sum_{k=1}^{\infty}\left(\frac{1}{8}L_B\left(\left\lfloor\frac{k}{2}\right\rfloor\right) - \frac{1}{8}L_B(k) + c_k\right)\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $c_0 = 0$, $c_{2k} = c_{\lfloor\frac{2k}{B}\rfloor} + k$, $k \geq 1$, $c_{2k+1} = c_{\lfloor\frac{2k+1}{B}\rfloor} + \frac{(-1)^{k-1}-1}{2}(2k+1)$, $k \geq 0$.

In particular, if $B = 2$ we get

$$\frac{G}{\pi} = \frac{11}{32} + \sum_{k=1}^{\infty}\frac{b_k}{2k(2k+1)(2k+2)},$$

where $b_1 = -\frac{9}{8}$, $b_{2k} = b_k + k$, $b_{2k+1} = b_k + \frac{(-1)^{k-1}-1}{2}(2k+1)$, $k \geq 1$.

6 Other generalized Euler constants

The purpose of this section is to draw attention to different generalizations of Euler's constant for which many interesting results remain to be discovered.

The simplest way to generalize Euler's constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right), \tag{41}$$

which is related to the digamma function by the equality $\gamma = -\psi(1)$, is to consider for $0 < \alpha \leq 1$,

$$\gamma(\alpha) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+\alpha} - \log n\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+\alpha} - \log(n+\alpha)\right).$$

Tasaka [27] proved that $\gamma(\alpha) = -\psi(\alpha)$. Its connection to the generalized Euler constant function $\gamma_{a,b}(z)$ is given by the formula

$$\gamma(\alpha) + \log\alpha = \gamma_{1,\alpha}(1).$$

Briggs [6] and Lehmer [20] studied the analog of γ corresponding to the arithmetical progression of positive integers $r,\ r+m,\ r+2m, \ldots,\ (r \leq m)$:

$$\gamma(r,m) = \lim_{n\to\infty}\left(H(n,r,m) - \frac{1}{m}\log n\right),$$

where $H(n,r,m) = \sum\limits_{\substack{0<k\leq n,\\ k\equiv r (\mathrm{mod}\ m)}} \frac{1}{k}$. Since $H(n,r,m) = \sum\limits_{0\leq k\leq (n-r)/m} \frac{1}{r+mk}$, it is easily seen that

$$m\gamma(r,m) = \gamma(r/m) - \log m = \gamma_{1,r/m}(1) - \log r.$$

Diamond and Ford [9] studied a family $\{\gamma(\mathcal{P})\}$ of generalized Euler constants arising from the sum of reciprocals of integers sieved by finite sets of primes $\mathcal{P}$. More precisely, if $\mathcal{P}$ represents a finite set of primes, let

$$1_{\mathcal{P}}(n) := \begin{cases} 1, & \text{if} \quad \gcd\left(n, \prod_{p\in\mathcal{P}} p\right) = 1; \\ 0, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad \delta_{\mathcal{P}} := \lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x} 1_{\mathcal{P}}(n).$$

A simple argument shows that $\delta_{\mathcal{P}} = \prod_{p\in\mathcal{P}}(1-1/p)$ and that the generalized Euler constant

$$\gamma(\mathcal{P}) := \lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1_{\mathcal{P}}(n)}{n} - \delta_{\mathcal{P}}\log x\right)$$

exists. Its connection to Euler's constant is given by the formula [9]

$$\gamma(\mathcal{P}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)\left\{\gamma + \sum_{p\in\mathcal{P}}\frac{\log p}{p-1}\right\}.$$

Another generalization of the Euler constant is connected with the well-known limit involving the Riemann zeta function:

$$\gamma = \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right). \tag{42}$$

Expanding the Riemann zeta function into Laurent series in a neighborhood of its simple pole at $s = 1$ gives

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(s-1)^k.$$

Stieltjes [26] pointed out that the coefficients γ_k can be expressed as

$$\gamma_k = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{\log^k j}{j} - \frac{\log^{k+1} n}{k+1} \right), \qquad k = 0, 1, 2, \ldots. \tag{43}$$

(In the case $k = 0$, the first summand requires evaluation of 0^0, which is taken to be 1.) The coefficients γ_k are usually called Stieltjes, or generalized Euler, constants (see [10, 11, 18]. In particular, the zero'th constant $\gamma_0 = \gamma$ is the Euler constant.

Hardy [13] gave an analog of the Vacca series (3) for γ_1 containing logarithmic coefficients:

$$\gamma_1 = \sum_{k=1}^{\infty} (-1)^k \frac{\log(k) \lfloor \log_2(k) \rfloor}{k} - \frac{\log 2}{2} \sum_{k=1}^{\infty} (-1)^k \frac{\lfloor \log_2(2k) \rfloor \lfloor \log_2(k) \rfloor}{k},$$

and Kluyver [16] presented more such series for higher-order constants.

The analog of γ_k corresponding to the arithmetical progression $r, r+m, r+2m, \ldots$ was studied by Knopfmacher [17], Kanemitsu [15], and Dilcher [10]:

$$\gamma_k(r, m) = \lim_{n\to\infty} \left(\sum_{\substack{0<j\le n \\ j\equiv r \ (\mathrm{mod}\ m)}} \frac{\log^k j}{j} - \frac{1}{m} \frac{\log^{k+1} n}{k+1} \right).$$

Another extension of γ_k can be derived from the Laurent series expansion of the Hurwitz zeta function:

$$\zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k(\alpha)}{k!} (s-1)^k.$$

Here $0 < \alpha \le 1$. Since $\zeta(s, 1) = \zeta(s)$, we have $\gamma_k(1) = \gamma_k$. Berndt [4] showed that

$$\gamma_k(\alpha) = \lim_{n\to\infty} \left(\sum_{j=0}^{n} \frac{\log^k (j+\alpha)}{(j+\alpha)} - \frac{\log^{k+1}(n+\alpha)}{k+1} \right),$$

which is equivalent to (43) when $\alpha = 1$. If $k = 0$ and $\alpha = r/m$, $r, m \in \mathbb{N}$, $r \le m$, then

$$\gamma_0(r/m) = m\gamma(r, m) + \log m = \gamma_{1,r/m}(1) - \log(r/m) = \gamma(r/m) = -\psi(r/m).$$

Recently, Lampret [19] considered the zeta-generalized Euler constant function

$$\Upsilon(s) := \sum_{j=1}^{\infty} \left(\frac{1}{j^s} - \int_j^{j+1} \frac{dx}{x^s} \right) \tag{44}$$

and its alternating version

$$\Upsilon^*(s) := \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{1}{j^s} - \int_j^{j+1} \frac{dx}{x^s} \right)$$

defined for $s \geq 0$. The name of the function $\Upsilon(s)$ comes from the fact that $\Upsilon(1) = \gamma$ and that the series $\sum_{j=1}^{\infty} 1/j^s$ defines the Riemann zeta function. Moreover, it is easily seen that $\Upsilon(1) = \gamma_{1,1}(1)$ and $\Upsilon^*(1) = \gamma_{1,1}(-1)$. In [19] it was shown that $\Upsilon(s)$ is infinitely differentiable on $\mathbb{R}^+$ and its k-th derivative $\Upsilon^{(k)}(s)$ can be obtained by termwise k-times differentiation of the series (44):

$$\Upsilon^{(k)}(s) = (-1)^k \sum_{j=1}^{\infty} \left(\frac{\log^k j}{j^s} - \int_j^{j+1} \frac{\log^k x}{x^s}\, dx \right). \tag{45}$$

Setting $s = 1$ in (45) we get the following relation between the zeta-generalized Euler constant function and Stieltjes constants (43):

$$\Upsilon^{(k)}(1) = (-1)^k \gamma_k.$$

The formula (41), as well as (44), can be further generalized to

$$\gamma_f = \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(k) - \int_1^n f(x)\, dx \right)$$

for some arbitrary positive decreasing function f (see [22]). For example, $f_n(x) = \frac{\log^n x}{x}$ gives rise to the Stieltjes constants, and $f_s(x) = x^{-s}$ gives $\gamma_{f_s} = \frac{(s-1)\zeta(s)-1}{s-1}$, where again the limit (42) appears.

There are other generalizations including a two-dimensional version of Euler's constant and a lattice sum form. For a survey of further results and an extended bibliography, see [11, Sections 1.5, 1.10, 2.21, 7.2].

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References

[1] A. W. Addison, A series representation for Euler's constant, *Amer. Math. Monthly* **74** (1967), 823–824.

[2] J.-P. Allouche, J. Shallit, J. Sondow, Summation of series defined by counting blocks of digits, *J. Number Theory* **123** (2007), no. 1, 133–143.

[3] J.-P. Allouche, J. Shallit, Sums of digits and the Hurwitz zeta function, *Analytic Number Theory* (Tokyo, 1988). Lecture Notes in Math., 1434, Springer, Berlin, 1990, pp. 19–30.

[4] B. C. Berndt, On the Hurwitz zeta-function, *Rocky Mount. J. Math.* **2** (1972), 151–157.

[5] B. C. Berndt, D. C. Bowman, Ramanujan's short unpublished manuscript on integrals and series related to Euler's constant, in *Constructive, Experimental and Nonlinear Analysis* (Limoges, 1999), CMS Conf. Proc., 27, Amer. Math. Soc., Providence, RI, 2000, pp. 19–27.

[6] W. E. Briggs, The irrationality of γ or sets of similar constants, *Norske Vid. Selsk. Forh. (Trondheim)* **34** (1961), 25–28.

[7] L. Carlitz, Advanced problem 5601, Solution by Heiko Harborth, *Amer. Math. Monthly* **76** (1969), 567–568.

[8] E. Catalan, Sur la constante d'Euler et la fonction de Binet, *J. Math. Pures Appl.* **1** (1875), 209–240.

[9] H. G. Diamond, K. Ford, Generalized Euler constants, *Math. Proc. Camb. Phil. Soc.* **145** (2008), 27–41.

[10] K. Dilcher, Generalized Euler constants for arithmetical progressions, *Math. Comp.* **59** (1992), 259–282.

[11] S. Finch, *Mathematical Constants,* Cambridge University Press, Cambridge, 2003.

[12] J. Guillera, J. Sondow, Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent, *Ramanujan J.* **16** (2008), no. 3, 247–270.

[13] G. H. Hardy, Note on Dr. Vacca's series for γ, *Quart. J. Pure Appl. Math.* **43** (1912), 215–216.

[14] Kh. Hessami Pilehrood, T. Hessami Pilehrood, Arithmetical properties of some series with logarithmic coefficients, *Math. Z.* **255** (2007), 117–131.

[15] S. Kanemitsu, On evaluation of certain limits in closed form, *Théorie des Nombres,* Proc. 1987 Québec conf., ed. J.-M. De Koninck and C. Levesque, Gruyter, 1989, pp. 459–474.

[16] J. C.Kluyver, On certain series of Mr. Hardy, *Quart. J. Pure Appl. Math.* **50** (1927), 185–192.

[17] J. Knopfmacher, Generalised Euler constants, *Proc. Edinburgh Math. Soc.* **21** (1978), 25–32.

[18] R. Kreminski, Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants, *Math. Comp.* **72**, no. 243, 1379–1397.

[19] V. Lampret, An accurate approximation of zeta-generalized-Euler-constant functions, *Cent. Eur. J. Math.* **8** (2010), no. 3, 488–499.

[20] D. H. Lehmer, Euler constants for arithmetical progressions, *Acta Arith.* **27** (1975), 125–142.

[21] S. Ramanujan, *Notebooks (*2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[22] J. Sándor, On generalized Euler constants and Schlömilch-Lemonnier type inequalities, *J. Math. Anal. Appl.* **328** (2007), 1336–1342.

[23] N. J. A. Sloane, *The on-line encyclopedia of integer sequences* (2005), published online at http://www.research.att.com/ njas/sequences/

[24] J. Sondow, New Vacca-type rational series for Euler's constant and its "alternating" analog $\log\frac{4}{\pi}$, *Additive Number Theory, Festschrift in Honor of the Sixtieth Birthday of M. B. Nathanson* (D. Chudnovsky and G. Chudnovsky, eds.), Springer, 2010. arXiv:0508042[math.NT]

[25] J. Sondow, P. Hadjicostas, The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant, *J. Math. Anal. Appl.* **332** (2007), no. 1, 292–314.

[26] T. J. Stieltjes, *Correspondance d'Hermite et de Stieltjes,* V.1 and 2. Gauthier-Villars, Paris, 1905.

[27] T. Tasaka, Note on the generalized Euler constants, *Math. J. Okayama Univ.* **36** (1994), 29–34.

[28] G. Vacca, A new series for the Eulerian constant $\gamma = .577\cdots$, *Quart. J. Pure Appl. Math.* **41**(1910), 363–364.

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