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On the Coefficients of the Asymptotic Expansion of $n!$

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Abstract

Applying a theorem of Howard to a formula recently proved by Brassesco and Méndez, we derive new simple explicit formulas for the coefficients of the asymptotic expansion of the sequence of factorials.

1 Introduction

It is well known that the factorial of a positive integer n has the asymptotic expansion

$$n! \sim n^n e^{-n} \sqrt{2\pi n} \sum_{k\geq 0} \frac{a_k}{n^k}, \tag{1}$$

known as Stirling's formula (see, e.g., [1, 3, 4]). The coefficients a_k in this series are usually called the Stirling coefficients [1, 6] (Sloane's A001163 and A001164) and can be computed from the sequence b_k defined by the recurrence relation

$$b_k = \frac{1}{k+1}\left(b_{k-1} - \sum_{j=2}^{k-1} j b_j b_{k-j+1}\right),\ b_0 = b_1 = 1, \tag{2}$$

since $a_k = (2k+1)!!b_{2k+1}$ [3, 4]. Here $(2k+1)!! = (2k+1)\cdot(2k-1)\cdots 5\cdot 3\cdot 1$ is the double factorial. It was pointed out by Paris and Kaminski [6] that "There is no known closed-form representation for the Stirling coefficients". However there is a closed-form expression that involves combinatorial quantities due to Comtet [5]:

$$a_k = \sum_{j=0}^{2k} (-1)^j \frac{d_3(2k+2j, j)}{2^{k+j}(k+j)!}, \tag{3}$$

where $d_3(p, q)$ is the number of permutations of p with q permutation cycles all of which are ≥ 3 (Sloane's A050211). Brassesco and Méndez [7] proved in a recent paper that

$$a_k = \sum_{j=0}^{2k} (-1)^j \frac{S_3(2k+2j, j)}{2^{k+j}(k+j)!}, \tag{4}$$

where $S_3(p, q)$ denotes the 3-associated Stirling numbers of the second kind (Sloane's A059022). We show that the Stirling coefficients a_k can be expressed in terms of the conventional Stirling numbers of the second kind (Sloane's A008277). A corollary of this result is an explicit, exact expression for the Stirling coefficients.

2 The formulas for coefficients

One of our main results is the following:

Theorem 1. *The Stirling coefficients have a representation of the form*

$$a_k = \frac{(2k)!}{2^k k!} \sum_{i=0}^{2k} \binom{k+i-1/2}{i} \binom{3k+1/2}{2k-i} 2^i \sum_{j=0}^{i} \binom{i}{j} (-1)^j j! \frac{S(2k+i+j, j)}{(2k+i+j)!}, \tag{5}$$

where $S(p, q)$ denotes the Stirling numbers of the second kind.

From the explicit formula

$$S(p, q) = \frac{1}{q!} \sum_{l=0}^{q} (-1)^l \binom{q}{l} (q-l)^p,$$

we immediately obtain our second main result.

Corollary 2. *The Stirling coefficients have an exact representation of the form*

$$a_k = \frac{(2k)!}{2^k k!} \sum_{i=0}^{2k} \binom{k+i-1/2}{i} \binom{3k+1/2}{2k-i} 2^i \sum_{j=0}^{i} \binom{i}{j} \frac{(-1)^j}{(2k+i+j)!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} (j-l)^{2k+i+j}. \tag{6}$$

To prove Theorem 1 we need some concepts. Let $r \geq 0$ and $a_r \neq 0$, let $F(x) = \sum_{j \geq r} a_j x^j / j!$ be a formal power series. The potential polynomials $F_n^{(z)}$ in the variable z are defined by the exponential generating function

$$\left(\frac{a_r x^r / r!}{F(x)} \right)^z = \sum_{n \geq 0} F_n^{(z)} \frac{x^n}{n!}. \tag{7}$$

For $r \geq 1$, the exponential Bell polynomials $B_{n,i}(0,\ldots,0,a_r,a_{r+1},\ldots)$ in an infinite number of variables $a_r,a_{r+1},\ldots$ can be defined by

$$(F(x))^i = i!\sum_{n\geq 0} B_{n,i}(0,\ldots,0,a_r,a_{r+1},\ldots)\frac{x^n}{n!}. \tag{8}$$

The following theorem is due to Howard [2].

Theorem 3. *If $F_n^{(z)}$ is defined by (7) and $B_{n,i}$ is defined by (8), then*

$$F_n^{(z)} = \sum_{i=0}^{n}(-1)^i\binom{z+i-1}{i}\binom{z+n}{n-i}\left(\frac{r!}{a_r}\right)^i\frac{n!i!}{(n+ri)!}B_{n+ri,i}(0,\ldots,0,a_r,a_{r+1},\ldots). \tag{9}$$

Now we prove Theorem 1.

Proof of Theorem 1. Brassesco and Méndez showed that if

$$G(x) = 2\frac{e^x-x-1}{x^2} = 2\sum_{j\geq 0}\frac{x^j}{(j+2)!}, \tag{10}$$

then

$$a_k = \frac{1}{2^k k!}\partial^{2k}\left(G^{-\frac{2k+1}{2}}\right)(0), \tag{11}$$

where $\partial^k f$ denotes the kth derivative of a function f. Define the polynomials $G_n^{(z)}$ in the variable z by the following exponential generating function:

$$\left(\frac{1}{2}\frac{x^2}{e^x-x-1}\right)^z = \sum_{j\geq 0} G_j^{(z)}\frac{x^j}{j!}. \tag{12}$$

Inserting $z=\frac{2k+1}{2}$ into this expression gives

$$\sum_{j\geq 0} G_j^{\left(\frac{2k+1}{2}\right)}\frac{x^j}{j!} = \left(\frac{1}{2}\frac{x^2}{e^x-x-1}\right)^{\frac{2k+1}{2}} = \left(2\frac{e^x-x-1}{x^2}\right)^{-\frac{2k+1}{2}} = G^{-\frac{2k+1}{2}}(x). \tag{13}$$

On the other hand we have by series expansion

$$G^{-\frac{2k+1}{2}}(x) = \sum_{j\geq 0}\partial^j\left(G^{-\frac{2k+1}{2}}\right)(0)\frac{x^j}{j!}. \tag{14}$$

Equating the coefficients in (13) and (14) gives

$$\partial^j\left(G^{-\frac{2k+1}{2}}\right)(0) = G_j^{\left(\frac{2k+1}{2}\right)} = G_j^{\left(k+\frac{1}{2}\right)}.$$

Now by comparing this with (11) yields

$$a_k = \frac{1}{2^k k!}G_{2k}^{\left(k+\frac{1}{2}\right)}. \tag{15}$$

Putting $r = 2$ an $a_r = a_{r+1} = \ldots = 1$ into the formal power series $F(x) = \sum_{j\geq r} a_j x^j/j!$ gives $F(x) = e^x - x - 1$. And therefore the generated potential polynomials are

$$\left(\frac{x^2/2!}{e^x - x - 1}\right)^z = \left(\frac{1}{2}\frac{x^2}{e^x - x - 1}\right)^z = \sum_{j\geq 0} G_j^{(z)} \frac{x^j}{j!}.$$

According to Howard's theorem we find

$$G_n^{(z)} = \sum_{i=0}^{n} (-1)^i \binom{z+i-1}{i} \binom{z+n}{n-i} 2^i \frac{n!i!}{(n+2i)!} B_{n+2i,i}(0,1,1,\ldots). \tag{16}$$

Now we derive an expression for the exponential Bell polynomials $B_{n,i}(0,1,1,\ldots)$ in terms of the Stirling numbers of the second kind:

$$\begin{aligned} i! \sum_{n\geq 0} B_{n,i}(0,1,1,\ldots) \frac{x^n}{n!} &= (F(x))^i = (e^x - x - 1)^i \\ &= \left(-x + \sum_{l\geq 1} \frac{x^l}{l!}\right)^i = \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} x^{i-j} \left(\sum_{l\geq 1} \frac{x^l}{l!}\right)^j \\ &= \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} x^{i-j} j! \sum_{n\geq 0} S(n,j) \frac{x^n}{n!} \\ &= \sum_{n\geq 0} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} j! S(n,j) \frac{x^{n+i-j}}{n!} \\ &= i! \sum_{n\geq 0} \left\{ \frac{n!}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} j! \frac{S(n-i+j,j)}{(n-i+j)!} \right\} \frac{x^n}{n!}. \end{aligned}$$

Hence

$$B_{n,i}(0,1,1,\ldots) = \frac{n!}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} j! \frac{S(n-i+j,j)}{(n-i+j)!}. \tag{17}$$

Thus we obtain

$$G_n^{(z)} = \sum_{i=0}^{n} \binom{z+i-1}{i} \binom{z+n}{n-i} 2^i n! \sum_{j=0}^{i} \binom{i}{j} (-1)^j j! \frac{S(n+i+j,j)}{(n+i+j)!}. \tag{18}$$

Substituting $z = k + 1/2$ and $n = 2k$ into this expression yields

$$G_{2k}^{\left(k+\frac{1}{2}\right)} = \sum_{i=0}^{2k} \binom{k+i-1/2}{i} \binom{3k+1/2}{2k-i} 2^i (2k)! \sum_{j=0}^{i} \binom{i}{j} (-1)^j j! \frac{S(2k+i+j,j)}{(2k+i+j)!}, \tag{19}$$

hence by (15) we finally have

$$a_k = \frac{(2k)!}{2^k k!} \sum_{i=0}^{2k} \binom{k+i-1/2}{i} \binom{3k+1/2}{2k-i} 2^i \sum_{j=0}^{i} \binom{i}{j} (-1)^j j! \frac{S(2k+i+j,j)}{(2k+i+j)!}. \tag{20}$$

This completes the proof of the theorem. □

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